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# SIMPLE AND MULTIPLE ANTISYMMETRY SPACE GROUPS DERIVED FROM CUBICAL $P$-SYMMETRY GROUPS

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**Abstract.** By use of antisymmetric characteristic method, from cubical $P$-symmetry groups $G_3^P$ ($P = 23, 26, \overline{4}3, \overline{4}6$) $(P, 2^l)$-symmetry three-dimensional space groups $G_3^{l,P}$ are derived.

Cubical $P$-symmetry three-dimensional space groups $G_3^P$ ($P = 23, 26, \overline{4}3, \overline{4}6$) are derived by Yu. S. Karpova (Yu. S. Chubarova) [1,2]. From space groups $G_3$ are derived 35 junior $G_3^{23}$, 53 $G_3^{26}$, 74 $G_3^{\overline{4}3}$ and 150 $G_3^{\overline{4}6}$. By the use of the generalized antisymmetric characteristic method ($AC$-method) introduced by the author [3], some of the results mentioned will be corrected, and derived all cubical $(P, 2^l)$-symmetry three-dimensional space groups $G_3^{l,P}$ ($P = 23, 26, \overline{4}3, \overline{4}6$).

## 1. Some general remarks on (23)-, (26)-, ($\overline{4}3$)- and ($\overline{4}6$)-symmetry

The cubical symmetries (23)-, (26)-, ($\overline{4}3$)- and ($\overline{4}6$)- (or ($23C_3$)-, ($26C_3$)-, ($\overline{4}3C_{3v}$)- and ($\overline{4}6C_{3v}$)-symmetry, according [2]) are particular cases of the general $P$-symmetry [2,4] with $P \simeq T, T_h, O$ and $O_h$ respectively.

In the case of (23)-symmetry $P \simeq A_4$ is the irregular permutation group, generated by the permutations $c_1 = (123)$ and $c_2 = (12)(34)$, satisfying the relations:

$$c_1^3 = c_2^2 = (c_1c_2)^3 = E.$$

In the case of (26)-symmetry $P \simeq T_h \simeq A_4 \times C_2$ is the irregular permutation group, generated by the permutations $c_1 = (123)(567)$, $c_2 = (12)(34)(56)(78)$ and $c_3 = (15)(26)(37)(48)$, satisfying the relations:

$$c_1^3 = c_2^2 = (c_1c_2)^3 = E \quad c_3^2 = E \quad c_1, c_2c_3.$$

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In the case of $(\overline{4}3)$-symmetry $P \simeq O \simeq S_4$ is the irregular permutation group, generated by the permutations $c_1 = (1234)$ and $c_2 = (34)$, satisfying the relations:

$$c_1^4 = c_2^2 = (c_1c_2)^3 = E.$$

In the case of $(\overline{4}6)$-symmetry $P \simeq O_h \simeq S_4 \times C_2$ is the irregular permutation group, generated by the permutations $c_1 = (1728)(3645)$, $c_2 = (17)(35)$ and $c_3 = (15)(26)(37)(48)$, satisfying the relations:

$$c_1^4 = c_2^2 = (c_1c_2)^3 = E \quad c_3^2 = E \quad c_1, c_2c_3.$$

By introducing $l$ antiidentity transformations $e_1, \ldots, e_l$ $(l \in N)$ [5,6] commuting between themeselves and with the generators of group $P$ we have $(P, 2^l)$-symmetry, with the group $P' = P \times C_2^l$.

**Definition 1.**

*(a) a $(P, 2^l)$-symmetry group is called the $M^m$-type $(P, 2^l)$-symmetry group, if it is a $M^m$-type group regarded as a $l$-multiple antisymmetry group;*

*(b) a $(P, 2^l)$-symmetry group $G^{l,P}$ is the group of the complete $(P, 2^l)$-symmetry, if every $e_i$-transformation can be obtained in the group $G^{l,P}$ as an independent $(P, 2^l)$-symmetry transformation. If the condition (b) it is not satisfied, such a group $G^{l,P}$ is the uncomplete $(P, 2^l)$-symmetry group;*

*(c) a complete $(P, 2^l)$-symmetry group $G^{l,P}$ is the junior $(P, 2^l)$-symmetry group if all the relations given in the presentation of its generating symmetry group $G$ remain satisfied after replacing the generators of the group $G$ by the corresponding $(P, 2^l)$-symmetry group generators.*

In this work will be considered only the junior $M^m$-type complete $(P, 2^l)$-symmetry groups.

From the reducibility of the group $T_h \simeq A_4 \times C_2 = \{c_1, c_2\} \times \{c_3\}$ results the equality of $(23, 2^l)$- and $(26, 2^{l-1})$-symmetry, and from the reducibility of the group $P \simeq S_4 \times C_2 = \{c_1, c_2\} \times \{c_3\}$ results the equality of $(\overline{4}3, 2^l)$- and $(\overline{4}6, 2^{l-1})$- symmetry. Using that equalities, the problem of the derivation of all cubical $(P, 2^l)$-symmetry groups $(P = 23, 26, \overline{4}3, \overline{4}6)$ is reduced to the derivation of $(P, 2^l)$-symmetry groups $(P = 23, \overline{4}3)$.

The derivation will be realized by the use of generalized $AC$:

**Definition 2.** *Let all the products of $P$-symmetry generators of a group $G^P$, within which every generator participates once at the most, be formed, and then subsets of transformations equivalent with regard to $P$-symmetry, be separated. The resulting system is called the antisymmetric characteristic of the group $G^P$ and denoted by $AC(G^P)$ [3].*

In the case of (23)-symmetry, in every transition from the antisymmetric characteristic $AC(G)$ of a generating group $G$ to the antisymmetric characteristic $AC(G^{23})$ of a (23)-symmetry group $G^{23}$ derived from it, the antisymmetric characteristic remains unchanged, and the conditions for $(23, 2^l)$-symmetry groups of the $M^m$-type are the same as the conditions for $l$-multiple antisymmetry groups of the $M^m$-type. Hence, from every (23)-symmetry group $G_3^{23}$ can be derived $N_m$ junior $(23, 2^l)$-symmetry groups of the $M^m$-type, where the number $N_m$ is the same as the number of $l$-multiple antisymmetry groups of the $M^m$-type derived from its generating symmetry group $G_3$. The corresponding data about $(23, 2^l)$-symmetry groups and $(26, 2^l)$-symmetry groups are given in Table 1.

**Table 1**

| | (23) | (23, 2) | $(23, 2^2)$ | $(23, 2^3)$ | (26) | (26, 2) | $(26, 2^2)$ |
|---|---|---|---|---|---|---|---|
| **38s** | 1 | 1 | | | | | |
| **39s** | 1 | 1 | | | 1 | | |
| **43s** | 1 | 2 | 3 | | 2 | 3 | |
| **49s** | 1 | 3 | 6 | | 3 | 6 | |
| **51s** | 1 | 2 | 3 | | 2 | 3 | |
| **52s** | 1 | 2 | 3 | | 2 | 3 | |
| **53s** | 1 | 5 | 24 | 84 | 5 | 24 | 84 |
| **59s** | 3 | 3 | | | 3 | | |
| **60s** | 1 | 1 | | | 1 | | |
| **61s** | 3 | | | | | | |
| **62s** | 3 | 6 | 9 | | 6 | 9 | |
| **63s** | 1 | 3 | 6 | | 3 | 6 | |
| **64s** | 3 | 3 | | | 3 | | |
| | 21 | 32 | 54 | 84 | 32 | 54 | 84 |
| **49h** | 1 | 2 | 3 | | 2 | 3 | |
| **50h** | 1 | 1 | | | | | |
| | 2 | 3 | 3 | | 2 | 3 | |
| **62a** | 1 | 1 | | | 1 | | |
| **69a** | 1 | 1 | | | 1 | | |
| **74a** | 1 | 1 | | | 1 | | |
| **75a** | 1 | 1 | | | 1 | | |
| **76a** | 1 | 3 | 6 | | 3 | 6 | |
| **77a** | 1 | 3 | 6 | | 3 | 6 | |
| **78a** | 1 | 1 | | | 1 | | |
| **81a** | 1 | 3 | 6 | | 3 | 6 | |
| **89a** | 1 | | | | | | |
| **90a** | 1 | 1 | | | 1 | | |
| **91a** | 1 | 1 | | | 1 | | |
| **92a** | 1 | 2 | 3 | | 2 | 3 | |
| | 12 | 18 | 21 | | 18 | 21 | |
| | 35 | 53 | 78 | 84 | 53 | 78 | 84 |

Having in mind the equality of $(\overline{4}3, 2^l)$- and $(\overline{4}6, 2^{l-1})$-symmetry, the only non-trivial problem is the derivation of $(\overline{4}3, 2^l)$-symmetry groups.

## 2. $(\overline{4}3, 2^l)$-symmetry three-dimensional space groups $G_3^{l,\overline{4}3}$

The application of the theoretical assumptions given above will be illustrated by example of junior $(\overline{4}3, 2^l)$-symmetry three-dimensional space groups of the $M^m$-type derived in the family with the common generating symmetry group $G =$ **68s** (P432), $\{a, b, c\}(3/4)$ with the $AC$:$\{4, 4a\}$ belonging to the $AC$-equivalency class 3.2 [3]. In the case of $(\overline{4}3)$-symmetry we have two junior $(\overline{4}3)$-symmetry groups [1]:

(1) $\{a, b, c\}(3^{3)}/4^{(\overline{4})}$,

(2) $\{a^{(2}, b^{(2'}, c^{(2"}\}(3^{3)}/4^{\underline{2}")})$.

The $AC$ of the first group is of the form $\{(\overline{4}, (\overline{4}\}$, so from it can be derived $N_1 = 1$ $(\overline{4}3, 2)$-symmetry groups. The $AC$ of the second is of the form $\{\underline{2}"), \underline{2}")(2\}$. Because of non-equivalence of the transformations $\underline{2}")$ and $\underline{2}")(2 \sim (\overline{4}$, in the sense of $(\overline{4}3)$-symmetry, this $AC$ results in the new $AC$:$\{\overline{4}\}\{a\}$ belonging to the $AC$-equivalence class 3.1 and giving $N_1 = 2$ $(\overline{4}3, 2)$-symmetry groups. Because of the equality between $(\overline{4}3, 2)$-symmetry and $(\overline{4}6)$-symmetry, to the three obtained $(\overline{4}3, 2)$-symmetry groups: $\{\underline{a}, \underline{b}, \underline{c}\}(3^{3)}/4^{(\overline{4})}$, $\{\underline{a}^{(2}, \underline{b}^{(2'}, \underline{c}^{(\underline{2}"}\}(3^3)/4^{\underline{2}")})$ and $\{\underline{a}, \underline{b}, \underline{c}\}(3^{3)}/\underline{4}^{\underline{2}")})$, correspond three $(\overline{4}6)$-symmetry groups: $\{a^{(2)}, b^{(2)}, c^{(2)}\}(3^{3)}/4^{(\overline{4})})$, $\{a^{(\underline{2}}, b^{(\underline{2}'}, c^{(\underline{2}"}\}(3^{3)}/4^{\underline{2}")})$ and $\{a^{(\underline{2}}, b^{(\underline{2}'}, c^{(\underline{2}"}\}(3^{3)}/4^{(\overline{4})})$.

In the same manner is realized the partial catalogue of all junior complete $(\overline{4}3, 2^l)$-symmetry three-dimensional space groups of the $M^m$-type $G_3^{l,3\overline{4}}$, making possibile their complete cataloguation [7] . The final results corresponding respectively to symmorphic, hemisymorphic and asymmorphic $(\overline{4}3, 2^l)$-symmetry groups are summarized in the catalogue, which can be obtained directly from the author. The enumeration results are given in Table 2:

**Table 2**

| | $(\overline{4}3)$ | $(\overline{4}3, 2)$ | $(\overline{4}3, 2^2)$ | $(\overline{4}3, 2^3)$ | $(\overline{4}6)$ | $(\overline{4}6, 2)$ | $(\overline{4}6, 2^2)$ |
|---|---|---|---|---|---|---|---|
| **40s** | 1 | 2 | | | 2 | | |
| **41s** | 1 | 2 | | | 2 | | |
| **42s** | 1 | 2 | | | 2 | | |
| **44s** | 1 | 1 | | | 1 | | |
| **45s** | 1 | 1 | | | 1 | | |
| **46s** | 1 | 1 | | | 1 | | |
| **47s** | 1 | 4 | 12 | | 4 | 12 | |
| **48s** | 1 | 4 | 12 | | 4 | 12 | |
| **50s** | 1 | 6 | 24 | | 6 | 24 | |
| **54s** | 1 | 4 | 12 | | 4 | 12 | |
| **55s** | 1 | 4 | 12 | | 4 | 12 | |
| **56s** | 1 | 4 | 12 | | 4 | 12 | |
| **57s** | 1 | 4 | 12 | | 4 | 12 | |
| **58s** | 1 | 10 | 96 | 672 | 10 | 96 | 672 |
| **65s** | 2 | 4 | | | 4 | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **66s** | 1 | 2 | | | 2 | | |
| **67s** | 2 | | | | | | |
| **68s** | 2 | 3 | | | 3 | | |
| **69s** | 1 | 2 | | | 2 | | |
| **70s** | 2 | | | | | | |
| **71s** | 2 | 8 | 24 | | 8 | 24 | |
| **72s** | 1 | 6 | 24 | | 6 | 24 | |
| **73s** | 3 | 6 | | | 6 | | |
| | 30 | 80 | 240 | 672 | 80 | 240 | 672 |
| **39h** | 1 | | | | | | |
| **40h** | 1 | | | | | | |
| **41h** | 1 | | | | | | |
| **42h** | 1 | 2 | | | 2 | | |
| **43h** | 1 | 2 | | | 2 | | |
| **44h** | 1 | 2 | | | 2 | | |
| **45h** | 1 | 2 | | | 2 | | |
| **46h** | 1 | 2 | | | 2 | | |
| **47h** | 1 | 2 | | | 2 | | |
| **48h** | 1 | 6 | 24 | | 6 | 24 | |
| **51h** | 1 | | | | | | |
| **52h** | 2 | | | | | | |
| **53h** | 1 | 2 | | | 2 | | |
| **54h** | 3 | 6 | | | 6 | | |
| | 17 | 26 | 24 | | 26 | 24 | |
| **70a** | 1 | 1 | | | 1 | | |
| **71a** | 1 | 1 | | | 1 | | |
| **72a** | 1 | 1 | | | 1 | | |
| **73a** | 1 | 1 | | | 1 | | |
| **79a** | 1 | 2 | | | 2 | | |
| **80a** | 1 | 2 | | | 2 | | |
| **82a** | 1 | 2 | | | 2 | | |
| **83a** | 1 | 2 | | | 2 | | |
| **84a** | 1 | 4 | 12 | | 4 | 12 | |
| **85a** | 1 | 4 | 12 | | 4 | 12 | |
| **86a** | 1 | 2 | | | 2 | | |
| **87a** | 1 | 6 | 24 | | 6 | 24 | |
| **88a** | 1 | 6 | 24 | | 6 | 24 | |
| **93a** | 1 | | | | | | |
| **94a** | 1 | | | | | | |
| **95a** | 1 | | | | | | |
| **96a** | 1 | 2 | | | 2 | | |
| **97a** | 3 | | | | | | |
| **98a** | 2 | 3 | | | 3 | | |
| **99a** | 1 | 2 | | | 2 | | |
| **100a** | 1 | 2 | | | 2 | | |
| **101a** | 1 | 2 | | | 2 | | |
| **102a** | 1 | 2 | | | 2 | | |
| **103a** | 1 | 4 | 12 | | 4 | 12 | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 27 | 51 | 84 | | | 51 | 84 | |
| 74 | 157 | 348 | 672 | | 157 | 348 | 672 |

From the results of the derivation of the $(\overline{4}3,1)$-symmetry groups we can first conclude that, because of the equality between $(\overline{4}3,1)$- and $(\overline{4}6)$-symmetry, there exist 157 (not 150) [1,2] $(\overline{4}3,1)$-symmetry groups and, consequently, the same number of $(\overline{4}6)$-symmetry groups. Such a disagreement with the results mentioned implies the necessary corrections for the $(\overline{4}6)$-symmetry groups in the families with the generating groups **46s** (R32), **65s** (P43m), **70a** ($P3_121$), **71a** ($P3_221$), **72a** ($P3_112$), **73a** ($P3_212$) and **98a** ($P4_232$). For these families, the correct numbers of $(\overline{4}6)$-symmetry groups are respectively, 1 (not 2), 4 (not 3), 1 (not 0), 1(not 0), 1(not 0), 1(not 0) and 3 (not 0).

## 3. Conclusion

As the final result, for the junior cubical complete $(P,2^l)$- symmetry three-dimensional space groups of the $M^m$-type the numbers $N_m^P$ ($P = 23, 26, \overline{4}3, \overline{4}6$) are the following:

| | |
|---|---|
| $N_0^{23} = 35$ | $N_0^{\overline{4}3} = 74$ |
| $N_1^{23} = N_0^{26} = 53$ | $N_1^{\overline{4}3} = N_0^{\overline{4}6} = 157$ |
| $N_2^{23} = N_1^{26} = 78$ | $N_2^{\overline{4}3} = N_1^{\overline{4}6} = 348$ |
| $N_3^{23} = N_2^{26} = 84$ | $N_3^{\overline{4}3} = N_2^{\overline{4}6} = 672$ |

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