FILOMAT (Niš) 11 (1997), 19–28

LIMITATIONS FOR THE NON-SYMMETRIC PART OF LATTICE CONVEX n-GONS WITH MINIMUM L_∞ DIAMETER

Dragan M. Acketa, Snežana Matić-Kekić and Joviša D. Žunić*

Abstract. Lattice convex n-gon ($n = 4k + b$) with minimum L_∞ diameter $MD(n)$ can be constructed as the Minkowski sum of a centrally symmetric lattice convex $4k$-gon and a non-symmetric part, so called Basic b-tuple.

This paper investigates the conditions by which a family of Basic b-tuples can and cannot be used to build optimum lattice convex n-gons for large classes of n. Solutions for five special small values of n are presented. It has been formerly shown that seventeen suitably chosen families of Basic b-tuples with $b \leq 11$ can cover all the remaining values.

1. Introduction

A *lattice convex polygon* (shortly *l.c.p.*) is a polygon, all the vertices of which are points on the integer grid and all the interior angles of which are strictly less than π radians. The L_∞ *diameter* of a lattice convex polygon is the minimum edge size of the enscribed lattice square with the edges parallel to the coordinate axes.

Let Δ_x and Δ_y of an edge denote the absolute values of differences of x- and y-coordinates of its endpoints. Edges od a l.c.p. can be naturally partitioned into four *arcs* that are separated from each other by the vertices (edges) corresponding to the maximum or minimum, x- or y-coordinates (the notion of arc has been made more precise in [2]).

Given an edge e of a l.c. polygon, the *edge slope* of e denotes the fraction Δ_x/Δ_y if e belongs to the north-east or south-west arc and the fraction Δ_y/Δ_x otherwise. The *bd-length* of e (shortly: $bdl(e)$ or only bdl) denotes the sum $\Delta_x + \Delta_y$; this is the edge length in the sense of the L_∞ (Manhattan, block distance) metrics.

Received August 8, 1996

1991 *Mathematics Subject Classification:* 52A43, 68C25, 10A30

Key words and phrases: lattice geometry, combinatorial optimization

$LS(\Delta_x, \Delta_y)$ will denote a lattice square with edge slopes Δ_y/Δ_x.

If the corresponding arcs of some two l.c. polygons P_1 and P_2 have no common edge slopes, then there exists the *Minkowski sum* of P_1 and P_2, which is the uniquely determined third l.c.p. P_3. Each arc of P_3 includes all the edges of the corresponding arcs of P_1 and P_2, sorted so that the convexity condition is preserved.

This paper is related to the problem of determining a lattice convex $n-$gon with the minimum L_∞ diameter $MD(n)$. This problem was primarily formulated in [4]; our main results related to it can be found in the papers [1], [2] and [3].

A sequence $P(j)$, $j = 1, 2, ...$ of optimum l.c. $v(j)-$gons is defined in the following manner: given a natural number j, each arc of $P(j)$ contains all the possible edge slopes Δ_y/Δ_x satisfying $\Delta_x + \Delta_y \leq j$, where Δ_y and Δ_x are relatively prime natural numbers, with the additional possibility that $\Delta_y/\Delta_x = 0/1$. In subsequent text, given the number n of edges of the l.c. polygon, which is to be determined, t will denote the natural number with the property $v(t-1) \leq n < v(t)$.

Greedy lower bound (shortly: $gdlb(n)$) for $MD(n)$ is defined as follows:

$$gdlb(n) \quad = \quad \text{diameter of } P(t-1) \quad + \quad \lceil \frac{(n - v(t-1)) \cdot t}{4} \rceil .$$

The expression for $gdlb(n)$ can be derived from the observation that the sum of bd-lengths of edges of a l.c. $n-$gon P is equal to the perimeter of the minimum enscribed rectangle with sides parallel to the coordinate axes. Namely, due to the convexity of P, the sides of this rectangle are exactly covered by the projections of the edges of P; it is assumed that the projections are made onto the sides that are not "hidden" by P.

The word "greedy" in the name of the bound is due to the fact that a greedy procedure for determining it is found. The weight function which is being minimized is the total edge length in the sense of the L_∞ metrics. In accordance with this, the notions "greedy approach" and "greedy choice" are used throughout the paper.

The edges of an optimum l.c.p. P with the property that their edge slopes are *not* used in each one of the four arcs constitute the *Basic b-tuple*. The remaining edges of P constitute the *Initial polygon*. The polygon P is the Minkowski sum of the Initial polygon and the Basic b-tuple, while the Initial polygon itself can be represented as the Minkowski sum of distinct squares $LS(\Delta_x, \Delta_y)$. The main problem with constructions for n mod 4 $\neq 0$ is to find a suitable Basic b-tuple, since this notion includes a deviation from the greedy approach.

It has been shown in [2] and [3] that $MD(n) \leq 1 + gdlb(n)$. Explicit constructions for l.c. n-gons reaching $MD(n)$ have been provided, as well as the proofs that
$MD(n) > gdlb(n)$ with six exceptional cases.

It is hoped that this paper elucidates and helps better understanding of some details related to the construction described in [2]. In particular, necessity for partitioning the constructions of optimum solutions into a considerably large number of cases is explained and the roles of *families* of Basic b-tuples are clarified.

2. Limitations for usage of Basic b-tuples

The limitations considered here are mainly related to the values of n that are close to members of the sequence $v(t)$.

2.1. Allowed and used tolerance. Basic b-tuples B allow some deviations from the greedy choice of edges. Two functions, $AT(n)$ and $UT(B)$, are related to the measure of this deviation.

The function $AT(n)$ (*allowed tolerance*) is equal to the rounding error in the formula for $gdlb(n)$ and is given by the expression $AT(n) = 4 \cdot \lceil \frac{n \cdot t}{4} \rceil - n \cdot t$. This function is the measure of allowed deviation from the greedy choice under the assumption that $gdlb(n)$ is reached. Allowed tolerance $AT(n)$ is a number from the set $\{0, 1, 2, 3\}$ that has the following geometrical sense: The minimum possible sum of bd-lengths of edges of a l.c. n-gon is equal to $4 \cdot gdlb(n) - AT(n)$. Thus the sum of bd-lengths of edges may be greater for $AT(n)$ than this theoretical minimum so that the l.c. n-gon still has the minimum possible diameter $gdlb(n)$.

The function $UT(B)$ (*used tolerance*) is the measure of deviation from the greedy choice corresponding to a Basic b-tuple B. Namely, when choosing the edges for B, it is optimal to use the edges with $bdl = t$:

Each edge e of B with $bdl(e) = t + i$, $i = 1, 2, ...$ corresponds to a deviation of size i from the greedy choice of edges. None of the n edges chosen in a greedy manner would have a bd-length greater than t, so the use of the edge e makes the minimum sum of bd-lengths by i greater.

On the other hand, a greedy choice of n edges also assumes that all possible edges with $bdl < t$ are included into B. Suppose that some $a(B, \Delta_x, \Delta_y)$ edges with edge slope Δ_y/Δ_x satisfying $\Delta_x + \Delta_y < t$ are used in B. This means that some $4 - a(B, \Delta_x, \Delta_y)$ edges with the same edge slope cannot be used at all. It would be optimal to substitute these edges by the same number of edges with $bdl = t$. Such a substitution of edges would make the minimum sum of bd-lengths by $(4-a(B,\Delta_x,\Delta_y))\cdot(t-(\Delta_x+\Delta_y))$ greater. Thus it holds:

$$UT(B) = \sum_{\substack{e\in B\\ bdl(e)>t}} (bdl(e)-t) + \sum_{\substack{\Delta_y/\Delta_x \text{ in } B\\ \Delta_x+\Delta_y<t}} (4-a(B,\Delta_x,\Delta_y))\cdot(t-(\Delta_x+\Delta_y)) ,$$

where e is an edge of B, Δ_y/Δ_x is an edge slope used in B, while $a(B,\Delta_x,\Delta_y)$ from the set $\{1,2,3\}$ denotes the number of arcs of B that contain an edge with the edge slope Δ_y/Δ_x.

We cite the obvious statement [3] that the used tolerance must not be greater than the allowed one:

Statement 1. If a Basic b-tuple B is used for the construction of a lattice convex n-gon with diameter equal to $gdlb(n)$, then the inequality $UT(B) \leq AT(n)$ is satisfied.

2.2. Notion of gaps. At first glance, it seems as if only three Basic b-tuples B (say, for $b = 1,2,3$) may be sufficient for all the constructions having n mod $4 \neq 0$. However, no fixed B can be in accordance with the requirements for general optimum solutions. Namely, greedy choice of edges with small bd-lengths requires bd-lengths of edges of B to be close to t, where $n \in (v(t-1), v(t))$. Therefore, fixed Basic b-tuples are replaced by *families* of Basic b-tuples depending on t.

A family $\{B_t\}$ of Basic b-tuples is *planned to cover* (the constructions of l.c. n-gons reaching $gdlb(n)$ with) all n with the same non-zero value of n mod 4. However, distinct families of Basic b-tuples are necessary depending on t mod 4. This follows from the fact that the allowed tolerance depends on $(n \cdot t)$ mod 4.

On the lower level, a member B_t from the family is planned to cover the corresponding values of n within a fixed interval $(v(t-1), v(t))$. These values constitute the set

$$Plan(B_t) = \{v(t-1) + (b \bmod 4) + 4\cdot i \quad | \quad i = 0,1,...,\varphi(t)-1\} ,$$

where $\varphi(t)$ denotes the Euler function; note that $v(t) = v(t-1)+4\cdot\varphi(t)$.

Remarks. The largest value of $Plan(B_t)$ is equal to $v(t)-4+(b \bmod 4)$. Multiple applications of the same Basic b-tuple B_t (for various n from $Plan(B_t)$) can be achieved by varying the number of edges of the underlying Initial polygon.

A family $\{B_t\}$ of Basic b-tuples is said to *leave some* **gaps** if it cannot be used for constructions of l.c. n-gons having diameter equal to $gdlb(n)$, with *all* planned values of n. Thus the gaps which are adjoined to a family are just those of the planned values of n, with which the family "fails".

Remark. If a gap is equal to a value of n which requires at least $1 + gdlb(n)$ edges, then the family $\{B_t\}$ covers n (does not leave a gap at n) provided that it can be used for constructions of l.c. n-gons having diameter equal to $1 + gdlb(n)$.

There exist three kinds of gaps: *left* , *right* and *small.* Left and right gaps are related to the left-hand side and right-hand side of the interval $(v(t-1), v(t))$.

Small gaps are some values of $n \in Plan(B_t)$, where t is so small that B_t is not well-defined. Namely, when substituting some small values of t into an edge slope (expressed as a function of t) of B_t, it may occur that either a negative edge slope is obtained or two edges with the same edge slope appear within the same arc. Therefore, *ad hoc* constructions of Basic b-tuples are necessary for these values of n.

Given the Basic b-tuple B_t, let e_1, e_2, e_3 denote the number of its edges having $\Delta_x + \Delta_y$ smaller than, equal to and greater than t, respectively. The corresponding three numbers of distinct edge *slopes* Δ_y/Δ_x used in B_t are denoted by s_1, s_2, s_3.

2.3. Left gaps. Given the Basic b-tuple B_t, let P^*_{t-1} denote the Minkowski sum of all squares $LS(\Delta_x, \Delta_y)$ satisfying that $\Delta_x + \Delta_y < t$ and that Δ_y/Δ_x is not an edge slope of B_t. Since B_t has s_1 edge slopes with $\Delta_x + \Delta_y < t$, it follows that P^*_{t-1} has $v(t-1) - 4s_1$ edges. Then the Minkowski sum of P^*_{t-1} and B_t is an optimum l.c. n-gon $O_{\min}$ with $n = v(t-1) - 4s_1 + b$. This is the smallest value of n that can be covered by using B_t. If it is greater than the smallest value $v(t-1) + (b \bmod 4)$ from $Plan(B_t)$, then B_t is said to leave *left gaps.* It can be also said that the *family* $\{B_t\}$ *leaves* left gaps in such a case, since the values of b and s_1 are the same for all its members.

We prove the following lemma by equalizing the smallest planned value of n and the smallest covered one:

Lemma 1. *A family $\{B_t\}$ of Basic b-tuples does not leave left gaps iff its members satisfy the equality $b = 4 \cdot s_1 + (b \bmod 4)$.*

If the left gaps exist, then they constitute the interval

$Left = [v(t-1) + (b \bmod 4),\ v(t-1) + (b \bmod 4) + 4,\ \dots,\ v(t-1) + b - 4s_1 - 4]$.

One can briefly say that left gaps arise in situations when the number b is too large. For example, if B_t does not use edges with $bdl < t$, then a left gap occurs whenever $b > b \bmod 4$.

2.4. Right gaps. We observe that there exist $\varphi(t) - s_2$ edge slopes Δ_y/Δ_x that are not used in B_t and which satisfy that $\Delta_x + \Delta_y = t$. Their corresponding squares $LS(\Delta_x, \Delta_y)$ are the only Minkowski summands which are allowed to be added to the n-gon $O_{\min}$ (that has been introduced in 2.3) and which are in accordance with the greedy minimization.

This implies that the largest value of n that can be covered by using B_t is equal to $(v(t-1) - 4s_1 + b) + 4(\varphi(t) - s_2)$. If it is smaller than the largest value $v(t-1) + (b \bmod 4) + 4(\varphi(t) - 1)$ from $Plan(B_t)$, then the b-tuple B_t (also the family $\{B_t\}$) is said to leave *right gaps.*

By equalizing the largest planned value of n and the largest covered one, we have:

Lemma 2. *A family $\{B_t\}$ of Basic b-tuples does not leave right gaps iff its members satisfy the equality $b = 4 \cdot (s_1 + s_2) + (b \bmod 4) - 4$.*

If the right gaps exist, then they constitute the interval

$Right = [v(t) - 4 \cdot (s_1 + s_2) + b + 4, ..., v(t) - 8 + (b \bmod 4), v(t) - 4 + (b \bmod 4)]$.

Right gaps arise in the situations when the number b is too small w.r.t. the maximum possible number of edges of the Initial polygon. In other words, the number of distinct edge slopes with $bdl \leq t$ used in Basic b-tuples is too large w.r.t. the number b.

Consequence of Lemmae 1 and 2: Members of a family $\{B_t\}$ of Basic b-tuples, which do not leave neither left nor right gaps, satisfy that $s_2 = 1$. This is a very restrictive constraint, which often implies necessity to solve the gap problem for the left and for the right gaps separately.

2.5. Upper bounds for number of edges of Basic b-tuples. There are $\varphi(t)$ distinct edge slopes having $\Delta_x + \Delta_y = t$. All these edge slopes and at most three edges with each one of them may be used within the Basic b-tuple B_t. Namely, the inequality with allowed tolerance (Statement 1) will not be violated, since none of these edges contributes to $UT(B_t)$. If there are no additional constraints, an upper bound for the number of edges of B_t should contain the summand $3 \cdot \varphi(t)$, which may be infinitely large.

However, if it is additionally required that B_t does not leave at least one kind of gaps (left or right), then there exists a fixed upper bound for the number of its edges:

Lemma 3. *The upper bounds for the number of edges of those Basic b-tuples, which do not leave either left gaps, or right gaps, or both kinds of gaps, are given in Table 1:*

b mod 4	max $AT(n)$	Upper bound for NO LEFT gaps	Upper bound for NO RIGHT gaps	Upper bound for NO GAPS
1	3	13	21	9
2	2	10	14	6
3	3	15	15	11

Table 1.

Proof. According to Lemmae 1 and 2, the maximization of the number of edges of a b-tuple B_t, which does not leave left and right gaps, respectively, requires the values of s_1 and s_1+s_2, respectively, to be as large as possible.

Statement 1 implies that $UT(B_t) \leq AT(n)$. Maximization of s_1 implies that the whole accessible tolerance should be used so that $AT(n)$ edge slopes having
$\Delta_y + \Delta_x = t-1$ and with three edges each should be used in B_t. Thus maximum possible values of s_1 and e_1 are equal to $AT(n)$ and $3 \cdot AT(n)$, respectively.

Using Lemma 1, the upper bound for the number of edges of a b-tuple which does not leave left gaps is equal to $4 \cdot AT(n) + (b \bmod 4)$. The proof for the left gaps is completed (the values 13, 10, 15 in Table 1 are found) by substituting the maximum possible values of $AT(n)$.

When the right gaps are considered, the inequality $b \leq 3 \cdot (s_1 + s_2) + e_3$ is used; it easily follows from the fact that B_t can contain at most three edges with the same edge slope. On the other hand, the maximum possible number of edges of the Initial polygon is equal to $v(t) - 4 \cdot (s_1 + s_2)$.

Summing up the last two expressions, ensues that the maximum possible number n of edges of a l.c. n-gon which is being constructed – is equal to $v(t) - (s_1 + s_2) + e_3$. If there are no right gaps, then this value should be equal to $v(t) - 4 + (b \bmod 4)$, which implies that $s_1 + s_2 = e_3 + 4 - (b \bmod 4)$. Thus the maximization of $s_1 + s_2$ requires the maximization of e_3.

The inequality $UT(B_t) \leq AT(n)$ will be used once more to reach the last goal; it implies that $e_3 \leq AT(n)$. The equality is satisfied iff B_t uses exactly $AT(n)$ edges having $\Delta_y + \Delta_x = t+1$. Thus the maximum possible value of $s_1 + s_2$ is equal to $AT(n) + 4 - (b \bmod 4)$.

This maximum value is substituted into the expression given in Lemma 2. This gives that the upper bound for the number of edges of a b-tuple which does not leave right gaps is equal to

$$4 \cdot (\ AT(n) + 4 - (b \bmod 4)\ ) - 4 + (b \bmod 4) = 12 + 4 \cdot AT(n) - 3 \cdot (b \bmod 4)\ .$$

The proof for the right gaps is completed (the values 21, 14, 15 in Table 1 are found) by substituting the maximum possible values of $AT(n)$.

Consequence of Lemmae 1 and 2 finally implies that B_t leaving no gaps has at most three edges having $\Delta_y + \Delta_x = t$. According to the above discussion, maximum possible values of e_1 and e_3 are equal to $3 \cdot AT(n)$ and $AT(n)$, respectively. Since at most one of these maximizations can be performed at the same time (both of them use the same tolerance $UT(B_t) = AT(n)$), it is preferable to maximize e_1. Such a maximization leads to the upper bound $3 + 3 \cdot AT(n)$ for the number b. Substituting the maximum values of $AT(n)$ in the last expression and taking into account the required values for b mod 4, the last column (with values 9, 6, 11) of Table 1 is derived. $\square$.

Remarks. Lemma 3 is the first step in proving that a finite number of families of Basic b-tuples is sufficient for a *general* construction of optimum l.c. n-gons (valid for all n).

Although the largest number in Table 1 is equal to 21, it turned out that Basic b-tuples with $b \leq 11$ were sufficient to cover all the cases.

Basic b-tuples with large values of b (that by Lemma 3 necessarily leave both left and right gaps) may be also of some interest, e.g., for the constructions of optimum l.c. n-gons with the largest possible degree of non-symmetry.

2.6. Complementary families of Basic b-tuples. A family of Basic b-tuples which does not leave gaps is sufficient to cover the construction of optimum l.c. n-gons with all planned values of n. The problem of gaps is overcome in the following manner: If a family of Basic b-tuples leaves only left (right) gaps, then another (auxiliary) family with the same set of planned values of n is found, which leaves only right (left) gaps.

Let be given a pair of complementary families of Basic b_l-tuples, which leave left gaps and Basic b_r-tuples, which leave right gaps. Then, generally speaking, it might occur that the planned values in the following subinterval are not covered:

$[v(t-1) + (b_l \bmod 4) + 4,\ v(t-1) + (b_l \bmod 4) + 8, \ldots,\ v(t-1) + b_l - 8,\ v(t-1) + b_l - 4]$.

Namely, it might occur that an optimum solution for $n = v(t-1) + (b_l \bmod 4)$ cannot be augmented by adding some $LS(\Delta_x, \Delta_y)$ with $\Delta_x + \Delta_y = t$, since there do not exist such "free" lattice squares (the edge slopes of which are not used in the Basic b_r-tuple). On the other hand, it is necessarily true that all the planned values in the following subinterval can be covered by diminishing Initial polygon corresponding to
$n = v(t) - 4 + (b_l \bmod 4)$:

$[v(t-1)+b_l,\ v(t-1)+b_l+4, ..., v(t)-12+(b_l \bmod 4),\ v(t)-8+(b_l \bmod 4)]$.

Each Basic b-tuple (and consequently a family of Basic b-tuples), which uses s distinct edge slopes, has its *dual* $(4s-b)$-tuple – it is defined to use the same edge slopes, but in the complementary arcs. Thus the Minkowski sum of two mutually dual tuples is equal to the Minkowski sum of s $LS(\Delta_x, \Delta_y)$'s, which correspond to the used edge slopes Δ_y/Δ_x. However, the dual of a Basic b-tuple cannot generally be used as a Basic $(4s-b)$-tuple for another optimum construction. The main reason lies in the fact that the dualization rapidly increases the used tolerance whenever the edge slopes with bd-lengths different from t are used. In particular, if more than one such edge slope is used, then dualization never succeeds.

As a consequence, one cannot translate the problem of left gaps to the right ones or vice versa.

2.7. Small gaps. A collection of 17 families $\{B_j(k)\}$, $1 \le j \le 17$ of b-tuples was presented in [2] (the parameter k is linearly related to t). These families are sufficient to cover all natural numbers $n \ge 3$, which are not divisible by 4, except for the values $n = 3, 7, 9, 13, 15$. It is exactly these five values that are small gaps; they arise from deffects of $B_{12}(0)$, $B_{14}(0)$, $B_7(0)$, $B_7(0)$ and $B_{17}(0)$, respectively.

These values of n can be covered by *ad hoc* constructed Basic b-tuples T_3, T_7, T_9, T_{13} and T_{15}, that are shown in Figure 1.

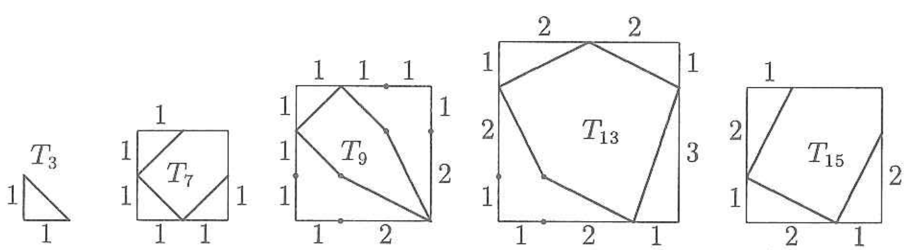

Figure 1. Basic b-tuples for small gaps

Refrences

[1] Acketa, D. M., Žunić, J., *On the maximal number of edges of digital convex polygons included into an $m \times m$ – grid*, Journal of Combin. Theory - Series A, Vol 69, No 2, 1995, 358-368.

[2] Matić-Kekić, S., Acketa, D. M., Žunić, J. D., *An exact construction of digital convex polygons with minimal diameter*, Discrete Mathematics 150 (1996), pp. 303-313.

[3] Matić-Kekić, S., Acketa, D. M., *Non-greedy optimal digital convex polygons*, submitted.

[4] Voss, K., Klette, R., *On the maximal number of edges of a convex digital polygon included into a square*, Pocitace a umela inteligencia, Vol.1., Bo.6, (1982), pp. 549-558.

Institute of Mathematics, University of Novi Sad Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia;

* Institute of Applied Basic Disciplines, Faculty of Engineering, Veljka Vlahovića 3, 21000 Novi Sad, Yugoslavia