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A New Approach to the Constructions of Braided T-Categories

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Abstract. The aim of this paper is to construct a new braided T-category via the generalized Yetter-Drinfel'd modules and Drinfel'd codouble over a Hopf algebra, an approach different from that proposed by Panaite and Staic [1]. Moreover, in the case of finite dimensional, we will show that this category coincides with the corepresentation of a certain coquasitriangular Turaev group algebra which we construct. Finally we apply our theory to the case of group algebra.

1. Introduction

Braided T-categories introduced by Turaev [2] are of interest due to their applications in homotopy quantum field theories, which are generalizations of ordinary topological quantum field theories. Braided T-category gives rise to 3-dimensional homotopy quantum field theory and plays a key role in the construction of Hennings-type invariants of flat group-bundles over complements of link in the 3-sphere, see [3]. As such, they are interesting to different research communities in mathematical physics (see [4, 5]).

The quantum double of Drinfel'd [6] is one of the most celebrated Hopf constructions, which associates to a Hopf algebra H a quasitriangular Hopf algebra $D(H)$. Unlike the Hopf algebra axioms themselves, the axioms of a dual quasitriangular (coquasitriangular) Hopf algebra are not self-dual. Thus the axioms and ways of working with these coquasitriangular Hopf algebras look somewhat different in practice and so it is surely worthwhile to study and write them out explicity in dual form. Moreover, the corepresentation categories of coquasitriangular Hopf algebras can give rise to a braided monoidal category which is different from one coming from the representation categories of quasitriangular Hopf algebras. It is these ideals which many authors studied these notions (cf.[7–17]).

In [1], the authors found a wise method to construct braided T-category $\mathcal{YD}(H)$ over the group $G = Aut_{Hopf}(H) \times Aut_{Hopf}(H)$, where H is a Hopf algebra. This category $\mathcal{YD}(H)$ is the disjoint union of all these categories ${}_H\mathcal{YD}^H(\alpha,\beta)$(the categories of (α,β)-Yetter-Drinfel'd modules) over H for all $\alpha,\beta \in Aut_{Hopf}(H)$. The authors also proved that, if H is finite dimensional, then $\mathcal{YD}(H)$ coincides with the representations of a certain quasitriangular T-coalgebra $DT(H)$.

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Our motivation is the following: Can we use (α, β)-Yetter-Drinfel'd modules and Drinfel'd codouble to construct a new braided T-category? And in the case of H being finite dimensional, can we prove that this new braided T-category is isomorphic to the corepresentation category of a certain coquasitriangular Turaev group algebra?

In this paper, we give a positive answer to the above question. The paper is organized as follows:

In section 1, we recall the notions of braided T-category, Turaev group algebra and generalized Yetter-Drinfel'd modules. In section 2, we introduce the diagonal crossed coproduct $H^{*op} \bowtie C$, where H is a Hopf algebra and C is an H-bimodule coalgebra. In section 3, we firstly recall the definition of (α, β)-Yetter-Drinfel'd module, then we construct braided T-category $\widehat{\mathcal{YD}(H)}$ over G whose multiplication is $(\alpha, \beta) * (\gamma, \delta) = (\delta\alpha\delta^{-1}\gamma, \delta\beta)$ for all $\alpha, \beta, \gamma, \delta \in Aut_{Hopf}(H)$. We also prove that category $\widehat{\mathcal{YD}(H)}$ coincides with the corepresentation category of a certain coquasitriangular crossed Turaev group algebra in the sense of [18].

2. Preliminary

Throughout this paper, let k be a fixed field, and all vector spaces and tensor product are over k. All vector spaces are assumed to be finite dimensional, although it should be clear when this restriction is not necessary.

In this section we recall some basic definitions and results related to our paper.

2.1. Crossed T-category

Let G be a group with the unit 1. Recall from [19–21] that a crossed category C (over G) is given by the following data:

• C is a strict monoidal category.

• A family of subcategory $\{C_\alpha\}_{\alpha \in G}$ such that C is a disjonit union of this family and that $U \otimes V \in C_{\alpha\beta}$ for any $\alpha, \beta \in G$, $U \in C_\alpha$ and $V \in C_\beta$.

• A group homomorphism $\varphi : G \to aut(C), \beta \mapsto \varphi_\beta$, the *conjugation*, where $aut(C)$ is the group of the invertible strict tensor functors from C to itself, such that $\varphi_\beta(C_\alpha) = C_{\beta\alpha\beta^{-1}}$ for any $\alpha, \beta \in G$.

We will use the left index notation in Turaev: Given $\beta \in G$ and an object $V \in C_\alpha$, the functor φ_β will be denoted by ${}^{\beta}(\cdot)$ or ${}^{V}(\cdot)$ and ${}^{\beta^{-1}}(\cdot)$ will be denoted by ${}^{\overline{V}}(\cdot)$. Since ${}^{V}(\cdot)$ is a functor, for any object $U \in C$ and any composition of morphism $g \circ f$ in C, we obtain ${}^{V}id_U = id_{{}^{V}U}$ and ${}^{V}(g \circ f) = {}^{V}g \circ {}^{V}f$. Since the conjugation $\varphi : \pi \to aut(C)$ is a group homomorphism, for any $V, W \in C$, we have ${}^{V\otimes W}(\cdot) = {}^{V}({}^{W}(\cdot))$ and ${}^{1}(\cdot) = {}^{V}({}^{\overline{V}}(\cdot)) = {}^{\overline{V}}({}^{V}(\cdot)) = id_C$. Since for any $V \in C$, the functor ${}^{V}(\cdot)$ is strict, we have ${}^{V}(f \otimes g) = {}^{V}f \otimes {}^{V}g$ for any morphism f and g in C, and ${}^{V}(1) = 1$.

A Turaev braided G-category is a crossed T-category C endowed with a braiding, i.e., a family of isomorphisms

$$c = \{c_{U,V} : U \otimes V \to {}^{V}U \otimes V\}_{U,V \in C}$$

obeying the following conditions:

• For any morphism $f \in Hom_{C_\alpha}(U, U')$ and $g \in Hom_{C_\beta}(V, V')$, we have

$$({}^{\alpha}g \otimes f) \circ c_{U,V} = c_{U',V'} \circ (f \otimes g),$$

• For all $U, V, W \in C$, we have

$$c_{U\otimes V,W} = (c_{U,{}^{V}W} \otimes id_V)(id_U \otimes c_{V,W}), \tag{2.1}$$

$$c_{U,V\otimes W} = (id_{{}^{U}V} \otimes c_{U,W})(c_{U,V} \otimes id_W). \tag{2.2}$$

• For any $U, V \in C$ and $\alpha \in G$, $\varphi_\alpha(c_{U,V}) = c_{{}^{\alpha}U,{}^{\alpha}V}$.

2.2. Turaev Group Algebras

Let G be a group with unit 1. Recall from [18, 22] that a G-algebra is a family $A = \{A_\alpha\}_{\alpha\in G}$ of k-spaces together with a family of k-linear maps $m = \{m_{\alpha,\beta} : A_\alpha \otimes A_\beta \to A_{\alpha\beta}\}_{\alpha,\beta\in G}$ (called multiplication) and a k-linear map $\eta : k \to A_1$ (called unit) such that m is associative in the sense that, for all $\alpha, \beta, \gamma \in G$

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes id) = m_{\alpha,\beta\gamma}(id \otimes m_{\beta,\gamma}),$$
$$m_{\alpha,1}(id \otimes \eta) = id = m_{1,\alpha}(\eta \otimes id).$$

A Turaev G-algebra is a G-algebra $H = \{H_\alpha\}_{\alpha\in G}$ such that each H_α is a coalgebra with comultiplication Δ_α and counit ε_α; the map $\eta : k \to H_1$ and the maps $m_{\alpha,\beta} : H_\alpha \otimes H_\beta \to H_{\alpha\beta}$ are coalgebra maps, with a family of k-linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in G}$ (called the antipode) such that for all $\alpha \in G$

$$m_{\alpha,\alpha^{-1}}(id \otimes S_\alpha)\Delta_\alpha = \varepsilon_\alpha 1 = m_{\alpha^{-1},\alpha}(S_\alpha \otimes id)\Delta_\alpha.$$

Furthermore, a crossed Turaev G-algebra is a Turaev G-algebra with a family of coalgebra isomorphisms $\psi = \{\psi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\beta\in G}$ (called crossing), satisfying the following conditions: for all $\alpha, \beta, \gamma \in G$

(i) ψ is multiplicative, i.e., $\psi_\alpha\psi_\beta = \psi_{\alpha\beta}$,

(ii) ψ is compatible with m, i.e., $m_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}(\psi_\gamma \otimes \psi_\gamma) = \psi_\gamma m_{\alpha,\beta}$,

(iii) ψ is compatible with η, i.e., $\eta = \psi_\gamma\eta$,

(iv) ψ preserves the antipode, i.e., $\psi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\psi_\beta$.

We use the Sweedlers notation for a comultiplication Δ_α on H_α: for all $h \in H_\alpha$

$$\Delta_\alpha(h) = h_1 \otimes h_2.$$

Recall from [18], a Turaev G-algebra H is called coquasitriangular if there exists a family of k-linear maps $\sigma = \{\sigma_{\alpha,\beta} : H_\alpha \otimes H_\beta \to k\}$ such that $\sigma_{\alpha,\beta}$ is convolution invertible for all $\alpha, \beta \in G$ and the following conditions are satisfied:

(TCT1) $\sigma_{\alpha\beta,\gamma}(xy, z) = \sigma_{\alpha,\gamma}(x, z_2)\sigma_{\beta,\gamma}(y, z_1)$,

(TCT2) $\sigma_{\alpha,\beta\gamma}(x, yz) = \sigma_{\alpha,\beta}(x_1, y)\sigma_{\beta^{-1}\alpha\beta,\gamma}(\psi_{\beta^{-1}}(x_2), z)$,

(TCT3) $\sigma_{\alpha,\beta}(x_1, y_1)y_2\psi_{\beta^{-1}}(x_2) = x_1y_1\sigma_{\alpha,\beta}(x_2, y_2)$,

(TCT4) $\sigma_{\alpha,\beta}(x, y) = \sigma_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}(\psi_\gamma(x), \psi_\gamma(y))$,

for all $x \in H_\alpha, y \in H_\beta, z \in H_\gamma$.

Note that if Turaev G-algebra H is coquasitriangular, then $(H_1, \sigma_{1,1})$ is a coquasitriangular Hopf algebra.

2.3. Yetter-Drinfel'd module

Let H be a Hopf algebra and C an H-bimodule coalgebra, with module structures $H\otimes C \to C$, $h\otimes c \mapsto h\cdot c$ and $C \otimes H \to C$, $c \otimes h \mapsto c \cdot h$. Recall from [23], we can consider the Yetter-Drinfel'd datum (H, C, H) and the Yetter-Drinfel'd category ${}_H\mathcal{YD}^C$, whose object M is a left H-module (with the action $h \otimes m \mapsto h \cdot m$) and right C-comodule (with the coaction $m \mapsto m_{(0)} \otimes m_{(1)}$) such that for all $h \in H, m \in M$,

$$h_1 \cdot m_{(0)} \otimes h_2 \cdot m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} \cdot h_1,$$

or equivalently

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes h_3 \cdot m_{(1)} \cdot S^{-1}(h_1).$$

3. Diagonal Crossed Coproduct

As the dual of diagonal crossed product (for details, see [1]), we have the following result.

Proposition 3.1. *Let H be a Hopf algebra with a bijective antipode S, and C a bimodule coalgebra with the actions $H\otimes C\to C$, $h\otimes c\mapsto h\cdot c$ and $C\otimes H\to C$, $c\otimes h\mapsto c\cdot h$. Then we have a coalgebra $H^{*op}\otimes C$ (denoted by $H^{*op}\bowtie C$) with the comultiplication and counit*

$$\bar{\Delta}(p\bowtie c)=\sum_{i,j}p_1\bowtie h_j\cdot c_1\cdot S^{-1}(h_i)\otimes h^ip_2h^j\bowtie c_2, \tag{3.1}$$

$$\bar{\varepsilon}(p\bowtie c)=p(1)\varepsilon(c), \tag{3.2}$$

for all $p\in H^{*op}, c\in C$, where $\{h_i\}$ and $\{h^i\}$ are basis and dual basis of H. $H^{*op}\bowtie C$ is called diagonal crossed coproduct.

Proof. For all $p\in H^{*op}, c\in C$, on one hand

$$\begin{aligned}(\bar{\Delta}\otimes id)\bar{\Delta}(p\bowtie c)&=\sum_{i,j}\bar{\Delta}(p_1\bowtie h_j\cdot c_1\cdot S^{-1}(h_i))\otimes h^ip_2h^j\bowtie c_2\\&=\sum_{i,j,s,t}p_1\bowtie h_s\cdot(h_j\cdot c_1\cdot S^{-1}(h_i))_1S^{-1}(h_t)\otimes h^tp_2h^s\bowtie(h_j\cdot c_1\cdot S^{-1}(h_i))_2\otimes h^ip_3h^j\bowtie c_2\\&=\sum_{i,j,s,t}p_1\bowtie h_sh_{j1}\cdot c_1\cdot S^{-1}(h_th_{i2})\otimes h^tp_2h^s\bowtie h_{j2}\cdot c_2\cdot S^{-1}(h_{i1})\otimes h^ip_3h^j\bowtie c_3.\end{aligned}$$

Evaluating the first, the third and the fifth factors at $h,h',h''\in H$ respectively, we have

$$\begin{aligned}&\sum_{i,j,s,t}p_1(h)h_sh_{j1}\cdot c_1\cdot S^{-1}(h_th_{i2})\otimes h^tp_2h^s(h')h_{j2}\cdot c_2\cdot S^{-1}(h_{i1})\otimes h^ip_3h^j(h'')c_3\\&=p_1(h)h'_3h''_4\cdot c_1\cdot S^{-1}(h'_1h''_2)\otimes p_2(h'_2)h''_5\cdot c_2\cdot S^{-1}(h''_1)\otimes p_3(h''_3)c_3\\&=p(hh'_2h''_3)h'_3h''_4\cdot c_1\cdot S^{-1}(h'_1h''_2)\otimes h''_5\cdot c_2\cdot S^{-1}(h''_1)\otimes c_3.\end{aligned}$$

On the other hand

$$\begin{aligned}(id\otimes\bar{\Delta})\bar{\Delta}(p\bowtie c)&=\sum_{i,j}p_1\bowtie h_j\cdot c_1\cdot S^{-1}(h_i)\otimes\bar{\Delta}(h^ip_2h^j\bowtie c_2)\\&=\sum_{i,j,s,t}p_1\bowtie h_j\cdot c_1\cdot S^{-1}(h_i)\otimes h^i_1p_2h^j_1\bowtie h_s\cdot c_2\cdot S^{-1}(h_t)\otimes h^th^i_2p_3h^j_2h^s\bowtie c_3.\end{aligned}$$

Evaluating the first, the third and the fifth factors at $h,h',h''\in H$ respectively, we have

$$\begin{aligned}&\sum_{i,j,s,t}p_1(h)h_j\cdot c_1\cdot S^{-1}(h_i)\otimes h^i_1p_2h^j_1(h')h_s\cdot c_2\cdot S^{-1}(h_t)\otimes h^th^i_2p_3h^j_2h^s(h'')c_3\\&=\sum_{i,j}p_1(h)h_j\cdot c_1\cdot S^{-1}(h_i)\otimes h^i_1(h'_1)p_2(h'_2)h^j_1(h'_3)h''_5\cdot c_2\cdot S^{-1}(h''_1)\otimes h^i_2(h''_2)p_3(h''_3)h^j_2(h''_4)c_3\\&=\sum_{i,j}p_1(hh'_2h''_3)h'_3h''_4\cdot c_1\cdot S^{-1}(h'_1h''_2)\otimes h''_5\cdot c_2\cdot S^{-1}(h''_1)\otimes c_3.\end{aligned}$$

Thus $\bar{\Delta}$ is coassociative. Easy to check that $\bar{\varepsilon}$ is counit. The proof is completed. □

Remark 3.2. *In particular when $C=H$ and the module action is multiplication, we can recover the Drinfel'd codouble $\widehat{D(H)}$ introduced in [12, Proposition 10.3.14].*

Proposition 3.3. *Diagonal crossed coproduct $H^{*op} \bowtie C$ is a $\widehat{D(H)}$-bimodule coalgebra with structures*

$$\widehat{D(H)} \otimes H^{*op} \bowtie C \to H^{*op} \bowtie C, \ (p \otimes h) \triangleright (q \bowtie c) = qp \bowtie h \cdot c, \tag{3.3}$$

$$H^{*op} \bowtie C \otimes \widehat{D(H)} \to H^{*op} \bowtie C, \ (q \bowtie c) \triangleleft (p \otimes h) = pq \bowtie c \cdot h, \tag{3.4}$$

for all $p, q \in H^{*op}, h \in H, c \in C$.

Proof. Obviously $H^{*op} \bowtie C$ is a left $\widehat{D(H)}$-module. And for all $p, q \in H^{*op}, h \in H, c \in C$,

$$\begin{aligned}
\bar{\Delta}((p \otimes h) \triangleright (q \bowtie c)) &= \bar{\Delta}(qp \bowtie h \cdot c) \\
&= \sum_{i,j} q_1 p_1 \bowtie h_j \cdot (h \cdot c)_1 \cdot S^{-1}(h_i) \otimes h^i q_2 p_2 h^j \bowtie (h \cdot c)_2 \\
&= \sum_{i,j} q_1 p_1 \bowtie h_j h_1 \cdot c_1 \cdot S^{-1}(h_i) \otimes h^i q_2 p_2 h^j \bowtie h_2 \cdot c_2 \\
&= \sum_{i,j} q_1 p_1 \bowtie h_i h_1 S^{-1}(h_j) h_s \cdot c_1 \cdot S^{-1}(h_t) \otimes h^t q_2 h^s h^j p_2 h^i \bowtie h_2 \cdot c_2 \\
&= (p \otimes h)_1 \triangleright (q \bowtie c)_1 \otimes (p \otimes h)_2 \triangleright (q \bowtie c)_2.
\end{aligned}$$

Thus $H^{*op} \bowtie C$ is a left $\widehat{D(H)}$-module coalgebra. Similarly one can check that $H^{*op} \bowtie C$ is also a right $\widehat{D(H)}$-module coalgebra. The proof is completed. □

4. The Construction of Braided *T*-Category $\widehat{\mathcal{YD}(H)}$

Definition 4.1. *[1, Definition 2.1] Let H be a Hopf algebra and $\alpha, \beta \in Aut_{Hopf}(H)$. An (α, β)-Yetter-Drinfel'd module over H is a vector space M such that M is a left H-module and right H-comodule with the following compatible condition*

$$h_1 \cdot m_{(0)} \otimes \beta(h_2) m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} \alpha(h_1),$$

for all $h \in H, m \in M$. We denote by ${}_H\mathcal{YD}^H(\alpha, \beta)$ the category of (α, β)-Yetter-Drinfel'd modules, morphisms being the H-linear and H-colinear.

Example 4.2. *For any Hopf algebra H and $\alpha, \beta \in Aut_{Hopf}(H)$, define $H_{\alpha,\beta}$ as follows: $H_{\alpha,\beta} = H$ with regular left H-module structure and right H-comodule structure given by*

$$\rho(h) = h_2 \otimes \beta(h_3) S^{-1} \alpha(h_1),$$

for all $h \in H$. Then $H_{\alpha,\beta} \in {}_H\mathcal{YD}^H(\alpha, \beta)$.

Let $\alpha, \beta \in Aut_{Hopf}(H)$. We define an H-bimodule coalgebra $H(\alpha, \beta)$ as follows: $H(\alpha, \beta) = H$ as coalgebra with module structures

$$H \otimes H(\alpha, \beta) \to H(\alpha, \beta), \quad h \otimes h' \mapsto \beta(h) h',$$
$$H(\alpha, \beta) \otimes H \to H(\alpha, \beta), \quad h' \otimes h \mapsto h' \alpha(h),$$

for all $h, h' \in H$.

Now consider the Yetter-Drinfel'd datum $(H, H(\alpha, \beta), H)$ and its Yetter-Drinfel'd category ${}_H\mathcal{YD}^{H(\alpha,\beta)}$.

Proposition 4.3. *With the above notations, we have the relation:*

$${}_H\mathcal{YD}^{H(\alpha,\beta)} = {}_H\mathcal{YD}^H(\alpha, \beta).$$

Consider now the diagonal crossed coproduct $C(\alpha,\beta) = H^{*op} \otimes H(\alpha,\beta)$ with the comultiplication

$$\bar{\Delta}(p \bowtie h) = \sum_{i,j} p_1 \bowtie \beta(h_j)h_1 S^{-1}\alpha(h_i) \otimes h^i p_2 h^j \bowtie h_2,$$

for all $p \in H^{*op}, h \in H$. Moreover $C(\alpha,\beta)$ is a $\widehat{D(H)}$-bimodule coalgebra with module structures

$$\widehat{D(H)} \otimes H^{*op} \bowtie H(\alpha,\beta) \to H^{*op} \otimes H(\alpha,\beta),\ p \otimes h \otimes q \bowtie h' \mapsto qp \bowtie \beta(h)h',$$
$$H^{*op} \bowtie H(\alpha,\beta) \otimes \widehat{D(H)} \to H^{*op} \otimes H(\alpha,\beta),\ q \bowtie h' \otimes p \otimes h \mapsto pq \bowtie h'\alpha(h).$$

Since H is finite dimensional, we have a category isomorphism ${}_H\mathcal{YD}^{H(\alpha,\beta)} \cong \mathcal{M}^{H^{*op}\bowtie H(\alpha,\beta)}$, hence ${}_H\mathcal{YD}^H(\alpha,\beta) \cong \mathcal{M}^{H^{*op}\bowtie H(\alpha,\beta)}$. The correspondence is given as follows. If $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, then $M \in \mathcal{M}^{H^{*op}\bowtie H(\alpha,\beta)}$ with structure

$$m_{[0]} \otimes m_{[1]} = \sum_i h_i \cdot m_{(0)} \otimes h^i \bowtie m_{(1)}.$$

Conversely if $M \in \mathcal{M}^{H^{*op}\bowtie H(\alpha,\beta)}$, then $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$ with structures

$$h \cdot m = m_{[0]}(h \otimes \varepsilon)m_{[1]},$$
$$m_{(0)} \otimes m_{(1)} = m_{[0]} \otimes (\varepsilon^* \otimes id)m_{[1]}.$$

Proposition 4.4. *Let H be a Hopf algebra and $\alpha,\beta,\gamma,\delta \in Aut_{Hopf}(H)$. If $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$, then $M \otimes N \in {}_H\mathcal{YD}^H(\delta\alpha\delta^{-1}\gamma, \delta\beta)$ with the following structures:*

$$h \cdot (m \otimes n) = h_2 \cdot m \otimes h_1 \cdot n,$$
$$(m \otimes n)_{(0)} \otimes (m \otimes n)_{(1)} = m_{(0)} \otimes n_{(0)} \otimes \delta(m_{(1)})\delta\alpha\delta^{-1}(n_{(1)}),$$

for all $h \in H, m \in M, n \in N$.

Proof. Clearly $M \otimes N$ is a left H-module and right H-comodule. We need only to verify the compatible condition.

$$\begin{aligned}
& h_1 \cdot (m \otimes n)_{(0)} \otimes \delta\beta(h_2)(m \otimes n)_{(1)} \\
&= h_2 \cdot m_{(0)} \otimes h_1 \cdot n_{(0)} \otimes \delta(\beta(h_3)m_{(1)})\delta\alpha\delta^{-1}(n_{(1)}) \\
&= (h_3 \cdot m)_{(0)} \otimes h_1 \cdot n_{(0)} \otimes \delta((h_3 \cdot m)_{(1)})\delta\alpha\delta^{-1}(\delta(h_2)n_{(1)}) \\
&= (h_3 \cdot m)_{(0)} \otimes (h_2 \cdot n)_{(0)} \otimes \delta((h_3 \cdot m)_{(1)})\delta\alpha\delta^{-1}((h_2 \cdot n)_{(1)}\gamma(h_1)) \\
&= (h_2 \cdot (m \otimes n))_{(0)} \otimes (h_2 \cdot (m \otimes n))_{(1)}\delta\alpha\delta^{-1}\gamma(h_1).
\end{aligned}$$

The proof is completed. □

Note that if $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$ and $P \in {}_H\mathcal{YD}^H(\mu,\nu)$, then $(M \otimes N) \otimes P = M \otimes (N \otimes P)$ as an object in ${}_H\mathcal{YD}^H(\nu\delta\alpha\delta^{-1}\gamma\nu^{-1}\mu, \nu\delta\beta)$.

Denote $G = Aut_{Hopf}(H) \times Aut_{Hopf}(H)$, a group with multiplication

$$(\alpha,\beta) * (\gamma,\delta) = (\delta\alpha\delta^{-1}\gamma, \delta\beta).$$

The unit is (id, id) and $(\alpha,\beta)^{-1} = (\beta^{-1}\alpha^{-1}\beta, \beta^{-1})$.

Proposition 4.5. *Let $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$ and $(\alpha,\beta) \in G$. Define ${}^{(\alpha,\beta)}N = N$ as vector space with structures*

$$h \rightharpoonup n = \alpha^{-1}\beta(h) \cdot n,$$
$$n_{<0>} \otimes n_{<1>} = n_{(0)} \otimes \beta^{-1}\delta\alpha\delta^{-1}(n_{(1)}).$$

Then ${}^{(\alpha,\beta)}N \in {}_H\mathcal{YD}^H(\beta^{-1}\delta\alpha\delta^{-1}\gamma\alpha^{-1}\beta, \beta^{-1}\delta\beta) = {}_H\mathcal{YD}^H((\alpha,\beta) * (\gamma,\delta) * (\alpha,\beta)^{-1})$.

Proof. Easy to see that ${}^{(\alpha,\beta)}N$ is a left H-module and right H-comodule. We check the compatible condition.

$$\begin{aligned}
&h_1 \rightharpoonup n_{<0>} \otimes \beta^{-1}\delta\beta(h_2)n_{<1>}\\
&= \alpha^{-1}\beta(h_1)\cdot n_{(0)} \otimes \beta^{-1}\delta\beta(h_2)\beta^{-1}\delta\alpha\delta^{-1}(n_{(1)})\\
&= (\alpha^{-1}\beta(h_2)\cdot n)_{(0)} \otimes \beta^{-1}\delta\alpha\delta^{-1}[(\alpha^{-1}\beta(h_2)\cdot n)_{(1)}\gamma\delta\alpha^{-1}\beta(h_1)]\\
&= (\alpha^{-1}\beta(h_2)\cdot n)_{(0)} \otimes \beta^{-1}\delta\alpha\delta^{-1}((\alpha^{-1}\beta(h_2)\cdot n)_{(1)})\beta^{-1}\delta\alpha\delta^{-1}\gamma\alpha^{-1}\beta(h_1)\\
&= (\alpha^{-1}\beta(h_2)\cdot n)_{<0>} \otimes (\alpha^{-1}\beta(h_2)\cdot n)_{<1>}\beta^{-1}\delta\alpha\delta^{-1}\gamma\alpha^{-1}\beta(h_1)\\
&= (h_2 \rightharpoonup n)_{<0>} \otimes (h_2 \rightharpoonup n)_{<1>}\beta^{-1}\delta\alpha\delta^{-1}\gamma\alpha^{-1}\beta(h_1).
\end{aligned}$$

The proof is completed. □

Remark 4.6. *Let $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$ and $(\mu,\nu)\in G$. We have*

$$^{(\alpha,\beta)*(\gamma,\delta)}N = {}^{(\alpha,\beta)}({}^{(\gamma,\delta)}N)$$

as an object in ${}_H\mathcal{YD}^H((\alpha,\beta)*(\mu,\nu)*(\gamma,\delta)*(\mu,\nu)^{-1}*(\alpha,\beta)^{-1})$. and

$$^{(\mu,\nu)}(M\otimes N) = {}^{(\mu,\nu)}M \otimes {}^{(\mu,\nu)}N$$

as an object in ${}_H\mathcal{YD}^H((\mu,\nu)*(\alpha,\beta)*(\gamma,\delta)*(\mu,\nu)^{-1})$.

Proposition 4.7. *Let $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$ and $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$. Denote ${}^MN = {}^{(\alpha,\beta)}N$ as an object in ${}_H\mathcal{YD}^H((\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1})$. Define the map*

$$c_{M,N}: M\otimes N \to {}^MN\otimes M, \quad m\otimes n \mapsto \alpha^{-1}(m_{(1)})\cdot n\otimes m_{(0)},$$

for all $m\in M, n\in N$. Then $c_{M,N}$ is H-linear H-colinear and satisfies the relations (1.1) and (1.2). And $c_{{}^PM,{}^PN} = c_{M,N}$. Moreover $c_{M,N}$ is bijective with inverse $c^{-1}_{M,N}(n\otimes m) = m_{(0)}\otimes\alpha^{-1}S(m_{(1)})\cdot n$.

Proof. We prove that $c_{M,N}$ is H-linear H-colinear. Indeed

$$\begin{aligned}
c_{M,N}(h\cdot(m\otimes n)) &= c_{M,N}(h_2\cdot m\otimes h_1\cdot n)\\
&= \alpha^{-1}((h_2\cdot m)_{(1)}\alpha(h_1))\cdot n\otimes(h_2\cdot m)_{(0)}\\
&= \alpha^{-1}(\beta(h_2)m_{(1)})\cdot n\otimes h_1\cdot m_{(0)}\\
&= h\cdot c_{M,N}(m\otimes n).
\end{aligned}$$

And

$$\begin{aligned}
&c_{M,N}(m\otimes n)_{(0)}\otimes c_{M,N}(m\otimes n)_{(1)}\\
&= (\alpha^{-1}(m_{(1)})\cdot n)_{<0>}\otimes m_{(0)(0)}\otimes\beta((\alpha^{-1}(m_{(1)})\cdot n)_{<1>})\delta\alpha\delta^{-1}\gamma\alpha^{-1}(m_{(0)(1)})\\
&= (\alpha^{-1}(m_{(1)2})\cdot n)_{(0)}\otimes m_{(0)}\otimes\delta\alpha\delta^{-1}((\alpha^{-1}(m_{(1)2})\cdot n)_{(1)}\gamma\alpha^{-1}(m_{(1)1}))\\
&= \alpha^{-1}(m_{(1)1})\cdot n_{(0)}\otimes m_{(0)}\otimes\delta(m_{(1)2})\delta\alpha\delta^{-1}(n_{(1)})\\
&= c_{M,N}((m\otimes n)_{(0)})\otimes(m\otimes n)_{(1)}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&(c_{M,{}^NP}\otimes id)(id\otimes c_{N,P})(m\otimes n\otimes p)\\
&= (c_{M,{}^NP}\otimes id)(m\otimes\gamma^{-1}(n_{(1)})\cdot p\otimes n_{(0)})\\
&= \alpha^{-1}(m_{(1)})\rightharpoonup(\gamma^{-1}(n_{(1)})\cdot p)\otimes m_{(0)}\otimes n_{(0)}\\
&= \gamma^{-1}\delta\alpha^{-1}(m_{(1)})\gamma^{-1}(n_{(1)})\cdot p\otimes m_{(0)}\otimes n_{(0)}\\
&= \gamma^{-1}\delta\alpha^{-1}\delta^{-1}((m\otimes n)_{(1)})\cdot p\otimes(m\otimes n)_{(0)}\\
&= c_{M\otimes N,P}(m\otimes n\otimes p).
\end{aligned}$$

Similarly we can prove (1.2). The proof is completed. □

Define $\widehat{\mathcal{YD}(H)}$ as the disjoint union of all ${}_H\mathcal{YD}^H(\alpha,\beta)$ with $(\alpha,\beta)\in G$. If we endow $\widehat{\mathcal{YD}(H)}$ with monoidal structure given in Proposition 4.4, then it becomes a strict monoidal category with the unit k as an object in ${}_H\mathcal{YD}^H$ (with trivial structure).

The group homomorphism $\psi: G\longrightarrow Aut(\widehat{\mathcal{YD}(H)})$, $(\alpha,\beta)\mapsto\psi_{(\alpha,\beta)}$ is defined on components as

$$\begin{aligned}\psi_{(\alpha,\beta)}:{}_H\mathcal{YD}^H(\gamma,\delta)&\longrightarrow{}_H\mathcal{YD}^H((\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}),\\ \psi_{(\alpha,\beta)}(N)&={}^{(\alpha,\beta)}N.\end{aligned}$$

and the functor acts on morphisms as identity. The braiding in $\widehat{\mathcal{YD}(H)}$ is given by the family $c=\{c_{M,N}\}$. Hence we have

Proposition 4.8. *$\widehat{\mathcal{YD}(H)}$ is a braided T-category over G.*

It is well known that for a Hopf algebra with a bijective antipode, the subcategory ${}_H\mathcal{YD}^H_{fd}$ of all finite dimensional objects in ${}_H\mathcal{YD}^H$ is rigid, i.e., every object has left and right dualities. For the category $\widehat{\mathcal{YD}(H)}$, we have the following result.

Proposition 4.9. *Let $M\in{}_H\mathcal{YD}^H(\alpha,\beta)$ and suppose that M is finite dimensional. Then $M^*=Hom(M,k)$ belongs to ${}_H\mathcal{YD}^H(\beta^{-1}\alpha^{-1}\beta,\beta^{-1})$ with*

$$\begin{aligned}&(h\cdot f)(m)=f(S^{-1}(h)\cdot m),\\ &f_{(0)}(m)f_{(1)}=f(m_{(0)})\beta^{-1}\alpha^{-1}S(m_{(1)}),\end{aligned}$$

for all $h\in H, m\in M$ and $f\in M^*$. Then M^* is a left dual of M. Similarly we can define the right dual ${}^*M=Hom(M,k)$ of M with

$$\begin{aligned}&(h\cdot f)(m)=f(S(h)\cdot m),\\ &f_{(0)}(m)f_{(1)}=f(m_{(0)})\beta^{-1}\alpha^{-1}S^{-1}(m_{(1)}).\end{aligned}$$

Therefore the category $\widehat{\mathcal{YD}(H)}_{fd}$, the subcategory of $\widehat{\mathcal{YD}(H)}$ consisting of finite dimensional objects, is rigid.

Proof. First of all, M^* is an object in ${}_H\mathcal{YD}^H(\beta^{-1}\alpha^{-1}\beta,\beta^{-1})$. Indeed, obviously M^* is a left H-module and right H-comodule. And

$$\begin{aligned}&(h_2\cdot f)_{(0)}(m)(h_2\cdot f)_{(1)}\beta^{-1}\alpha^{-1}\beta(h_1)\\ &=(h_2\cdot f)(m_{(0)})S(m_{(1)})\beta^{-1}\alpha^{-1}\beta(h_1)\\ &=f(S^{-1}(h_2)\cdot m_{(0)})\beta^{-1}\alpha^{-1}S(m_{(1)})\beta^{-1}\alpha^{-1}\beta(h_1)\\ &=f(S^{-1}(h_2)\cdot m_{(0)})S(\beta^{-1}\alpha^{-1}(\beta S^{-1}(h_1)m_{(1)}))\\ &=f((S^{-1}(h_1)\cdot m)_{(0)})S(\beta^{-1}\alpha^{-1}((S^{-1}(h_1)\cdot m)_{(1)})\beta^{-1}S^{-1}(h_2))\\ &=f((S^{-1}(h_1)\cdot m)_{(0)})\beta^{-1}(h_2)S(\beta^{-1}\alpha^{-1}((S^{-1}(h_1)\cdot m)_{(1)}))\\ &=f_{(0)}(S^{-1}(h_1)\cdot m)\beta^{-1}(h_2)f_{(1)}\\ &=(h_1\cdot f_{(0)})(m)\beta^{-1}(h_2)f_{(1)},\end{aligned}$$

as required. Define maps

$$\begin{aligned}&b_M:k\to M\otimes M^*,\quad 1\mapsto\sum_i m_i\otimes m^i,\\ &d_M:M^*\otimes M\to k,\quad f\otimes m\mapsto f(m),\end{aligned}$$

where $\{m_i\}$ and $\{m^i\}$ are basis and dual basis of M. We need to prove that b_M and d_M are H-linear. We compute

$$\begin{aligned}(h\cdot b_M(1))(m) &= (h\cdot \sum_i m_i\otimes m^i)(m)\\ &= (\sum_i h_2\cdot m_i\otimes h_1\cdot m^i)(m)\\ &= \sum_i h_2\cdot m_i m^i(S^{-1}(h_1)\cdot m)\\ &= h_2S^{-1}(h_1)\cdot m\\ &= \varepsilon(h)b_M(1)(m),\end{aligned}$$

and

$$\begin{aligned}d_M(h\cdot(f\otimes m)) &= d_M(h_2\cdot f\otimes h_1\cdot m)\\ &= (h_2\cdot f)(h_1\cdot m)\\ &= f(S^{-1}(h_2)h_1\cdot m)\\ &= \varepsilon(h)f(m)\\ &= h\cdot d_M(f\otimes m).\end{aligned}$$

They are also H-colinear. Indeed,

$$\begin{aligned}b_M(1)_{(0)}(m)\otimes b_M(1)_{(1)} &= \sum_i m_{i(0)}m^i_{(0)}(m)\otimes\beta^{-1}(m_{i(1)})\beta^{-1}\alpha\beta(m^i_{(1)})\\ &= \sum_i m_{i(0)}m^i(m_{(0)})\otimes\beta^{-1}(m_{i(1)})\beta^{-1}(S(m_{(1)}))\\ &= m_{(0)}\otimes\beta^{-1}(m_{(1)1})S(m_{(1)2})\\ &= b_M(1)(m)\otimes 1,\end{aligned}$$

and

$$\begin{aligned}d_M((f\otimes m)_{(0)})\otimes(f\otimes m)_{(1)} &= d_M(f_{(0)}\otimes m_{(0)})\otimes\beta(f_{(1)})\alpha^{-1}(m_{(1)})\\ &= f_{(0)}(m_{(0)})\beta(f_{(1)})\alpha^{-1}(m_{(1)})\\ &= f(m_{(0)})\alpha^{-1}(S(m_{(1)1})m_{(1)2})\\ &= d_M(f\otimes m)_{(0)}\otimes d_M(f\otimes m)_{(1)}.\end{aligned}$$

It is straightforward to verify that
$(id_M\otimes d_M)(b_M\otimes id_M) = id_M$ and $(d_M\otimes id_{M^*})(id_{M^*}\otimes b_M) = id_{M^*}$.
Similarly we can prove that ${}^*M$ is a right dual of M. The proof is completed. □

Now we are in a position to construct a coquasitriangular Turaev group algebra over G, denoted by $CT(H)$ such that the T-category $Corep(CT(H))$ of corepresentation of $CT(H)$ is isomorphic to $\widehat{\mathcal{YD}(H)}$ as braided T-categories.

For $(\alpha,\beta)\in G$, the (α,β)-component $CT(H)_{\alpha,\beta}$ will be the diagonal crossed coproduct $H^{*op}\bowtie H(\alpha,\beta)$. Define multiplication by

$$\begin{aligned}m_{(\alpha,\beta),(\gamma,\delta)}: &H^{*op}\bowtie H(\alpha,\beta)\otimes H^{*op}\bowtie H(\gamma,\delta)\longrightarrow H^{*op}\bowtie H((\alpha,\beta)*(\gamma,\delta)),\\ &(p\bowtie h)\otimes(q\bowtie h')\mapsto qp\bowtie\delta(h)\delta\alpha\delta^{-1}(h'). \qquad (4.1)\end{aligned}$$

Then we have the following result.

Proposition 4.10. *$CT(H)$ becomes a Turaev G-algebra under the diagonal crossed coproduct and multiplication (4.1). The antipode is given by*

$$S_{(\alpha,\beta)} : H^{*op} \bowtie H(\alpha,\beta) \longrightarrow H^{*op} \bowtie H((\alpha,\beta)^{-1}),$$
$$p \bowtie h \mapsto \sum_{i,j} h^i S^{-1*}(p) S^{-1*}(h^j) \bowtie \beta^{-1}(h_j)\beta^{-1}\alpha^{-1}S(h_1)\beta^{-1}\alpha^{-1}\beta(h_i).$$

Proof. The multiplication is associative. For all $f \bowtie h \in H^{*op} \bowtie H(\alpha,\beta), p \bowtie h' \in H^{*op} \bowtie H(\gamma,\delta), q \bowtie h'' \in H^{*op} \bowtie H(\mu,\nu)$, we compute

$$\begin{aligned}
[(f \bowtie h)(p \bowtie h')](q \bowtie h'') &= (pf \bowtie \delta(h)\delta\alpha\delta^{-1}(h'))(q \bowtie h'') \\
&= qpf \bowtie \nu\delta(h)\nu\delta\alpha\delta^{-1}(h')\nu\delta\alpha\delta^{-1}\gamma\nu^{-1}(h'') \\
&= (f \bowtie h)(qp \bowtie \nu(h')\nu\gamma\nu^{-1}(h'')) \\
&= (f \bowtie h)[(p \bowtie h')(q \bowtie h'')],
\end{aligned}$$

as claimed. Next we prove that $m_{(\alpha,\beta),(\gamma,\delta)}$ is a coalgebra map. Indeed,

$$\begin{aligned}
& m_{(\alpha,\beta),(\gamma,\delta)}((p \bowtie h)_1 \otimes (q \bowtie h')_1) \otimes m_{(\alpha,\beta),(\gamma,\delta)}((p \bowtie h)_2 \otimes (q \bowtie h')_2) \\
&= \sum_{i,j,s,t} m_{(\alpha,\beta),(\gamma,\delta)}(p_1 \bowtie \beta(h_j)h_1\alpha S^{-1}(h_i) \otimes q_1 \bowtie \delta(h_s)h'_1\gamma S^{-1}(h_t)) \\
&\otimes m_{(\alpha,\beta),(\gamma,\delta)}(h^i p_2 h^j \bowtie h_2 \otimes h^t q_2 h^s \bowtie h'_2) \\
&= \sum_{i,j,s,t} q_1 p_1 \bowtie \delta\beta(h_j)\delta(h_1)\delta\alpha S^{-1}(h_i)\delta\alpha(h_s)\delta\alpha\delta^{-1}(h'_1)\delta\alpha\delta^{-1}\gamma S^{-1}(h_t) \\
&\otimes h^t q_2 h^s h^i p_2 h^j \bowtie \delta(h_2)\delta\alpha\delta^{-1}(h'_2) \\
&= \sum_{j,t} q_1 p_1 \bowtie \delta\beta(h_j)\delta(h_1)\delta\alpha\delta^{-1}(h'_1)\delta\alpha\delta^{-1}\gamma S^{-1}(h_t) \otimes h^t q_2 p_2 h^j \bowtie \delta(h_2)\delta\alpha\delta^{-1}(h'_2) \\
&= (qp \bowtie \delta(h)\delta\alpha\delta^{-1}(h'))_1 \otimes (qp \bowtie \delta(h)\delta\alpha\delta^{-1}(h'))_2 \\
&= m_{(\alpha,\beta),(\gamma,\delta)}(p \bowtie h \otimes q \bowtie h')_1 \otimes m_{(\alpha,\beta),(\gamma,\delta)}(p \bowtie h \otimes q \bowtie h')_2,
\end{aligned}$$

as required. Easy to see that $(\varepsilon \bowtie 1)_1 \otimes (\varepsilon \bowtie 1)_2 = \varepsilon \bowtie 1 \otimes \varepsilon \bowtie 1$.

We now check that S is the antipode of $CT(H)$.

$$\begin{aligned}
& S_{(\alpha,\beta)}((p \bowtie h)_1)(p \bowtie h)_2 \\
&= \sum_{i,j} S_{(\alpha,\beta)}(p_1 \bowtie \beta(h_j)h_1\alpha S^{-1}(h_i))(h^i p_2 h^j \bowtie h_2) \\
&= \sum_{i,j,s,t} (h^s S^{-1*}(p_1) S^{-1*}(h^t) \bowtie \beta^{-1}(h_t h_i)\beta^{-1}\alpha^{-1}S(h_1)\beta^{-1}\alpha^{-1}\beta S(h_j)\beta^{-1}\alpha^{-1}\beta(h_s))(h^i p_2 h^j \bowtie h_2) \\
&= \sum_{i,j,s,t} h^i p_2 h^j h^s S^{-1*}(p_1) S^{-1*}(h^t) \bowtie h_t h_i \alpha^{-1}S(h_1)\alpha^{-1}\beta(S(h_j)h_s)\alpha^{-1}(h_2) \\
&= \sum_{i,j,t} h^i p_2 h^j S^{-1*}(p_1) S^{-1*}(h^t) \bowtie h_t h_i \alpha^{-1}S(h_1)\alpha^{-1}\beta(S(h_{i1})h_{i2})\alpha^{-1}(h_2) \\
&= \sum_{i,j,t} h^i p_2 S^{-1*}(p_1) S^{-1*}(h^t) \bowtie h_t h_i \alpha^{-1}S(h_1)\alpha^{-1}(h_2) \\
&= p(1)\varepsilon(h)\varepsilon \bowtie 1.
\end{aligned}$$

Thus $S_{(\alpha,\beta)} * id_{(\alpha,\beta)} = \varepsilon_{(\alpha,\beta)}\varepsilon \bowtie 1$. Similarly one can verify that $id_{(\alpha,\beta)} * S_{(\alpha,\beta)} = \varepsilon_{(\alpha,\beta)}\varepsilon \bowtie 1$. S is the antipode of $CT(H)$. The proof is completed. □

Proposition 4.11. *Moreover $CT(H)$ is a crossed Turaev G-algebra with the crossing ψ given by*

$$\begin{aligned}\psi_{(\alpha,\beta)} :& H^{*op}\bowtie H(\gamma,\delta)\longrightarrow H^{*op}\bowtie H((\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}),\\ & p\bowtie h\mapsto p\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\alpha\delta^{-1}(h).\end{aligned}$$

Proof. First of all $\psi_{(\alpha,\beta)}$ is bijective and for all $p\in H^*, h\in H$,

$$\begin{aligned}&\psi_{(\alpha,\beta)}(p\bowtie h)_1\otimes\psi_{(\alpha,\beta)}(p\bowtie h)_2\\ &=(p\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\alpha\delta^{-1}(h))_1\otimes(p\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\alpha\delta^{-1}(h))_2\\ &=\sum_{i,j}p_1\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\beta(h_j)\beta^{-1}\delta\alpha\delta^{-1}(h_1)\beta^{-1}\delta\alpha\delta^{-1}\gamma\alpha^{-1}\beta S^{-1}(h_i)\otimes h^i(p_2\circ\alpha^{-1}\beta)h^j\bowtie\beta^{-1}\delta\alpha\delta^{-1}(h_2)\\ &=\sum_{i,j}p_1\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\alpha(h_j)\beta^{-1}\delta\alpha\delta^{-1}(h_1)\beta^{-1}\delta\alpha\delta^{-1}\gamma S^{-1}(h_i)\otimes(h^ip_2h^j)\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\alpha\delta^{-1}(h_2)\\ &=\sum_{i,j}\psi_{(\alpha,\beta)}(p_1\bowtie\delta(h_j)h_1\gamma S^{-1}(h_i))\otimes\psi_{(\alpha,\beta)}(h^ip_2h^j\bowtie h_2)\\ &=\psi_{(\alpha,\beta)}((p\bowtie h)_1)\otimes\psi_{(\alpha,\beta)}((p\bowtie h)_2).\end{aligned}$$

Thus $\psi_{(\alpha,\beta)}$ is a coalgebra isomorphism. And

(i) ψ is multiplicative, since for $h\in H(\mu,\nu)$

$$\begin{aligned}\psi_{(\alpha,\beta)}\psi_{(\gamma,\delta)}(p\bowtie h)&=\psi_{(\alpha,\beta)}(p\circ\gamma^{-1}\delta\bowtie\delta^{-1}\nu\gamma\nu^{-1}(h))\\ &=p\circ\gamma^{-1}\delta\alpha^{-1}\beta\bowtie\beta^{-1}\delta^{-1}\nu\delta\alpha\delta^{-1}\gamma\nu^{-1}(h)\\ &=\psi_{(\delta\alpha\delta^{-1}\gamma,\delta\beta)}(p\bowtie h)\\ &=\psi_{(\alpha,\beta)*(\gamma,\delta)}(p\bowtie h).\end{aligned}$$

Obviously $\psi_{(1,1)}(CT(\alpha,\beta))=id_{(\alpha,\beta)}$.

(ii) For $p,q\in H^*$ and $h\in H(\gamma,\delta), h'\in H(\mu,\nu)$,

$$\begin{aligned}\psi_{(\alpha,\beta)}(p\bowtie h)\psi_{(\alpha,\beta)}(q\bowtie h')&=(p\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\alpha\delta^{-1}(h))(q\circ\alpha^{-1}\beta\bowtie\beta^{-1}\nu\alpha\nu^{-1}(h'))\\ &=qp\circ\alpha^{-1}\beta\bowtie\beta^{-1}\nu\delta\alpha\delta^{-1}(h)\beta^{-1}\nu\delta\alpha\delta^{-1}\gamma\nu^{-1}(h')\\ &=qp\circ\alpha^{-1}\beta\bowtie\beta^{-1}\nu\delta\alpha\delta^{-1}\nu^{-1}(\nu(h)\nu\gamma\nu^{-1}(h'))\\ &=\psi_{(\alpha,\beta)}(qp\bowtie\nu(h)\nu\gamma\nu^{-1}(h'))\\ &=\psi_{(\alpha,\beta)}((p\bowtie h)(q\bowtie h')).\end{aligned}$$

(iii) $\psi_{(\alpha,\beta)}(\varepsilon\bowtie 1)=\varepsilon\bowtie 1$.

(iv)

$$\begin{aligned}\psi_{(\alpha,\beta)}S_{(\gamma,\delta)}(p\bowtie h)&=\sum_{i,j}\psi_{(\alpha,\beta)}(h^iS^{-1*}(p)S^{-1*}(h^j)\bowtie\delta^{-1}(h_j)\delta^{-1}\gamma^{-1}(S(h))\delta^{-1}\gamma^{-1}\delta(h_i))\\ &=\sum_{i,j}(h^iS^{-1*}(p)S^{-1*}(h^j))\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta^{-1}\alpha\delta(\delta^{-1}(h_j)\delta^{-1}\gamma^{-1}(S(h))\delta^{-1}\gamma^{-1}\delta(h_i))\\ &=\sum_{i,j}(h^iS^{-1*}(p)S^{-1*}(h^j))\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta^{-1}\alpha(h_j\gamma^{-1}(S(h))\gamma^{-1}\delta(h_i))\\ &=\sum_{i,j}h^iS^{-1*}(p\circ\alpha^{-1}\beta)S^{-1*}(h^j)\bowtie\beta^{-1}\delta^{-1}\beta(h_j)\beta^{-1}\delta^{-1}\alpha\gamma^{-1}S(h)\beta^{-1}\delta^{-1}\alpha\gamma^{-1}\delta\alpha^{-1}\beta(h_i)\\ &=S_{(\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}}(p\circ\alpha^{-1}\beta\bowtie\beta^{-1}\delta\alpha\delta^{-1}(h))\\ &=S_{(\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}}\psi_{(\alpha,\beta)}(p\bowtie h).\end{aligned}$$

The proof is completed. □

Proposition 4.12. *$CT(H)$ is coquasitriangular with the structure*

$$\sigma_{(\alpha,\beta),(\gamma,\delta)}(p\bowtie h,q\bowtie h')=p(\delta^{-1}(h'))q(1)\varepsilon(h).$$

Proof. For all $f,p,q\in H^*, h\in H(\alpha,\beta), h'\in H(\gamma,\delta), h''\in H(\mu,\nu)$,
For (TCT1):

$$\begin{aligned}\sigma_{(\alpha,\beta)*(\gamma,\delta),(\mu,\nu)}((f\bowtie h)(p\bowtie h'),(q\bowtie h''))&=\sigma_{(\alpha,\beta)*(\gamma,\delta),(\mu,\nu)}(pf\bowtie\delta(h)\delta\alpha\delta^{-1}(h'),(q\bowtie h''))\\&=pf(\nu^{-1}(h''))q(1)\varepsilon(hh')\\&=p(\nu^{-1}(h''_1))f(\nu^{-1}(h''_2))q(1)\varepsilon(hh'),\end{aligned}$$

and

$$\begin{aligned}&\sigma_{(\alpha,\beta),(\mu,\nu)}(f\bowtie h,(q\bowtie h'')_2)\sigma_{(\gamma,\delta),(\mu,\nu)}(p\bowtie h',(q\bowtie h'')_1)\\&=\sum_{i,j}\sigma_{(\alpha,\beta),(\mu,\nu)}(f\bowtie h,h^iq_2h^j\bowtie h''_2)\sigma_{(\gamma,\delta),(\mu,\nu)}(p\bowtie h',q_1\bowtie\nu(h_j)h''_1\mu S^{-1}(h_i))\\&=\sum_{i,j}f(\nu^{-1}(h''_2))h^j(1)q_2(1)h^i(1)\varepsilon(h)p(h_j\nu^{-1}(h''_1)\nu^{-1}\mu S^{-1}(h_i))\\&=f(\nu^{-1}(h''_2))p(\nu^{-1}(h''_1))\varepsilon(hh')q(1).\end{aligned}$$

For (TCT2):

$$\begin{aligned}\sigma_{(\alpha,\beta),(\gamma,\delta)*(\mu,\nu)}(f\bowtie h,(p\bowtie h')(q\bowtie h''))&=\sigma_{(\alpha,\beta),(\gamma,\delta)*(\mu,\nu)}(f\bowtie h,qp\bowtie\nu(h')\nu\gamma\nu^{-1}(h''))\\&=f(\delta^{-1}(h'\gamma\nu^{-1}(h'')))qp(1)\varepsilon(h),\end{aligned}$$

and

$$\begin{aligned}&\sigma_{(\alpha,\beta),(\gamma,\delta)}((f\bowtie h)_1,p\bowtie h')\sigma_{(\gamma,\delta)^{-1}*(\alpha,\beta)*(\gamma,\delta),(\mu,\nu)}(\psi_{(\gamma,\delta)^{-1}}((f\bowtie h)_2),q\bowtie h'')\\&=\sum_{i,j}\sigma_{(\alpha,\beta),(\gamma,\delta)}(f_1\bowtie\beta(h_j)h_1\alpha S^{-1}(h_i),p\bowtie h')\\&\quad\sigma_{(\gamma,\delta)^{-1}*(\alpha,\beta)*(\gamma,\delta),(\mu,\nu)}(\psi_{(\gamma,\delta)^{-1}}(h^if_2h^j\bowtie h_2),q\bowtie h'')\\&=f_1(\delta^{-1}(h'))p(1)\sigma_{(\gamma,\delta)^{-1}*(\alpha,\beta)*(\gamma,\delta),(\mu,\nu)}(f_2\circ\delta^{-1}\gamma\bowtie\delta\beta\delta^{-1}\gamma^{-1}\delta\beta^{-1}(h_2),q\bowtie h'')\\&=f_1(\delta^{-1}(h'))qp(1)f_2(\delta^{-1}\gamma\nu^{-1}(h''))\varepsilon(h)\\&=f(\delta^{-1}(h')\delta^{-1}\gamma\nu^{-1}(h''))qp(1)\varepsilon(h).\end{aligned}$$

For (TCT3):

$$\begin{aligned}&\sigma_{(\alpha,\beta),(\gamma,\delta)}((f\bowtie h)_1,(p\bowtie h')_1)(p\bowtie h')_2\psi_{(\gamma,\delta)^{-1}}((f\bowtie h)_2)\\&=\sum_{i,j,s,t}\sigma_{(\alpha,\beta),(\gamma,\delta)}(f_1\bowtie\beta(h_j)h_1\alpha S^{-1}(h_i),p_1\bowtie\delta(h_s)h'_1\gamma S^{-1}(h_t))(h^tp_2h^s\bowtie h'_2)\psi_{(\gamma,\delta)^{-1}}(h^if_2h^j\bowtie h_2)\\&=\sum_{s,t}f_1(h_s\delta^{-1}(h'_1)\delta^{-1}\gamma S^{-1}(h_t))p_1(1)(h^tp_2h^s\bowtie h'_2)\psi_{(\gamma,\delta)^{-1}}(f_2\bowtie h)\\&=\sum_{s,t}f_1(h_s\delta^{-1}(h'_1)\delta^{-1}\gamma S^{-1}(h_t))p_1(1)(h^tp_2h^s\bowtie h'_2)(f_2\circ\delta^{-1}\gamma\bowtie\delta\beta\delta^{-1}\gamma^{-1}\delta\beta^{-1}(h))\\&=\sum_{s,t}f_1(h_s\delta^{-1}(h'_1)\delta^{-1}\gamma S^{-1}(h_t))(f_2\circ\delta^{-1}\gamma)h^tph^s\bowtie\delta\beta\delta^{-1}(h'_2)\delta(h)\\&=\sum_{s,t}f_2(\delta^{-1}(h'_1))(f_4\circ\delta^{-1}\gamma)(f_3\circ\delta^{-1}\gamma S^{-1})pf_1\bowtie\delta\beta\delta^{-1}(h'_2)\delta(h)\\&=f_2(\delta^{-1}(h'_1))pf_1\bowtie\delta\beta\delta^{-1}(h'_2)\delta(h),\end{aligned}$$

and

$$\begin{aligned}
&(f \bowtie h)_1(p \bowtie h')_1\sigma_{(\alpha,\beta),(\gamma,\delta)}((f \bowtie h)_2, (p \bowtie h')_2)\\
&= \sum_{i,j,s,t}(f_1 \bowtie \beta(h_j)h_1\alpha S^{-1}(h_i))(p_1 \bowtie \delta(h_s)h'_1\gamma S^{-1}(h_t))\sigma_{(\alpha,\beta),(\gamma,\delta)}(h^i f_2 h^j \bowtie h_2, h^t p_2 h^s \bowtie h'_2)\\
&= \sum_{i,j,s,t} p_1 f_1 \bowtie \delta\beta(h_j)\delta(h_1)\delta\alpha S^{-1}(h_i)\delta\alpha(h_s)\delta\alpha\delta^{-1}(h'_1)\delta\alpha\delta^{-1}\gamma S^{-1}(h_t)h^i f_2 h^j(\delta^{-1}(h'_2))\varepsilon(h_2)h^t p_2 h^s(1)\\
&= p f_1 \bowtie \delta\beta\delta^{-1}(h'_4)\delta(h)\delta\alpha\delta^{-1}S^{-1}(h'_2)\delta\alpha\delta^{-1}(h'_1)f_2(\delta^{-1}(h'_3))\\
&= p f_1 \bowtie \delta\beta\delta^{-1}(h'_2)\delta(h)f_2(\delta^{-1}(h'_1)).
\end{aligned}$$

For (TCT4):

$$\begin{aligned}
&\sigma_{(\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1},(\alpha,\beta)*(\mu,\nu)*(\alpha,\beta)^{-1}}(\psi_{(\alpha,\beta)}(p \bowtie h'), \psi_{(\alpha,\beta)}(q \bowtie h''))\\
&= \sigma_{(\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1},(\alpha,\beta)*(\mu,\nu)*(\alpha,\beta)^{-1}}(p \circ \alpha^{-1}\beta \bowtie \beta^{-1}\delta\alpha\delta^{-1}(h'), q \circ \alpha^{-1}\beta \bowtie \beta^{-1}\nu\alpha\nu^{-1}(h''))\\
&= p(\nu^{-1}(h''))q(1)\varepsilon(h')\\
&= \sigma_{(\gamma,\delta),(\mu,\nu)}(p \bowtie h', q \bowtie h'').
\end{aligned}$$

The proof is completed. □

By the arguments after Proposition 4.3 we obtain the main result:

Theorem 4.13. *Corep(CT(H)) and $\widehat{\mathcal{YD}(H)}$ are isomorphic as braided T-categories over G.*

Example 4.14. *Let π be a group, then we have a group algebra $k(\pi)$. It is well known that the group $Aut_{Hopf}(k(\pi))$ of Hopf automorphisms of $k(\pi)$ is equal to the group $Aut(\pi)$ of automorphisms of π. Let $\alpha, \beta \in Aut(\pi)$. An (α,β)-Yetter-Drinfel'd module is a left π-module M with a decomposition $M = \bigoplus_{a\in\pi} M_a$, where $M_a = \{m \in M | m_{(0)}\otimes m_{(1)} = m\otimes a\}$.*

If $\alpha,\beta,\gamma,\delta \in Aut(\pi)$, $M \in {}_{k(\pi)}\mathcal{YD}^{k(\pi)}(\alpha,\beta)$ and $N \in {}_{k(\pi)}\mathcal{YD}^{k(\pi)}(\gamma,\delta)$, then $M\otimes N \in {}_{k(\pi)}\mathcal{YD}^{k(\pi)}(\delta\alpha\delta^{-1}\gamma, \delta\beta)$ with action $a\cdot(m\otimes n) = a\cdot m\otimes a\cdot n$ for all $a \in \pi, m \in M, n \in N$, and decomposition $M\otimes N = \bigoplus_{c\in\pi}(\bigoplus_{ab=c} M_{\delta^{-1}(a)} \otimes N_{\delta\alpha^{-1}\delta^{-1}(b)})$.

If $\alpha,\beta \in Aut(\pi)$ and $N \in {}_{k(\pi)}\mathcal{YD}^{k(\pi)}(\gamma,\delta)$, then ${}^{(\alpha,\beta)}N = N$ as vector space with action $a \rightharpoonup n = \alpha^{-1}\beta(a)\cdot n$ for all $a \in \pi, n \in N$, and decomposition ${}^{(\alpha,\beta)}N = \bigoplus_{a\in\pi} N_{\delta\alpha^{-1}\delta^{-1}\beta(a)}$.

With the above notations, the braiding $c_{M,N} : M\otimes N \to {}^{M}N \otimes M$ acts on homogeneous elements $m \in M_a, n \in N_b$ as $c_{M,N}(m\otimes n) = \alpha^{-1}(a)\cdot n\otimes m_{(0)}$. Therefore $M_\alpha \otimes N_\beta$ is sent to $N_{\delta\alpha^{-1}(a)b\gamma\alpha^{-1}(a^{-1})} \otimes M_a$.

Now assume that $M \in {}_{k(\pi)}\mathcal{YD}^{k(\pi)}(\alpha,\beta)$ is finite dimensional. Since $S = S^{-1}$ for $k(\pi)$, we have $M^* = {}^*M$, and for all $a \in \pi, m \in M, f \in M^*$, $(a\cdot f)(m) = f(a^{-1}\cdot m)$ with decomposition $M^* = \bigoplus_{a\in\pi}(M_{\beta^{-1}\alpha^{-1}(a)})^*$.

Let π be a finite group and $\{p_a\}_{a\in\pi}$ the dual of $k(\pi)$. For $\alpha,\beta \in Aut(\pi)$, the component $CT(k(\pi))(\alpha,\beta) = k(\pi)^{*op} \bowtie k(\pi)$ with comultiplication

$$\bar{\Delta}(p_c \bowtie d) = \sum_{ab=c} p_a \bowtie \beta(b)d\alpha(b^{-1}) \otimes p_b \bowtie d,$$

for all $c,d \in \pi$. Furthermore for $a \in k(\pi)(\alpha,\beta)$ and $b \in k(\pi)(\gamma,\delta)$,

$$\begin{aligned}
&(p_c \bowtie a)(p_d \bowtie b) = \delta_{c,d}p_c \bowtie \delta(a)\delta\alpha\delta^{-1}(b),\\
&1_{CT(k(\pi))(id,id)} = \sum_{a\in\pi} p_a \otimes 1,\\
&\psi_{(\alpha,\beta)}(p_c \bowtie d) = p_{\beta^{-1}\alpha(c)} \otimes \beta^{-1}\delta\alpha\delta^{-1}(d),\\
&S_{(\alpha,\beta)}(p_c \bowtie a) = p_{c^{-1}} \bowtie \beta^{-1}(c)\beta^{-1}\alpha^{-1}(a^{-1})\beta^{-1}\alpha^{-1}\beta(c^{-1}),\\
&\sigma_{(\alpha,\beta),(\gamma,\delta)}((p_c \bowtie a), (p_d \bowtie b)) = \delta_{b,\delta(c)}\delta_{1,d}.
\end{aligned}$$

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