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A COMMON FIXED POINTS THEOREM FOR CONTRACTION TYPE MAPPINGS ON MENGER SPACES

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Abstract. In this paper we prove a common fixed point theorem for selfmaps satisfying a new contraction type condition in Menger spaces.

1. Introduction

A Menger space is a space in which the concept of distance is considered to be probabilistic. The fixed point theory in Menger spaces has been extensively developing in the last thirty years. Sehgal and Bharucha-Reid [5] introduced the notion of contraction for the mappings in Menger spaces and proved fixed-point theorems which are extensions of the classical Banach's fixed point principle. After that, various generalizations of the contractions were considered. Lj. Ćirić introduced in [2] a notion of a generalized contraction map on a probabilistic metric space and proved fixed point theorems which are generalizations of the theorem of Sehgal and Bharucha-Reid [5] and extensions of some his results from [1]. The study of common fixed points of mappings satisfying some conctractive type condition has been at the centre of vigorous research activity. Vasuki [7] proved a fixed point theorem for a sequence of selfmaps satisfying a new contraction type condition in Menger spaces.

The purpose of this paper is to define and investigate a class of mappings which is more general than the corresponding class of mappings considered by Vasuki. So, we proved a common fixed point theorem which contains the fixed point theorem of Vasuki as a special case.

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2. Preliminaries

Probabilistic metric spaces were introduced by K. Menger [3]. A probabilistic metric space (briefly a Pm-space) is an ordered pair $(X, \mathcal{F})$ where X is an nonempty set and $\mathcal{F}$ is a mapping of $X \times X$ into a collection $\mathcal{L}$ of all distribution functions F (a distribution function F is a nondecreasing and left continuous mapping of reals into $[0, 1]$ with $\inf F(x) = 0$ and $\sup F(x) = 1$). The value of $\mathcal{F}$ at $(u, v) \in X \times X$ will be denoted by $F_{u,v}$. The functions $F_{u,v}$ $(u, v \in X)$ are assumed to satisfy the following conditions:

(a) $F_{u,v}(x) = 1$ for all $x > 0$ if and only if $u = v$,

(b) $F_{u,v}(0) = 0$,

(c) $F_{u,v} = F_{v,u}$,

(d) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ imply $F_{u,w}(x + y) = 1$.

The value $F_{u,v}(x)$ of $F_{u,v}$ at $x \in R$ may be interpreted as the probability that the distance between u and v is less than x.

A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if it satisfies:

1. $t(a, 1) = a$, $t(0, 0) = 0$,
2. $t(a, b) = t(b, a)$,
3. $t(c, d) \geq t(a, b)$ for $c \geq a$, $d \geq b$,
4. $t(t(a, b), c) = t(a, t(b, c))$.

A Menger space is a triplet $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a Pm-space and t-norm t is such that the Menger's triangle inequality

$F_{u,w}(x + y) \geq t[F_{u,v}(x), F_{v,w}(y)]$

is satisfied for all $u, v, w \in X$ and for all $x \geq 0, y \geq 0$. A topology in $(X, \mathcal{F}, t)$ is introduced by the family $\{U_v(\varepsilon, \lambda) \mid v \in X, \varepsilon > 0, \lambda > 0\}$, where the set

$U_v(\varepsilon, \lambda) = \{u \in X \mid F_{u,v}(\varepsilon) > 1 - \lambda,\ \varepsilon > 0,\ \lambda > 0\}$

is called an (ϵ, λ) - neighborhood of $v \in X$. For details of the topological preliminaries refer to Schweizer and Sklar [4].

3. Results

We need the following Lemma, which can be selected from the proof of Theorem 3 of Sehgal and Bharucha-Reid [5] (see also Singh and Pant [6]).

Lemma. *Let $\{y_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, t)$, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. If there exists $\alpha \in (0, 1)$ such that $F_{y_n, y_{n+1}}(\alpha p) \geq F_{y_{n-1}, y_n}(p)$, $n = 1, 2, \ldots$, for all $p > 0$, then $\{y_n\}$ is a Cauchy sequence in X.*

Theorem 1. *Let* (T_n), $n = 1, 2, ...$ *be a sequence of mappings of a complete Menger space* $(X, \mathcal{F}, t)$ *into itself and* $S : X \to X$ *continuous mapping such that* $T_n(X) \subseteq S(X)$ *and* S *is commuting with each* T_n*. Let* $t(r, s) = \min(r, s)$ *for every* $r, s \in [0, 1]$*. Suppose that there exists a costant* $\alpha \in [0, 1)$ *such that for any two maps* T_i *and* T_j *and for every* $x, y \in X$*:*

$$(1) \quad F^2_{T_i x, T_j y}(\alpha p) \geq \min\{F^2_{Sx, T_i x}(p),\ F^2_{Sy, T_j y}(p),\ F^2_{Sx, Sy}(p),\ F_{Sx, T_j y}(2p)F_{Sy, T_i x}(p),\ F_{Sx, T_j y}(2p)F_{Sx, T_i x}(p)\}$$

holds for all $p > 0$*, then there exists a unique common fixed point for all* T_n *and* S*.*

Proof. Let x_0 be an arbitrary point in X and $\{x_n\}$ a sequence defined by

$$Sx_n = T_n x_{n-1}, \quad (n = 1, 2, ...).$$

From (1), for each $p > 0$ and $0 \leq \alpha < 1$, we have

$$\begin{aligned} F^2_{Sx_1, Sx_2}(\alpha p) &= F^2_{T_1 x_0, T_2 x_1}(\alpha p) \\ &\geq \min\{F^2_{Sx_0, T_1 x_0}(p),\ F^2_{Sx_1, T_2 x_1}(p),\ F^2_{Sx_0, Sx_1}(p), \\ &\quad F_{Sx_0, T_2 x_1}(2p)F_{Sx_1, T_1 x_0}(p),\ F_{Sx_0, T_2 x_1}(2p)F_{Sx_0, T_1 x_0}(p)\}. \end{aligned}$$

Hence,

$$F^2_{Sx_1, Sx_2}(\alpha p) \geq \min\{F^2_{Sx_0, Sx_1}(p),\ F^2_{Sx_1, Sx_2}(p),\ F^2_{Sx_0, Sx_1}(p),\ F_{Sx_0, Sx_2}(2p)F_{Sx_1, Sx_1}(p),\ F_{Sx_0, Sx_2}(2p)F_{Sx_0, Sx_1}(p)\}.$$

Using (a) and the Menger's triangle inequality for $t = \min$ we get:

$$\begin{aligned} F_{Sx_0, Sx_2}(2p)F_{Sx_1, Sx_1}(p) = F_{Sx_0, Sx_2}(2p) &\geq \min\{F_{Sx_0, Sx_1}(p), F_{Sx_1, Sx_2}(p)\} \\ &\geq \min\{F^2_{Sx_0, Sx_1}(p), F^2_{Sx_1, Sx_2}(p)\}; \end{aligned}$$

$$\begin{aligned} F_{Sx_0, Sx_2}(2p)F_{Sx_0, Sx_1}(p) &\geq \min\{F_{Sx_0, Sx_1}(p), F_{Sx_1, Sx_2}(p)\} \cdot F_{Sx_0, Sx_1}(p) \\ &= \min\{F^2_{Sx_0, Sx_1}(p), F_{Sx_0, Sx_1}(p)F_{Sx_1, Sx_2}(p)\}. \end{aligned}$$

So, we obtain

$$F^2_{Sx_1, Sx_2}(\alpha p) \geq \min\{F^2_{Sx_0, Sx_1}(p), F^2_{Sx_1, Sx_2}(p),\ F_{Sx_0, Sx_1}(p)F_{Sx_1, Sx_2}(p)\},$$

which implies

$$(2) \quad F_{Sx_1, Sx_2}(\alpha p) \geq \min\{F_{Sx_0, Sx_1}(p), F_{Sx_1, Sx_2}(p)\}.$$

Suppose that $p > 0$ is such that $F_{Sx_0,Sx_1}(p) > F_{Sx_1,Sx_2}(p) = r$. Put $q = \sup\{t > 0 : F_{Sx_1,Sx_2}(t) = r\}$. Then there exists $p_1 > q$ such that $\alpha p_1 < q$. So we have $r = F_{Sx_1,Sx_2}(\alpha p_1) < F_{Sx_1,Sx_2}(p_1)$. From (2) follows

$$F_{Sx_1,Sx_2}(\alpha p_1) \geq \min\{F_{Sx_0,Sx_1}(p_1), F_{Sx_1,Sx_2}(p_1)\}.$$

Hence $F_{Sx_1,Sx_2}(\alpha p_1) \geq F_{Sx_0,Sx_1}(p_1)$. Now we have

$$F_{Sx_1,Sx_2}(\alpha p) = F_{Sx_1,Sx_2}(\alpha p_1) \geq F_{Sx_0,Sx_1}(p_1) \geq F_{Sx_0,Sx_1}(p),$$

which is a contradiction with $F_{Sx_0,Sx_1}(p) > F_{Sx_1,Sx_2}(p)$.

So (2) implies that $F_{Sx_1,Sx_2}(\alpha p) \geq F_{Sx_0,Sx_1}(p)$ for all $p > 0$ and $0 \leq \alpha < 1$. By induction,

$$F_{Sx_n,Sx_{n+1}}(\alpha p) \geq F_{Sx_{n-1},Sx_n}(p), \; n = 1, 2, ... \tag{3}$$

From (3) we conclude that the sequence $Sx_n = y_n$ satisfies the condition of the *Lemma*. Therefore, $\{Sx_n\}$ is a Cauchy sequence in X. X being complete, there is some $u \in X$ such that $Sx_n \to u$. As $Sx_n = T_n x_{n-1}$, it follows that $\{T_n x_{n-1}\}$ also converges to u. Since S commutes with each T_n, from (1), we have:

$$\begin{aligned} F^2_{SSx_n,T_k u}(\alpha p) &= F^2_{ST_n x_{n-1},T_k u}(\alpha p) = F^2_{T_n Sx_{n-1},T_k u}(\alpha p) \\ &\geq \min\{F^2_{SSx_{n-1},SSx_n}(p), F^2_{Su,T_k u}(p),\; F^2_{SSx_{n-1},Su}(p), \\ &\qquad F_{SSx_{n-1},T_k u}(2p) F_{Su,SSx_n}(p), \\ &\qquad F_{SSx_{n-1},T_k u}(2p) F_{SSx_{n-1},SSx_n}(p)\}. \end{aligned}$$

Taking the limit as $n \to \infty$ and using the continuity of S, we get:

$$\begin{aligned} &F^2_{Su,T_k u}(\alpha p) \geq \min\{F^2_{Su,Su}(p),\; F^2_{Su,T_k u}(p),\; F^2_{Su,Su}(p) \\ &F_{Su,T_k u}(2p) F_{Su,Su}(p),\; F_{Su,T_k u}(2p) F_{Su,Su}(p)\} \;=\; F^2_{Su,T_k u}(p). \end{aligned}$$

So, we obtain $F_{Su,T_k u}(\alpha p) \geq F^2_{Su,T_k u}(p)$ for each $p > 0$ and $0 \leq \alpha < 1$. Hence we conclude that $Su = T_k u$ for any fixed integer k. Further,

$$\begin{aligned} F^2_{Sx_n,T_k u}(\alpha p) &\geq F^2_{T_n x_{n-1},T_k u}(\alpha p) \\ &\geq \min\{F^2_{Sx_{n-1},Sx_n}(p),\; F^2_{Su,T_k u}(p), F^2_{Sx_n,Su}(p), \\ &\qquad F_{Sx_{n-1},T_k u}(2p) F_{Su,Sx_n}(p), \\ &\qquad F_{Sx_{n-1},T_k u}(2p) F_{Sx_{n-1},Sx_n}(p)\}. \end{aligned}$$

This inequality on letting $n \to \infty$ implies that

$$\begin{aligned} F^2_{u,T_k u}(\alpha p) \geq\ & \min\{F^2_{u,u}(p),\ F^2_{T_k u,T_k u}(p),\ F^2_{u,T_k u}(p) \\ & F_{u,T_k u}(2p)F_{T_k u,u}(p),\ F_{u,T_k u}(2p)F_{u,u}(p)\} \ =\ F^2_{u,T_k u}(p). \end{aligned}$$

Hence, we have $F_{u,T_k u}(\alpha p) \geq F_{u,T_k u}(p)$ for each $p > 0$ and $0 \leq \alpha < 1$. This implies $u = T_k u = Su$ for any fixed integer k. Thus, u is a common fixed point of S and T_n for $n = 1, 2, \ldots$.

Now we shall show that u is the unique common fixed point. Assume that u' is also common fixed point of S and T_n. Then from (1), for each $p > 0$:

$$\begin{aligned} F^2_{u,u'}(\alpha p) \ &=\ F^2_{T_i u,T_j u'}(\alpha p) \\ &\geq\ \min\{F^2_{Su,u}(p),\ F^2_{Su',u'}(p),\ F^2_{Su,Su'}(p) \\ &\qquad F_{Su,u'}(2p)F_{Su',u}(p),\ F_{Su,u'}(2p)F_{Su,u}(p)\} \\ &=\ \min\{F^2_{u,u'}(p),\ F^2_{u',u'}(p),\ F^2_{u,u'}(p), \\ &\qquad F_{u,u'}(2p)F_{u',u}(p),\ F_{u,u'}(2p)F_{u,u}(p)\} \\ &=\ F^2_{u,u'}(p). \end{aligned}$$

This implies $F_{u,u'}(\alpha p) \geq F_{u,u'}(p)$ for each $p > 0$ and $0 \leq \alpha < 1$. Therefore $u' = u$ and so, u is the unique common fixed point for all T_n and S. This completes the proof of Theorem.

If S is the identity mapping on X, we obtain the following result.

Corollary 1. *Let (T_n), $n = 1, 2, \ldots$ be a sequence of selfmaps of a complete Menger space (X, F, t) into itself with $t(x, y) = \min(x, y)$ for every $x, y \in [0, 1]$. If for any two maps T_i and T_j the following inequality:*

$$\text{(4)} \qquad \begin{aligned} F^2_{T_i x,T_j y}(\alpha p) \geq \min\{& F^2_{x,T_i x}(p),\ F^2_{y,T_j y}(p),\ F^2_{x,y}(p), \\ & F_{x,T_j y}(2p)F_{y,T_i x}(p),\ F_{x,T_j y}(2p)F_{x,T_i x}(p)\} \end{aligned}$$

holds for all $x, y \in X$, where $0 \leq \alpha < 1$ and $p > 0$, then for any $x_0 \in X$ the sequence $x_n = T_n x_{n-1}$ $(n = 1, 2, \ldots)$ converges and its limit is the unique common fixed point for all T_n.

Proof. Existence and uniquess of common fixed point follows from *Theorem 1.* Convergece of sequence $\{x_n\}$ can be proved as in *Theorem 1.*

Corollary 2. *(R. Vasuki [7]) Let* (T_n), $n = 1, 2, \ldots$ *be a sequence of self-maps of a complete Menger space* (X, F, t) *into itself with* $t(x, y) = \min(x, y)$ *for every* $x, y \in [0, 1]$. *If for any two maps* T_i *and* T_j *the following inequality:*

$$(5) \qquad F^2_{T_i x, T_j y}(\alpha p) \geq \min\{F_{x,y}(p) F_{x,T_i x}(p),\ F_{x,y}(p) F_{y,T_j y}(p),\ F_{x,T_i x}(p) F_{y,T_j y}(p),\ F_{x,T_j y}(2p) F_{y,T_i x}(p)\}$$

holds for all $x, y \in X$, *where* $0 \leq \alpha < 1$ *and* $p > 0$, *then for any* $x_0 \in X$ *the sequence* $x_n = T_n x_{n-1}$ *for all* $(n = 1, 2, \ldots)$ *converges and its limit is the unique common fixed point for all* T_n.

Proof. Since $\min\{a^2, b^2, c^2\} \leq \min\{ab, bc, ca\}$ $a, b, c \geq 0$ we have

$$\min\{F_{x,y}(p) F_{x,T_i x}(p),\ F_{x,y}(p) F_{y,T_j y}(p),\ F_{x,T_i x}(p) F_{y,T_j y}(p)\}, \geq \min\{F^2_{x,T_i x}(p),\ F^2_{y,T_j y}(p),\ F^2_{x,y}(p)\}.$$

From (5) we get

$$\begin{aligned} &F^2_{T_i x, T_j y}(\alpha p) \\ &\geq \min\{F_{x,y}(p) F_{x,T_i x}(p),\ F_{x,y}(p) F_{y,T_j y}(p), \\ &F_{x,T_i x}(p) F_{y,T_j y}(p),\ F_{x,T_j y}(2p) F_{y,T_i x}(p)\} \\ &\geq \min\{F^2_{x,T_i x}(p),\ F^2_{y,T_j y}(p),\ F^2_{x,y}(p), \\ &F_{x,T_j y}(2p) F_{y,T_i x}(p),\ F_{x,T_j y}(2p) F_{x,T_i x}(p)\}. \end{aligned}$$

So it follows that if sequence (T_n) satisfies conditions (5) then it also satisfies our condition (4) of *Corollary 1.*

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References

[1] Lj. Ćirić, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. **12 (26)** (1971), 19–26.

[2] Lj. Ćirić, *On fixed points of generalized contractions on probabilistic metric spaces*, Publ. Inst. Math. **18 (32)** (1975), 71–78.

[3] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 535–537.

[4] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 313–334.

[5] V. Sehgal and A. Bharucha-Reid, *Fixed points of contractions mappings on probabilstic metric spaces*, Math. systems theory **6** (1972), 97–102.

[6] S. L. Singh and B. D. Pant paper Fixed point theorems for commuting mappings in probabilistic metric spaces, Honam J. Math. **5** (1983), 139–150.

[7] R. Vasuki, *A fixed point theorem for a sequence of maps satisfying a new contraction type condition in Menger spaces*, Math. Japonica **35** (6) (1990), 1099–1102.

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