Filomat 31:20 (2017), 6269–6280
https://doi.org/10.2298/FIL1720269G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nonlinear Singular One Dimensional Thermo-Elasticity Coupled System and Double Laplace Decomposition Methods

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Abstract. In this work, combined double Laplace transform and Adomian decomposition method is presented to solve nonlinear singular one dimensional thermo-elasticity coupled system. Moreover, the convergence proof of the double Laplace transform decomposition method applied to our problem. By using one example, our proposed method is illustrated and the obtained results are confirmed.

1. Introduction

The one dimensional thermo-elasticity coupled system was one of the first domains in coupled field theory that attracted the mathematicians. The thermo-elasticity problems occur in different fields of science and engineering. linear and nonlinear problems, in physics, biology. Recently, many methods have been used for solution of linear and nonlinear problem, for example, Adomian decomposition method (ADM) see: [1–3, 7, 8]. In [9], the authors have solved a particular case of the given nonlinear problem by combining a functional analysis and iteration method. The convergence of the decomposition method has been studied by several authors see: [10–14]. The aim of this paper is to solve nonlinear singular one dimensional thermo-elasticity coupled system by using the combine domain decomposition techniques and double laplace transform methods and in addition, we will study the convergence analysis. Now, we recall the following definitions which are given by [4–6]. The double Laplace transform is defined as:

$$L_x L_t\left[f(x,s)\right] = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)dt\, dx, \tag{1}$$

where $x, t > 0$ and p, s are complex values and further double Laplace transform of the first order partial derivatives is given by

$$L_x L_t\left[\frac{\partial f(x,t)}{\partial x}\right] = pF(p,s) - F(0,s). \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 35A44, 65M44, 35A22
Keywords. double Laplace transform, inverse double Laplace transform, nonlinear singular system single Laplace transform, decomposition methods
Received: 18 January 2016; Revised: 27 October 2016; Accepted: 29 October 2016
Communicated by Maria Alessandra Ragusa
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Similarly, the double Laplace transform for second partial derivative with respect to x and t is defined as follows

$$\begin{aligned} L_xL_t\left[\frac{\partial^2 f(x,t)}{\partial^2 x}\right] &= p^2F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x}, \\ L_xL_t\left[\frac{\partial^2 f(x,t)}{\partial^2 t}\right] &= s^2F(p,s) - sF(p,0) - \frac{\partial F(p,0)}{\partial t}, \end{aligned} \tag{3}$$

where L_xL_t double Laplace transform with respect to x, t. The following lemma is used in this paper.

Lemma 1.1. *Double Laplace transform of the non constant coefficient first and second order partial derivative*

$$x\frac{\partial v}{\partial t},\ x^2\frac{\partial v}{\partial t},\ x\frac{\partial^2 u}{\partial t^2},\ x^2\frac{\partial^2 u}{\partial t^2},\ x^n\frac{\partial^2 u}{\partial t^2},$$

and the function $xf(x,t)$, $x^2f(x,t)$ and $x^nf(x,t)$ are given by

$$\begin{aligned} -\frac{d}{dp}\left[sV(p,s) - V(p,0)\right] &= L_xL_t\left(x\frac{\partial v}{\partial t}\right), \\ \frac{d^2}{dp^2}\left[sV(p,s) - V(p,0)\right] &= L_xL_t\left(x^2\frac{\partial v}{\partial t}\right), \\ (-1)^n\frac{d^n}{dp^n}\left[sV(p,s) - V(p,0)\right] &= L_xL_t\left(x^n\frac{\partial v}{\partial t}\right), \end{aligned} \tag{4}$$

$$\begin{aligned} -\frac{d}{dp}\left[s^2U(p,s) - sU(p,0) - \frac{\partial U(p,0)}{\partial t}\right] &= L_xL_t\left(x\frac{\partial^2 u}{\partial t^2}\right), \\ (-1)^2\frac{d^2}{dp^2}\left[s^2U(p,s) - sU(p,0) - \frac{\partial U(p,0)}{\partial t}\right] &= L_xL_t\left(x^2\frac{\partial^2 u}{\partial t^2}\right), \\ (-1)^n\frac{d^n}{dp^n}\left[s^2U(p,s) - sU(p,0) - \frac{\partial U(p,0)}{\partial t}\right] &= L_xL_t\left(x^n\frac{\partial^2 u}{\partial t^2}\right) \end{aligned} \tag{5}$$

and

$$\begin{aligned} L_xL_t\left(xf(x,t)\right) &= -\frac{dF(p,s)}{dp}, \\ L_xL_t\left(x^2f(x,t)\right) &= (-1)^2\frac{d^2F(p,s)}{dp^2}, \\ L_xL_t\left(x^nf(x,t)\right) &= (-1)^n\frac{d^nF(p,s)}{dp^n}. \end{aligned} \tag{6}$$

One can prove this lemma by using the definition of double Laplace transform in Eq.(1), Eq.(2) and Eq.(3).

2. Nonlinear Singular One Dimensional Thermo-Elasticity Coupled System

The main aim of this section is to discuss the use of modified double Laplace decomposition method for solving singular one dimensional thermo-elasticity coupled system. We consider nonlinear singular one dimensional thermo-elasticity coupled system with initial conditions in the form

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(xu\frac{\partial u}{\partial x}\right) + x\frac{\partial v}{\partial x} &= f(x,t), \\ \frac{\partial v}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(xv\frac{\partial v}{\partial x}\right) + x\frac{\partial^2 u}{\partial x\partial t} &= g(x,t), \quad t > 0 \end{aligned} \tag{7}$$

subject to

$$\begin{aligned} u(x,0) &= f_1(x), \ \frac{\partial u(x,0)}{\partial t} = f_2(x), \\ v(x,0) &= g_1(x), \end{aligned} \tag{8}$$

where, the nonlinear term, $\frac{1}{x}\left(xu\frac{\partial u}{\partial x}\right)_x$ and $\frac{1}{x}\left(xv\frac{\partial v}{\partial x}\right)_x$ are called Bessel's operators and $f(x,t)$, $g(x,t)$, $f_1(x)$, $f_2(x)$ and $g_1(x)$ are known function. Multiplying both sides of Eq.(7) by x we have

$$\begin{aligned} x\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(xu\frac{\partial u}{\partial x}\right) + x^2\frac{\partial v}{\partial x} &= xf(x,t), \\ x\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(xv\frac{\partial v}{\partial x}\right) + x^2\frac{\partial^2 u}{\partial x\partial t} &= xg(x,t). \quad t>0 \end{aligned} \tag{9}$$

Particular cases of equations belonging to the class of the model Eq.(9) can be found in [15, 16]. The method consists of first applying the double Laplace transform (denoted by L_xL_t) to both sides of equation Eq.(9) and single Laplace transform for Eq.(8) and by using the differentiation property of double Laplace transform and lemma1, we obtain:

$$\frac{dU(p,s)}{dp} = \frac{dF_1(p)}{sdp} + \frac{dF_2(p)}{s^2dp} + \frac{dF(p,s)}{s^2dp} - \frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(xu\frac{\partial u}{\partial x}\right) - x^2\frac{\partial v}{\partial x}\right], \tag{10}$$

and

$$\frac{dV(p,s)}{dp} = \frac{dG_1(p)}{sdp} + \frac{dG(p,s)}{sdp} - \frac{1}{s}L_xL_t\left[\frac{\partial}{\partial x}\left(xv\frac{\partial v}{\partial x}\right) - x^2\frac{\partial^2 u}{\partial x\partial t}\right], \tag{11}$$

by applying the integral for both sides of Eq.(10) and Eq.(11) from 0 to p with respect p, we have

$$U(p,s) = \frac{F_1(p)}{s} + \frac{F_2(p)}{s^2} + \frac{1}{s^2}\int_0^p\left(\frac{dF(p,s)}{dp}\right)dp - \frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}(xN_1) - x^2\frac{\partial v}{\partial x}\right]dp, \tag{12}$$

and

$$V(p,s) = \frac{G_1(p)}{s} + \frac{1}{s}\int_0^p\left(\frac{dG(p,s)}{dp}\right)dp - \frac{1}{s}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}(xN_2) - x^2\frac{\partial^2 u}{\partial x\partial t}\right]dp, \tag{13}$$

where $N_1 = u\frac{\partial u}{\partial x}$ and $N_2 = v\frac{\partial v}{\partial x}$, the modified double Laplace decomposition methods (MDLDM) defines the solution of the nonlinear singular one dimensional thermo-elasticity coupled as $u(x,t)$ and $v(x,t)$ by the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{14}$$

The nonlinear operators can be defined as follows

$$N_1 = \sum_{n=0}^{\infty} A_n, \quad N_2 = \sum_{n=0}^{\infty} B_n, \tag{15}$$

where A_n and B_n are denoted by:

$$A_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\sum_{i=0}^{\infty}(\lambda^n u_n)\right]\right)_{\lambda=0}, \quad B_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_2\sum_{i=0}^{\infty}(\lambda^n u_n)\right]\right)_{\lambda=0}. \tag{16}$$

Here some a few term of a domain's polynomials A_n and B_n are given by:

$$\begin{aligned} A_0 &= u_0 u_{0x} \\ A_1 &= u_0 u_{1x} + u_1 u_{0x} \\ A_3 &= u_0 u_{2x} + u_1 u_{1x} + u_2 u, \end{aligned} \tag{17}$$

and

$$\begin{aligned} B_0 &= v_0 v_{0x} \\ B_1 &= v_0 v_{1x} + v_1 v_{0x} \\ B_3 &= v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x}, \end{aligned} \tag{18}$$

by using double inverse Laplace transform for Eq.(12), Eq.(13) and use Eq.(14) and Eq.(15) we have

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= f_1(x) + t f_2(x) + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p \left(\frac{dF(p,s)}{dp}\right) dp\right] \\ &\quad - L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty} A_n\right)\right] dp\right] \\ &\quad + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2} L_x L_t\left[\int_0^p \left(x^2 \frac{\partial}{\partial x}\sum_{n=0}^{\infty} v_n\right) dp\right]\right] \end{aligned} \tag{19}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} v_n(x,t) &= g_1(x) + L_p^{-1} L_s^{-1}\left[\frac{1}{s}\int_0^p \left(\frac{dG(p,s)}{dp}\right) dp\right] \\ &\quad - L_p^{-1} L_s^{-1}\left[\frac{1}{s}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty} B_n\right)\right] dp\right] \\ &\quad + L_p^{-1} L_s^{-1}\left[\frac{1}{s} L_x L_t\left[\int_0^p \left(x^2 \frac{\partial^2}{\partial x \partial t}\sum_{n=0}^{\infty} u_n\right) dp\right]\right]. \end{aligned} \tag{20}$$

we define the following recursively formula:

$$\begin{aligned} u_0 &= f_1(x) + t f_2(x) + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p \left(\frac{dF(p,s)}{dp}\right) dp\right], \\ v_0 &= g_1(x) + L_p^{-1} L_s^{-1}\left[\frac{1}{s}\int_0^p \left(\frac{dG(p,s)}{dp}\right) dp\right]. \end{aligned} \tag{21}$$

and

$$\begin{aligned} u_{n+1}(x,t) &= -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty} A_n\right)\right] dp\right] \\ &\quad + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[x^2 \frac{\partial}{\partial x}\sum_{n=0}^{\infty} v_n\right] dp\right] \end{aligned} \tag{22}$$

$$\begin{aligned} v_{n+1}(x,t) &= -L_p^{-1} L_s^{-1}\left[\frac{1}{s}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty} B_n\right)\right] dp\right] \\ &\quad + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[x^2 \frac{\partial^2}{\partial x \partial t}\sum_{n=0}^{\infty} u_n\right] dp\right]. \end{aligned} \tag{23}$$

where L_xL_t double Laplace transform with respect to x, t and double inverse laplace transform denoted by $L_p^{-1}L_s^{-1}$ with respect to p, s. Here we Provide double inverse laplace transform with respect to p and s exist for each terms in the right hand side of Eqs. (19) and (20). To confirm our method for solving the nonlinear singular one dimensional thermo-elasticity coupled system, we consider the following example:

Example 2.1. *Consider the following nonlinear singular one dimensional thermo-elasticity coupled system:*

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - \frac{1}{x}\left(xu\frac{\partial u}{\partial x}\right)_x + x\frac{\partial v}{\partial x} &= -6x^2t^2,\\ \frac{\partial v}{\partial t} - \frac{1}{x}\left(xv\frac{\partial v}{\partial x}\right)_x + x\frac{\partial^2 u}{\partial x\partial t} &= 2x^2t - 8x^2t^4 + 2x^2,\end{aligned} \tag{24}$$

subject to

$$u(x,0) = 0,\ \frac{\partial u(x,0)}{\partial t} = x^2,\quad v(x,0) = 0. \tag{25}$$

By multiplying Eq.(24) by x and using Eq. (10), Eq. (11),Eq. (12)and Eq. (13) we obtain

$$U(p,s) = \frac{2}{p^3s^2} - \frac{24}{p^3s^5} - \frac{1}{s^2}\int_0^p L_xL_t\left[\left(xu\frac{\partial u}{\partial x}\right)_x - \left(x^2\frac{\partial v}{\partial x}\right)\right]dp, \tag{26}$$

and

$$V(p,s) = \frac{4}{p^3s^3} - \frac{16\,4!}{p^3s^6} + \frac{4}{p^3s^2} - \frac{1}{s}\int_0^p L_xL_t\left[\left(xv\frac{\partial v}{\partial x}\right)_x - \left(x^2\frac{\partial^2 u}{\partial x\partial t}\right)\right]dp. \tag{27}$$

On using double inverse Laplace transform, we have

$$u(x,t) = x^2t - \frac{1}{2}x^2t^4 - L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p L_xL_t\left[\left(xu\frac{\partial u}{\partial x}\right)_x - \left(x^2\frac{\partial v}{\partial x}\right)\right]dp\right), \tag{28}$$

and

$$v(x,t) = x^2t^2 - \frac{8}{5}x^2t^5 + 2x^2t - L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_xL_t\left[\left(xv\frac{\partial v}{\partial x}\right)_x - \left(x^2\frac{\partial^2 u}{\partial x\partial t}\right)\right]dp\right). \tag{29}$$

By using equation Eq.(19) and Eq.(20), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) &= x^2t - \frac{1}{2}x^2t^4 - L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}A_n\right)\right]dp\right)\\ &\quad + L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p L_xL_t\left[x^2\frac{\partial}{\partial x}\sum_{n=0}^{\infty}v_n\right]dp\right)\end{aligned} \tag{30}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} v_n(x,t) &= x^2t^2 - \frac{8}{5}x^2t^5 + 2x^2t - L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}B_n\right)\right]dp\right)\\ &\quad + L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_xL_t\left[\left(x^2\frac{\partial^2}{\partial x\partial t}\sum_{n=0}^{\infty}u_n\right)\right]dp\right),\end{aligned} \tag{31}$$

where A_n and B_n are A domain polynomials given by equations Eq.(16). By using equations Eq.(21),Eq.(22) and Eq.(23) we have

$$\begin{aligned}u_0 &= x^2t - \frac{1}{2}x^2t^4\\ v_0 &= x^2t^2 - \frac{8}{5}x^2t^5 + 2x^2t.\end{aligned} \tag{32}$$

The other components are given by

$$\begin{aligned} u_{n+1} &= -L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}A_n\right)\right]dp\right) \\ &+L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p L_xL_t\left[x^2\frac{\partial}{\partial x}\sum_{n=0}^{\infty}v_n\right]dp\right) \end{aligned} \tag{33}$$

and

$$\begin{aligned} v_{n+1} &= -L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}B_n\right)\right]dp\right) \\ &+L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p L_xL_t\left[\left(x^2\frac{\partial^2}{\partial x\partial t}\sum_{n=0}^{\infty}u_n\right)\right]dp\right). \end{aligned} \tag{34}$$

By applying Eq. Eq.(17), Eq.(18) , we obtain

$$\begin{aligned} u_1 &= \frac{1}{2}x^2t^4-\frac{4}{35}x^2t^7+\frac{1}{45}x^2t^{10}-\frac{2}{3}x^2t^3 \\ v_1 &= \frac{8}{5}x^2t^5-\frac{16}{5}x^2t^8+9x^2t^4-\frac{256}{35}x^2t^7+\frac{32}{3}x^2t^3+\frac{512}{275}x^2t^{11}-2x^2t, \end{aligned} \tag{35}$$

and

$$u_2=\frac{4}{35}x^2t^7+\frac{2}{315}x^2t^{10}+\frac{10628}{39}x^2t^{13}-\frac{43}{45}x^2t^6+\frac{262}{945}x^2t^9-\frac{16}{15}x^2t^5+\frac{2}{3}x^2t^3$$

and

$$\begin{aligned} v_2 &= \frac{16}{5}x^2t^8-\frac{2304}{275}x^2t^{11}+\frac{248}{7}x^2t^7-\frac{70883}{1575}x^2t^{10}+\frac{688}{9}x^2t^6+\frac{3072}{385}x^2t^{14} \\ &-9x^2t^4+\frac{59392}{1925}x^2t^8-\frac{53248}{945}x^2t^9-20x^2t^3+\frac{1024}{15}x^2t^5-\frac{65536}{23375}x^2t^{17}. \end{aligned}$$

It is obvious that the self-cancelling some terms appear between various components and connected by coming terms, we have

$$u(x,t)=u_0+u_1+...+u_n.\ \text{and}\ v(x,t)=v_0+v_1+...+v_n.$$

Therefore, the exact solution is given by

$$u(x,t)=x^2t,\ \ v(x,t)=x^2t^2.$$

By extending Eq.(7) as follows

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}-\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2u\frac{\partial u}{\partial x}\right)+x\frac{\partial v}{\partial x} &= f(u), \\ \frac{\partial v}{\partial t}-\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2v\frac{\partial v}{\partial x}\right)+x\frac{\partial^2 u}{\partial x\partial t} &= g(v), \end{aligned} \tag{36}$$

subject to Eq.(8) and where $f(u)$, $g(v)$ are nonlinear functions. By using our method, we have

$$\begin{aligned} U(p,s) &= \frac{1}{s}\int_0^p\int_0^p\frac{d^2F_1(p)}{dp^2}dpdp+\frac{1}{s^2}\int_0^p\int_0^p\frac{d^2F_2(p)}{dp^2}dpdp \\ &+\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[f(u)\right]dpdp+\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x^2u\frac{\partial u}{\partial x}\right)\right]dpdp \\ &-\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[\left(x^3\frac{\partial v}{\partial x}\right)\right]dpdp \end{aligned} \tag{37}$$

and

$$\begin{aligned} v(p,s) &= \frac{1}{s}\int_0^p\int_0^p \frac{d^2G_1(p)}{dp^2}dpdp + \frac{1}{s}\int_0^p\int_0^p L_xL_t\left[g(v)\right]dpdp \\ &+\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x^2 v\frac{\partial v}{\partial x}\right)\right]dpdp \\ &-\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[\left(x^3\frac{\partial^2 u}{\partial x\partial t}\right)\right]dpdp, \end{aligned} \tag{38}$$

by taking double inverse Laplace transform for Eq.(37) and Eq.(38) we get

$$\begin{aligned} u(x,t) &= L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p \frac{d^2F_1(p)}{dp^2}dpdp\right) + L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p \frac{d^2F_2(p)}{dp^2}dpdp\right) \\ &+L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[f(u)\right]dpdp\right) \\ &+L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x^2 u\frac{\partial u}{\partial x}\right)\right]dpdp\right) \\ &-L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[\left(x^3\frac{\partial v}{\partial x}\right)\right]dpdp\right) \end{aligned} \tag{39}$$

and

$$\begin{aligned} v(x,t) &= L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p \frac{d^2G_1(p)}{dp^2}dpdp\right) + L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[g(v)\right]dpdp\right) \\ &+L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x^2 v\frac{\partial v}{\partial x}\right)\right]dpdp\right) \\ &-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[\left(x^3\frac{\partial^2 u}{\partial x\partial t}\right)\right]dpdp\right) \end{aligned} \tag{40}$$

then the solution of Eq.(36) is given by

$$\begin{aligned} u_0(x,t) &= L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p \frac{d^2F_1(p)}{dp^2}dpdp\right) + L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p \frac{d^2F_2(p)}{dp^2}dpdp\right) \\ &+L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[f(u)\right]dpdp\right) \\ u_{n+1}(x,t) &= L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x}\sum_{n=0}^{\infty}A_n\right)\right]dpdp\right) \\ &-L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}\int_0^p\int_0^p L_xL_t\left[\left(x^3\frac{\partial}{\partial x}\sum_{n=0}^{\infty}v_n\right)\right]dpdp\right), \end{aligned} \tag{41}$$

and

$$\begin{aligned} v_0(x,t) &= L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p \frac{d^2G_1(p)}{dp^2}dpdp\right) + L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[g(v)\right]dpdp\right) \\ v_{n+1}(x,t) &= L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x}\sum_{n=0}^{\infty}B_n\right)\right]dpdp\right) \\ &-L_p^{-1}L_s^{-1}\left(\frac{1}{s}\int_0^p\int_0^p L_xL_t\left[\left(x^3\frac{\partial^2}{\partial x\partial t}\sum_{n=0}^{\infty}u_n\right)\right]dpdp\right). \end{aligned} \tag{42}$$

3. Convergence Analysis of the Method for the Nonlinear Singular One Dimensional Thermo-Elasticity Coupled System

The aim of this section is to discuss our method for the nonlinear singular one dimensional thermo-elasticity coupled system. We consider the general form of nonlinear singular one dimensional thermo-elasticity coupled system with initial conditions is given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(xu\frac{\partial u}{\partial x}\right) + \frac{\partial v}{\partial x} = f(u) \\ \frac{\partial v}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(xv\frac{\partial v}{\partial x}\right) + \frac{\partial^2 u}{\partial x \partial t} = g(v) \\ u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = 0, \ v(x,0) = 0 \\ u(x,T) = v(x,T) = 0 \end{cases} \tag{43}$$

The operators T, M, N, P, Ψ and K which are continuous and self-adjoin satisfies:

$$\begin{aligned} (-Tu, u) &\geq 0, \ (-Mu, u) \geq 0, \ (-Nv, u) \geq 0 \\ (-Pu, v) &\geq 0, \ (-Fv, v) \geq 0, \ (-\Psi v, v) \geq 0, \end{aligned}$$

and

$$\begin{aligned} (-Tu, u) &= 0, \ (-Mu, u) = 0, \ (-Nv, v) = 0 \\ (-Pu, v) &= 0, \ (-Fv, v) = 0, \ (\Psi v, v) = 0, \end{aligned} \tag{44}$$

if and only if $u, v = 0$, There exists a numbers δ, α, β and η such that:

$$\begin{aligned} (-Tu, u) &= \left(-\frac{1}{2}\frac{\partial u^2}{\partial x}, u\right)_{L^1} \geq \alpha \|u\|^2, (-Mu, u) = \left(-\frac{1}{2}x\frac{\partial^2 u^2}{\partial x^2}, u\right)_{L^2} \geq \delta a \|u\|^2, \\ (-\Psi v, v) &= \left(-\frac{1}{2}\frac{\partial v^2}{\partial x}, v\right)_{L^1} \geq \eta \|v\|^2, \ (Pu, v) = \left(x\frac{\partial^2 u}{\partial x \partial t}, v\right)_{L^2} \geq \delta\beta a \|u\| \|v\|, \\ (-Fv, v) &= \left(-\frac{1}{2}x\frac{\partial^2 v^2}{\partial x^2}, v\right)_{L^2} \geq a\rho \|v\|^2, \\ (Nv, u) &= \left(x\frac{\partial v}{\partial x}, u\right)_{L^1} \geq \eta a \|v\| \|u\|, \end{aligned} \tag{45}$$

for all $u, v \in H$. We define H as $H = L^2_\mu((a,b) \times [0,T])$, where $a \ll 0$ and

$$\begin{aligned} (u, v) &: (a,b) \times [0,T] \to \mathbb{R} \times \mathbb{R}, \text{ with } \|u\|^2_H = \int_Q xu^2(x,t)\,dxdt \\ (u, v) &= \int_Q xu(x,t)\,v(x,t)\,dxdt \end{aligned}$$

where $Q = (a,b) \times [0,T]$ and

$$H = \left\{ \begin{array}{c} (u,v) : (a,b) \times [0,T], \text{ with} \\ L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t [u(x,t)](p,s)\,dp\right](x,t) < \infty \end{array} \right\}.$$

Now we rewrite Eq.(43) as follows

$$\begin{aligned} x\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\frac{\partial u^2}{\partial x} - \frac{1}{2}x\frac{\partial^2 u^2}{\partial x^2} + x\frac{\partial v}{\partial x} &= xf(u), \\ x\frac{\partial v}{\partial t} - \frac{1}{2}\frac{\partial v^2}{\partial x} - \frac{1}{2}x\frac{\partial^2 v^2}{\partial x^2} + x\frac{\partial^2 u}{\partial x \partial t} &= xg(v). \end{aligned} \tag{46}$$

For L and R are hemicontinuous operator, consider the following hypothesis:
1. (H1)

$$\begin{aligned}(L(u,v)-L(u,w),u-w) &\geq k\|u-w\|^2;\\ (R(u,v)-R(v,w),v-w) &\geq k\|v-w\|^2;\ \ k>0,\forall u,v,w\in H\end{aligned}$$

2. (H2) whatever may be $m>0$, there exist a constant $C(m)>0$ such that for $u,w\in H$ with $\|u\|\leq m$, $\|w\|\leq m$ we have:

$$\begin{aligned}(L(u,v)-L(u,w),z) &\leq C(m)\|u-w\|\|z\|,\\ (R(u,v)-R(v,w),z) &\leq C(m)\|v-w\|\|z\|\end{aligned}$$

for every $z\in H$

Theorem 3.1. *(Sufficient condition of convergence)*
Modified double Laplace decomposition methods for the nonlinear singular one dimensional thermo-elasticity coupled system as follows.

$$\begin{aligned}x\frac{\partial^2 u}{\partial t^2} &= \frac{1}{2}\frac{\partial u^2}{\partial x}+\frac{1}{2}x\frac{\partial^2 u^2}{\partial x^2}-x\frac{\partial v}{\partial x}+xf(u)\\ x\frac{\partial v}{\partial t} &= \frac{1}{2}\frac{\partial v^2}{\partial x}+\frac{1}{2}x\frac{\partial^2 v^2}{\partial x^2}-x\frac{\partial^2 u}{\partial x\partial t}+xg(v)\end{aligned}\tag{47}$$

with homogenous initial condition, converges towards a particular solution.

Proof. For the equation Eq.(47) Let

$$\begin{aligned}S(u,v) &= x\frac{\partial^2 u}{\partial t^2},-T(u)=-\frac{1}{2}\frac{\partial u^2}{\partial x},-M(u)=-\frac{1}{2}x\frac{\partial^2 u^2}{\partial x^2},N(v)=x\frac{\partial v}{\partial x}\\ D(u,v) &= x\frac{\partial v}{\partial t},-\Psi(v)=-\frac{1}{2}\frac{\partial v^2}{\partial x},-F(v)=-\frac{1}{2}x\frac{\partial^2 v^2}{\partial x^2},P(u)=x\frac{\partial^2 u}{\partial x\partial t}\end{aligned}$$

we have

$$\begin{aligned}L(u,v) &= -\frac{1}{2}\frac{\partial u^2}{\partial x}-\frac{1}{2}x\frac{\partial^2 u^2}{\partial x^2}+x\frac{\partial v}{\partial x}-xf(u)\\ R(u,v) &= -\frac{1}{2}\frac{\partial v^2}{\partial x}-\frac{1}{2}x\frac{\partial^2 v^2}{\partial x^2}+x\frac{\partial^2 u}{\partial x\partial t}-xg(v)\end{aligned}$$

We will start to verify the convergence hypotheses (H1), we use the definition of our operator L and R, and the conditions in Eq.(45) we have the following form

$$\begin{aligned}L(u,v)-L(u,w) &= -\frac{1}{2}\left(\frac{\partial}{\partial x}\left(u^2-w^2\right)\right)-\frac{1}{2}x\frac{\partial^2}{\partial x^2}\left(u^2-w^2\right)+x\left(\frac{\partial}{\partial x}(v-w)\right)-x\left(f(u)-f(w)\right)\\ R(u,v)-R(v,w) &= -\frac{1}{2}\left(\frac{\partial}{\partial x}\left(v^2-w^2\right)\right)-\frac{1}{2}x\frac{\partial^2}{\partial x^2}\left(v^2-w^2\right)+x\left(\frac{\partial^2}{\partial x\partial t}(u-w)\right)-x\left(g(v)-g(w)\right)\end{aligned}\tag{48}$$

then we get

$$\begin{aligned}(L(u,v)-L(u,w),u-w) &= \left(-\frac{1}{2}\frac{\partial}{\partial x}\left(u^2-w^2\right),u-w\right)+\left(-\frac{1}{2}x\frac{\partial^2}{\partial x^2}\left(u^2-w^2\right),u-w\right)\\ &\quad+\left(x\left(\frac{\partial}{\partial x}(v-w)\right),u-w\right)+\left(-x\left(f(u)-f(w)\right),u-w\right)\end{aligned}\tag{49}$$

and

$$
\begin{aligned}
(R(u,v)-R(v,w),v-w) &= \left(-\frac{1}{2}\left(\frac{\partial}{\partial x}\left(v^2-w^2\right)\right),v-w\right)+\left(-\frac{1}{2}x\frac{\partial^2}{\partial x^2}\left(v^2-w^2\right),v-w\right)\\
&\quad+\left(x\left(\frac{\partial^2}{\partial x\partial t}(u-w)\right),v-w\right)+\left(-x\left(g(v)-g(w)\right),v-w\right)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
\left(-\frac{1}{2}\frac{\partial}{\partial x}\left(u^2-w^2\right),u-w\right) &\geq \alpha\left\|u-w\right\|^2,\\
\left(-\frac{1}{2}x\frac{\partial^2}{\partial x^2}\left(u^2-w^2\right),u-w\right) &\geq \delta a\left\|u-w\right\|^2
\end{aligned} \tag{51}
$$

and

$$
\left(x\left(\frac{\partial}{\partial x}(v-w)\right),u-w\right)\geq \eta a\left\|v-w\right\|\left\|u-w\right\| \tag{52}
$$

where $\sigma>0$ as f is Lipschitzian function, we have:

$$
\begin{aligned}
\left(x\left(f(u)-f(w)\right),u-w\right) &\leq \left\|x\left(f(u)-f(w)\right)\right\|\left\|u-w\right\|\\
&\leq \left\|x\right\|\left\|f(u)-f(w)\right\|\left\|u-w\right\|\\
&\leq a\sigma\left\|u-w\right\|^2 \Leftrightarrow\\
\left(-x\left(f(u)-f(w)\right),u-w\right) &\geq -a\sigma\left\|u-w\right\|^2
\end{aligned} \tag{53}
$$

substituting Eq.(51), Eq.(52) and Eq.(53) into equation Eq.(49) gives

$$
\begin{aligned}
(L(u,v)-L(u,w),u-w) &\geq (\alpha+a\delta-a\sigma)\left\|u-w\right\|^2+a\eta\left\|v-w\right\|\left\|u-w\right\|\\
&\geq (\alpha+a\delta-a\sigma)\left\|u-w\right\|^2\\
&= k\left\|u-w\right\|^2
\end{aligned}
$$

where $k=\alpha+a\delta-a\sigma>0\Rightarrow\sigma>\frac{\alpha+a\delta}{a}$. In the same manner we get

$$
\begin{aligned}
\left(-\frac{1}{2}\left(\frac{\partial}{\partial x}\left(v^2-w^2\right)\right),v-w\right) &\geq \eta\left\|v-w\right\|^2\\
\left(-\frac{1}{2}x\frac{\partial^2}{\partial x^2}\left(v^2-w^2\right),v-w\right) &\geq a\rho\left\|v-w\right\|^2
\end{aligned} \tag{54}
$$

and

$$
\left(x\left(\frac{\partial^2}{\partial x\partial t}(u-w)\right),v-w\right)\geq \delta\beta a\left\|u-w\right\|\left\|v-w\right\| \tag{55}
$$

where $\delta_1>0$ as g is Lipschitzian function, we have:

$$
\begin{aligned}
\left(x\left(g(v)-g(w)\right),v-w\right) &\leq \left\|x\left(g(v)-g(w)\right)\right\|\left\|v-w\right\|\\
&\leq \left\|x\right\|\left\|\left(g(v)-g(w)\right)\right\|\left\|v-w\right\|\\
&\leq a\delta_1\left\|v-w\right\|^2 \Leftrightarrow\\
\left(-x\left(g(v)-g(w)\right),v-w\right) &\geq -a\delta_1\left\|v-w\right\|^2
\end{aligned} \tag{56}
$$

substituting Eq.(54), Eq.(55) and Eq.(56) into equation Eq.(50) gives

$$\begin{aligned}(R(u,v)-R(v,w),v-w) &\geq (\eta+a\rho-a\delta_1)\|v-w\|^2+\delta\beta a\|u-w\|\|v-w\| \\ &\geq (\eta+a\rho-a\delta_1)\|v-w\|^2 \\ &= k\|v-w\|^2,\end{aligned}$$

where $k=(\eta+a\rho-a\delta_1)>0 \Rightarrow \frac{\eta+a\rho}{a}>\delta_1$. Now we verify the convergence hypotheses (H2) for the operator $L(u,v)$, for every $m>0$, there exist a constant $C(m)>0$ such that for $u,v\in H$

$$(L(u,v)-L(u,w),z)\leq C(M)\|u-w\|\|z\|$$

for every $w\in H$. We now verify hypothesis H2 for the operator $L(u,v)$ and $R(u,v)$. Using the Schwartz inequality and the fact that u and v are bounded, we obtain the following:

$$(L(u,v)-L(u,w),z) = \left(\frac{\partial}{\partial x}\left(\frac{1}{2}x\left(\frac{\partial}{\partial x}u^2-w^2\right)\right),z\right)-\left(x\left(\frac{\partial v}{\partial x}-\frac{\partial w}{\partial x}\right),z\right)+(x(f(u)-f(w)),z) \quad (57)$$

$$\begin{aligned}\left(\frac{\partial}{\partial x}\left(\frac{1}{2}x\left(\frac{\partial}{\partial x}u^2-w^2\right)\right),z\right) &\leq \left\|\left(\frac{1}{2}x\left(\frac{\partial}{\partial x}u^2-w^2\right)\right)_x\right\|\|z\| \\ &\leq \left\|\frac{1}{2}\frac{\partial}{\partial x}\left(u^2-w^2\right)\right\|+\left\|\frac{1}{2}x\frac{\partial^2}{\partial x^2}\left(u^2-w^2\right)\right\|\|z\| \\ &\leq \frac{1}{2}\alpha\left\|\left(u^2-w^2\right)\right\|+\frac{1}{2}a\delta\left\|\left(u^2-w^2\right)\right\|\|z\| \\ &\leq \frac{1}{2}\alpha\|(u-w)\|\|(u+w)\|\|z\| \\ &\quad +\frac{1}{2}a\delta\|(u-w)\|\|(u+w)\|\|z\| \\ &\leq (m\alpha+ma\delta)\|(u-w)\|\|z\|\end{aligned}$$

and

$$\begin{aligned}\left(x\left(\frac{\partial v}{\partial x}-\frac{\partial w}{\partial x}\right),z\right) &\leq a\eta\|v-w\|\|z\|, \\ (x(f(u)-f(w)),u-w) &\leq \left\|x(f(u)-f(w))\right\|\|z\| \\ &\leq a\sigma\|u-w\|\|z\|\end{aligned}$$

Again, by the Schwartz inequality we get

$$\begin{aligned}(L(u,v)-L(u,w),z) &\leq (m\alpha+ma\delta+a\sigma)\|u-w\|\|z\|-a\eta\|v-w\|\|z\| \\ &\leq (m\alpha+ma\delta+a\sigma)\|u-w\|\|z\| \\ &= C(M)\|u-w\|\|z\|\end{aligned}$$

where

$$C(M)=m\alpha+ma\delta+a\sigma$$

in the same manner we have

$$\begin{aligned}(R(u,v)-R(v,w),z) &\leq (\eta+a\rho+a\delta_1)\|v-w\|\|z\|-\delta\beta a\|v-w\|\|z\| \\ &\leq (m\eta+ma\rho+a\delta_1)\|v-w\|\|z\| \\ &= C(M)\|v-w\|\|z\|\end{aligned}$$

where

$$C(M) = m\eta + ma\rho + a\delta_1$$

This completes the proof. □

Conclusion 3.2. *In this work ,firstly the double Laplace transform which is based on the Adomian decomposition method is used for solving the nonlinear singular one dimensional thermo-elasticity coupled system. The results show that the new modification of double Laplace decomposition method is a powerful mathematical tool for solving nonlinear singular one dimensional thermo-elasticity coupled system. finally, we presented a convergence proof of the (MDLDM) applied to the nonlinear singular one dimensional pesudo thermo-elasticity coupled system.*

Acknowledgement: The author also thank the referee for very constructive comments and suggestions.

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