Filomat 31:19 (2017), 6101–6112
https://doi.org/10.2298/FIL1719101D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterization of Matrix Classes Involving Some Sets of Sequences of Fuzzy Numbers

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Abstract. In 1996, M. Stojaković and Z. Stojaković examined the convergence of a sequence of fuzzy numbers via Zadeh's Extension Principle, which is quite difficult for practical use. In this paper, we utilize the notion $\lambda-$level sets to deal with convergence and summable related notions and adopted a relatively new approach to characterize matrix classes involving some sets of single sequences of fuzzy numbers. The approach is expected to be useful in dealing with characterization of several other matrix classes involving different kinds of sets of sequences of fuzzy numbers, single or multiple.

1. Introduction and Definitions

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [23] as an extension of the classical notion of set and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy possibility theory, fuzzy measures of fuzzy events, fuzzy mathematical programming, etc. Working as a powerful mathematical tool for approximate reasoning, they play a significant role in decision making in complex phenomena which are difficult to describe by traditional mathematics. Matloka [9] introduced bounded and convergent sequences of fuzzy numbers and studied their properties. Later on sequences of fuzzy numbers have been discussed by various authors. For further relevant studies related to various operations and notions involving fuzzy sets, and different sets of sequences of fuzzy numbers, we refer to [1–6, 8, 10–12, 14–17, 19–22].

Definition 1.1. (Goetschel and Voxman [7]) A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \longrightarrow [0, 1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e., $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), v(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R}(x) > 0\}}$ is compact, where $\overline{\{x \in \mathbb{R}(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R}(x) > 0\}$ in the usual topology of $\mathbb{R}$.

2010 *Mathematics Subject Classification*. Primary 03E72, 40C05, 40D05; Secondary 40A05, 26A03, 46A45
Keywords. Fuzzy numbers, $\lambda-$level set, set of sequences of fuzzy numbers, convergence sequence, matrix transformations
Received: 13 April 2017; Accepted: 03 September 2017
Communicated by Ljubiša D.R. Kočinac
Research is supported by University Grants commission, New Delhi, India, Award letter number F./2015-16/NFO-2015-17-OBC-ASS-36722/(SA-III/Website)
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We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it as the space of fuzzy numbers. $\lambda-$level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \{t \in \mathbb{R} : u(t) \geq \lambda\}, \ \ (0 < \lambda \leq 1),$$

$$= \overline{\{t \in \mathbb{R} : u(t) > \lambda\}}, \ \ (\lambda = 0).$$

The set $[u]_\lambda$ is a closed, bounded and non-empty interval for each $\lambda \in [0, 1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number

$$\overline{r}(t) = 1, \quad t = r,$$

$$= 0, \quad t \neq r.$$

Definition 1.2. (Talo and Başar [18]) Let W be the set of all closed bounded intervals A of real numbers such that $A = [A_1, A_2]$. Define the relation d on W as follows:

$$d(A, B) = \max\{|A_1 - B_1|, |A_2 - B_2|\}.$$

Then (W, d) is a complete metric space (see Diamond and Kloeden [4], Nanda [10]). Then Talo and Başar [18] defined the metric D on E^1 by means of Hausdorff metric d as

$$D(u, v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-\lambda|, |u^+(\lambda) - v^+(\lambda)|\}.$$

The partial ordering relation on E^1 is defined as follows:

$$u \preccurlyeq v \Leftrightarrow [u]_\lambda \preccurlyeq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda) \text{ and } u^+(\lambda) \leq v^+(\lambda) \text{ for all } \lambda \in [0, 1].$$

Definition 1.3. (Talo and Başar [18])Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on E^1 by

$u + v = w \Leftrightarrow [w]_\lambda = [u]_\lambda + [v]_\lambda$ for all $\lambda \in [0, 1] \Leftrightarrow w^-(\lambda) = u^-(\lambda) - v^-(\lambda)$ and $w^+(\lambda) = u^+(\lambda) + v^+(\lambda)$ for all $\lambda \in [0, 1]$,

$[ku]_\lambda = k[u]_\lambda$ for all $\lambda \in 0, 1]$

and

$uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda[v]_\lambda$ for all $\lambda \in [0, 1]$,

where it is immediate that

$$w^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

and

$$w^+(\lambda) = \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\},$$

for all $\lambda \in [0, 1]$.

Definition 1.4. (Talo and Başar [18]) $u \in E^1$ is a non-negative fuzzy number if and only if $u(x_0) = 0$ for all $x_0 < 0$. It is immediate that $u \succcurlyeq \overline{0}$ if x is a non-negative fuzzy number.

One can see that

$$D(u, \overline{0}) = \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda)|, |u^+(\lambda)|\} = \max\{|u^-(0)|, |u^+(0)|\}. \tag{1}$$

Lemma 1.5. (Talo and Başar [18]) *Let $x, y, z, u \in E^1$ and $k \in \mathbb{R}$. Then:*
(i) *(E^1, D) is a complete metric space.*
(ii) *$D(kx, ky) = |k|\, D(x, y)$.*
(iii) *$D(x + y, z + y) = D(x, z)$.*
(iv) *$D(x + y, z + u) \leq D(x, z) + D(y, u)$.*
(v) *$|D(x, \overline{0}) - D(y, \overline{0})| \leq D(x, y) \leq D(x, \overline{0}) + D(y, \overline{0})$.*

Lemma 1.6. (Talo and Başar [18]) *The following statements hold:*
(i) *$D(xy, \overline{0}) \leq D(x, \overline{0}) D(y, \overline{0})$ for all $x, y \in E^1$.*
(ii) *If $x_k \longrightarrow x$, as $k \longrightarrow \infty$ then $D(x_k, \overline{0}) \longrightarrow D(x, \overline{0})$ as $k \longrightarrow \infty$.*

By w^F we denote the set of all single sequences of fuzzy numbers on $\mathbb{R}$. Matloka [9] introduced bounded and convergent sequences of fuzzy numbers and studied their properties. We now quote the following definitions given by Talo and Başar [18] which we will use in later part of this paper.

Definition 1.7. A sequence of fuzzy numbers (x_k) is said to be bounded if the set of fuzzy numbers consisting of the terms of the sequence (x_k) is a bounded set. That is to say that a sequence $(x_k) \in w^F$ is bounded if and only if there exist two fuzzy numbers m and M such that $m \preccurlyeq x_k \preccurlyeq M$ for all $k \in \mathbb{N}$. This means that $m^-(\lambda) \leq x_k^-(\lambda) \leq M^-(\lambda)$ and $m^+(\lambda) \leq x_k^+(\lambda) \leq M^+(\lambda)$ for all $\lambda \in [0, 1]$.

The fact that the boundedness of the sequence $(x_k) \in w^F$ is equivalent to the uniform boundedness of the functions $x_k^-(\lambda)$ and $x_k^+(\lambda)$ on $[0, 1]$. Therefore, one can say by using relation (1) that the boundedness of the sequence $(x_k) \in w^F$ is equivalent to the fact that

$$\sup_{k \in \mathbb{N}} D(x_k, \overline{0}) = \sup_{k \in \mathbb{N}} \sup_{\lambda \in [0,1]} \max\left(|x_k^-(\lambda)|, |x_k^+(\lambda)|\right) < \infty.$$

Definition 1.8. A sequence of fuzzy numbers $(x_k) \in w^F$ is called convergent with limit $x \in E^1$, if and only if for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $D(x_k, x) < \epsilon$ for all $k \geq n_0$.

If the sequence $(x_k) \in w^F$ converges to a fuzzy number x then by the definition of D the sequence of functions $\left\{x_k^-(\lambda)\right\}$ and $\left\{x_k^+(\lambda)\right\}$ are uniformly convergent to $x^-(\lambda)$ and $x^+(\lambda)$ in $[0, 1]$, respectively.

Definition 1.9. Let $(x_k) \in w^F$. Then the expression $\sum_k x_k$ is called a series corresponding to the sequence (x_k) of fuzzy numbers. We denote

$$s_n = \sum_{k=1}^{n} x_k \text{ for all } n \in \mathbb{N}.$$

If the sequence (s_n) converges to a fuzzy number x, then we say that the series $\sum_k x_k$ converges to x and write $\sum_k x_k = x$, which implies as $n \longrightarrow \infty$ that

$$\sum_{k=1}^{n} x_k^-(\lambda) \longrightarrow x^-(\lambda) \ \textit{and} \sum_{k=1}^{n} x_k^+(\lambda) \longrightarrow x^+(\lambda),$$

uniformly in $\lambda \in [0, 1]$. Conversely, if the fuzzy numbers $x_k = \{(x_k^-(\lambda), x_k^+(\lambda)) : \lambda \in [0, 1]\}$, $\sum_k x_k^-(\lambda) = x^-(\lambda)$ and $\sum_k x_k^+(\lambda) = x^+(\lambda)$ converge uniformly in $\lambda \in [0, 1]$, then $x = \{(x^-(\lambda), x^+(\lambda)) : \lambda \in [0, 1]\}$ defines a fuzzy number such that $x = \sum_k x_k$. The proof is due to Talo and Başar [18] in the form of the following lemma.

Otherwise, we say the series of fuzzy numbers diverges. Additionally, if the sequence (s_n) is bounded then we say that the series $\sum_k x_k$ of fuzzy numbers is bounded.

Lemma 1.10. *If the fuzzy numbers* $x_k = \{(x_k^-(\lambda), x_k^+(\lambda)) : \lambda \in [0,1]\}$, $\sum_k x_k^-(\lambda) = x^-(\lambda)$ *and* $\sum_k x_k^+(\lambda) = x^+(\lambda)$ *converge uniformly in* $\lambda \in [0,1]$, *then* $x = \{(x^-(\lambda), x^+(\lambda)) : \lambda \in [0,1]\}$ *defines a fuzzy number such that* $x = \sum_{k=0}^{\infty} x_k$.

Throughout the paper, the summations without limit run from 1 to ∞, for example, $\sum_k x_k$ means that $\sum_{k=1}^{\infty} x_k$. We also suppose that $1 \le p < \infty$ with $p^{-1} + q^{-1} = 1$ and $\mathbb{N} = \{1, 2, 3, ...\}$.

Definition 1.11. We have the sets $\ell_1^F, \ell_p^F, \ell_\infty^F, c^F, c_0^F$ consisting of the absolutely summable, p–absolutely summable, bounded, convergent and convergent to $\overline{0}$ sequences of fuzzy numbers (Talo and Başar [18]) as follows:

$$\ell_1^F = \{(x_k) \in w^F : \sum_k D(x_k, \overline{0}) < \infty\},$$

$$\ell_p^F = \{(x_k) \in w^F : \sum_k D(x_k, \overline{0})^p < \infty\},$$

$$\ell_\infty^F = \{(x_k) \in w^F : \sup_k D(x_k, \overline{0}) < \infty\},$$

$$c^F = \{(x_k) \in w^F : \textit{there exists } l \in E^1 \textit{ such that } \lim_{k \longrightarrow \infty} D(x_k, l) = 0\},$$

$$c_0^F = \{(x_k) \in w^F : \lim_{k \longrightarrow \infty} D(x_k, \overline{0}) = 0\}.$$

We denote by cs^F and bs^F, the set of all convergent and bounded series of fuzzy numbers respectively. Now we define α–, β– and γ–duals of a set $\mu^F \subset w^F$ which are respectively denoted by $\{\mu^F\}^\alpha$, $\{\mu^F\}^\beta$ and $\{\mu^F\}^\gamma$ as follows:

$$\{\mu^F\}^\alpha = \{(u_k) \in w^F : (u_k v_k) \in \ell_1^F, \ for\ all\ (v_k) \in \mu^F\},$$

$$\{\mu^F\}^\beta = \{(u_k) \in w^F : (u_k v_k) \in cs^F, \ for\ all\ (v_k) \in \mu^F\},$$

$$\{\mu^F\}^\gamma = \{(u_k) \in w^F : (u_k v_k) \in bs^F, \ for\ all\ (v_k) \in \mu^F\}.$$

2. Matrix Transformations Between Some Sets of Sequences of Fuzzy Numbers

An infinite matrix is one of the most general linear operators between two sequence spaces. The study of theory of matrix transformations has always been of great interest to mathematicians in the study of sequence spaces, which is motivated by special results in summability theory. Talo and Başar [18] gave some matrix transformations between some sets of sequences of fuzzy numbers. We try to give some results characterizing matrix transformations involving some classes of sequences of fuzzy numbers whose classical counterparts can be found in Nanda [10].

Definition 2.1. Let $\mu_1^F, \mu_2^F \subset w^F$ and $A = (a_{nk})$ be any two dimensional matrix of fuzzy numbers. Then we say that A defines a mapping from μ_1^F into μ_2^F, denote it by writing $A : \mu_1^F \longrightarrow \mu_2^F$ if for every sequence $x = (x_k) \in \mu_1^F$, the $A - transform$ of x, $Ax = \{(Ax)_n\}$ given by

$$(Ax)_n = \sum_k a_{nk} x_k \tag{2}$$

exists for each $n \in \mathbb{N}$ and is in μ_2^F.

$A \in (\mu_1^F : \mu_2^F)$ if and only if the series on the right hand side of (2) converges for each $n \in \mathbb{N}$ and every $x = (x_k) \in \mu_1^F$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu_2^F$. A sequence x is said to be $A-$*summable* to α if Ax converges to α which is called the $A-$ *limit* of x. Also by $A \in (\mu_1^F : \mu_2^F; P)$ we mean that A preserves the limit that is $A-$ *limit* of x is equal to limit of x for all $x = (x_k) \in \mu_1^F$.

Talo and Başar [18] characterized the following classes $(\mu^F : \ell_\infty^F)$, $(c_0^F : c^F)$, $(c_0^F : c_0^F)$, $(c^F : c^F; P)$, $(\ell_p^F : c^F)$, $(\ell_p^F : c_0^F)$ and $(\ell_\infty^F : c_0^F)$ of infinite matrices of fuzzy numbers, where $\mu^F = \{\ell_\infty^F, c^F, c_0^F, \ell_p^F\}$.

Theorem 2.2. (Talo and Başar [18]) *Let $A = (a_{nk})$ be any two dimensional infinite matrix of fuzzy numbers. Then*

(*i*) *$A = (a_{nk}) \in (\ell_\infty^F : \ell_\infty^F)$ if and only if*

$$M = \sup_n \sum_k D(a_{nk}, \overline{0}) < \infty. \tag{3}$$

(*ii*) *$A = (a_{nk}) \in (c^F : \ell_\infty^F)$ if and only if (3) holds.*

(*iii*) *$A = (a_{nk}) \in (c_0^F : \ell_\infty^F)$ if and only if (3) holds.*

(*iv*) *$A = (a_{nk}) \in (\ell_p^F : \ell_\infty^F)$ if and only if*

$$C = \sup_n \sum_k [D(a_{nk}, \overline{0})]^q < \infty.$$

Theorem 2.3. (Theorem 4.6, Talo and Başar [18]) *Let $A = (a_{nk})$ be a two dimensional infinite matrix of fuzzy numbers with $a_{nk} \geqslant \overline{0}$ for all $n, k \in \mathbb{N}$. Then $A \in (c^F : c^F; P)$ if and only if (3) holds and*

$$\lim_{n\longrightarrow\infty} a_{nk} = \overline{0}, \tag{4}$$

$$\lim_{n\longrightarrow\infty} \sum_k a_{nk} = \overline{1}, \tag{5}$$

for all $k \in \mathbb{N}$.

The proof of the following theorem follows using similar arguments applied in the proof of above Theorem 2.3. However, we give the detailed proof for the benefit of new readers in the field of our paper.

Theorem 2.4. *Let $A = (a_{nk})$ be a two dimensional matrix of fuzzy numbers with $a_{nk} \geqslant \overline{0}$ for all $n, k \in \mathbb{N}$. Then $A \in (c^F : c^F)$ if and only if (3) holds and*

$$\text{there exists } \bar{\alpha}_k \in E^1 \text{ such that } \lim_{n\longrightarrow\infty} a_{nk} = \bar{\alpha}_k, \tag{6}$$

$$\text{there exists } \bar{\alpha} \in E^1 \text{ such that } \lim_{n\longrightarrow\infty} \sum_k a_{nk} = \bar{\alpha}, \tag{7}$$

for all $k \in \mathbb{N}$.

Proof. Let us suppose that $A = (a_{nk}) \in (c^F : c^F)$ and $x = (x_k) \in c^F$. Since the inclusion $c^F \subset \ell_\infty^F$ holds, the inclusion $(c^F : c^F) \subset (c^F : \ell_\infty^F)$ also hold. Thus, the necessity of (3) holds.

We define the sequence $x = (x_k) \in c^F$ by

$$\begin{aligned} x_k &= \overline{1}, \quad n = k, \\ &= \overline{0}, \quad n \neq k, \end{aligned}$$

for all $n \in \mathbb{N}$. Then

$$(Ax)_n = (a_{nk})_{n=1}^{\infty} \in c^F.$$

Thus, as $n \longrightarrow \infty$, $(Ax)_n$ tends to a limit say, $\bar{\alpha}_k \in E^1$. So, (6) holds.

Similarly, taking $u = (u_k) := (\overline{1}) \in c^F$, we get that (7) holds.

For the converse part, let us consider that the conditions (3), (6) and (7) hold. Let $(x_k) \in c^F$. Then since Ax exists, the series $\sum_k a_{nk}x_k$ converges for each fixed $n \in \mathbb{N}$. Hence, $A_n \in \{c^F\}^\beta$ for all $n \in \mathbb{N}$.

It is obvious that (3) holds if and only if

$$\sup_n \sum_k \sup_{\lambda\in[0,1]} |a^-_{nk}(\lambda)| < \infty,$$

and

$$\sup_n \sum_k \sup_{\lambda\in[0,1]} |a^+_{nk}(\lambda)| < \infty.$$

(6) holds if and only if

$$\lim_{n\longrightarrow\infty} \sup_{\lambda\in[0,1]} |a^-_{nk}(\lambda) - \alpha^-_k(\lambda)| = 0,$$

and

$$\lim_{n\longrightarrow\infty} \sup_{\lambda\in[0,1]} |a^+_{nk}(\lambda) - \alpha^+_k(\lambda)| = 0.$$

Similarly, (7) holds if and only if

$$\lim_{n\longrightarrow\infty} \sup_{\lambda\in[0,1]} |\sum_k a^-_{nk}(\lambda) - \alpha^-(\lambda)| = 0,$$

and

$$\lim_{n\longrightarrow\infty} \sup_{\lambda\in[0,1]} |\sum_k a^+_{nk}(\lambda) - \alpha^+(\lambda)| = 0.$$

Now, suppose that $x_k \longrightarrow x$ as $k \longrightarrow \infty$. This implies that, $x^-_k(\lambda) \longrightarrow x^-(\lambda)$ as $k \longrightarrow \infty$ and $x^+_k(\lambda) \longrightarrow x^+(\lambda)$ as $k \longrightarrow \infty$, uniformly in λ's.
Since,

$$\begin{aligned}
&|\sum_k a^-_{nk}(\lambda)x^-_k(\lambda) - \alpha^-(\lambda)x^-(\lambda)| \\
&= |\sum_k a^-_{nk}(\lambda)x^-_k(\lambda) - x^-(\lambda)\sum_k a^-_{nk}(\lambda) + x^-(\lambda)\sum_k a^-_{nk}(\lambda) - \alpha^-(\lambda)x^-(\lambda)| \\
&\leq |\sum_k a^-_{nk}(\lambda)x^-_k(\lambda) - x^-(\lambda)\sum_k a^-_{nk}(\lambda)| + |x^-(\lambda)\sum_k a^-_{nk}(\lambda) - \alpha^-(\lambda)x^-(\lambda)| \\
&\leq \sum_k |a^-_{nk}(\lambda)||x^-_k(\lambda) - x^-(\lambda)| + |x^-(\lambda)||\sum_k a^-_{nk}(\lambda) - \alpha^-(\lambda)| \\
&\leq \sum_k \sup_{\lambda\in[0,1]} |a^-_{nk}(\lambda)| \sup_{\lambda\in[0,1]} |x^-_k(\lambda) - x^-(\lambda)| + \sup_{\lambda\in[0,1]} |x^-(\lambda)| \sup_{\lambda\in[0,1]} |\sum_k a^-_{nk}(\lambda) - \alpha^-(\lambda)|,
\end{aligned}$$

We have

$$\sup_{\lambda\in[0,1]} |\sum_k a^-_{nk}(\lambda)x^-_k(\lambda) - \alpha^-(\lambda)x^-(\lambda)| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since, $a_{nk} \geqslant \overline{0}$ for all $n, k \in \mathbb{N}$ and $x^-_k(\lambda) \leq x^+_k(\lambda)$ for all $\lambda \in [0,1]$,we have

$a^-_{nk}(\lambda)x^-_k(\lambda) \leq a^-_{nk}(\lambda)x^+_k(\lambda)$ and $a^+_{nk}(\lambda)x^-_k(\lambda) \leq a^+_{nk}(\lambda)x^+_k(\lambda)$,
which implies,

$$\begin{aligned}(a_{nk}x_k)^-(\lambda) &= \min\{a^-_{nk}(\lambda)x^-_k(\lambda), a^-_{nk}(\lambda)x^+_k(\lambda), a^+_{nk}(\lambda)x^-_k(\lambda), a^+_{nk}(\lambda)x^+_k(\lambda)\} \\ &= \min\{a^-_{nk}(\lambda)x^-_k(\lambda), a^+_{nk}(\lambda)x^-_k(\lambda)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}(a_{nk}x_k)^+(\lambda) &= \max\{a^-_{nk}(\lambda)x^-_k(\lambda), a^-_{nk}(\lambda)x^+_k(\lambda), a^+_{nk}(\lambda)x^-_k(\lambda), a^+_{nk}(\lambda)x^+_k(\lambda)\} \\ &= \max\{a^-_{nk}(\lambda)x^+_k(\lambda), a^+_{nk}(\lambda)x^+_k(\lambda)\}.\end{aligned}$$

Consequently,

$$\lim_{n\longrightarrow\infty}\sum_k (a_{nk}x_k)^-(\lambda) = \lim_{n\longrightarrow\infty}\sum_k \min\{a^-_{nk}(\lambda)x^-_k(\lambda), a^+_{nk}(\lambda)x^-_k(\lambda)\} = \alpha^-(\lambda)x^-(\lambda),$$

and

$$\lim_{n\longrightarrow\infty}\sum_k (a_{nk}x_k)^+(\lambda) = \lim_{n\longrightarrow\infty}\sum_k \max\{a^-_{nk}(\lambda)x^+_k(\lambda), a^+_{nk}(\lambda)x^+_k(\lambda)\} = \alpha^+(\lambda)x^+(\lambda),$$

uniformly in λ's.

Hence $\sum_k a_{nk}x_k \longrightarrow \bar{\alpha}x$ as $n \longrightarrow \infty$.

So, $A \in (c^F : c^F)$.

This step completes the proof. □

Theorem 2.5. *Let $A = (a_{nk})$ be a two dimensional matrix of fuzzy numbers with $a_{nk} \succcurlyeq \overline{0}$ for all $n, k \in \mathbb{N}$. Then $A \in (\ell^F_1 : \ell^F_p)$ if and only if for all $k \in \mathbb{N}$,*

$$M = \sup_k \sum_n (D(a_{nk}, \overline{0}))^p < \infty, \quad (1 \leq p < \infty), \tag{8}$$

$$\sup_{n,k} (D(a_{nk}, \overline{0}))^p < \infty, \quad (p = \infty). \tag{9}$$

Proof. Since the proofs are similar, we give the proof only for (8). Suppose that the condition (8) holds and let $x = (x_k) \in \ell^F_1$.

Now,

$$(\sum_n D((Ax)_n, \overline{0})^p)^{1/p} = (\sum_n (D(\sum_k a_{nk}x_k, \overline{0}))^p)^{1/p}$$

We have,

$$(D(\sum_k a_{nk}x_k, \overline{0}))^p \leq (\sum_k D(a_{nk}x_k, \overline{0}))^p.$$

Then

$$\sum_n (D(\sum_k a_{nk}x_k, \overline{0}))^p \leq \sum_n (\sum_k D(a_{nk}x_k, \overline{0}))^p$$

$$\Rightarrow (\sum_n (D(\sum_k a_{nk}x_k, \overline{0}))^p)^{1/p} \leq (\sum_n (\sum_k D(a_{nk}x_k, \overline{0}))^p)^{1/p}.$$

Using Minkowski's inequality, we have,

$(\sum_n(\sum_k D(a_{nk}x_k,\overline{0}))^p)^{1/p} \leq (\sum_n D(a_{n1}x_1,\overline{0})^p)^{1/p} + (\sum_n D(a_{n2}x_2,\overline{0})^p)^{1/p} + ...$

$= \sum_k(\sum_n D(a_{nk}x_k,\overline{0})^p)^{1/p}$

$\leq \sum_k(\sum_n D(a_{nk},\overline{0})^p D(x_k,\overline{0})^p)^{1/p}$

$= \sum_k(D(x_k,\overline{0})^p)^{1/p}(\sum_n D(a_{nk},\overline{0})^p)^{1/p}$

$\leq M^{1/p}\sum_k(D(x_k,\overline{0})^p)^{1/p} < \infty.$

Thus, $Ax \in \ell_p^F$, i.e., $A \in (\ell_1^F : \ell_p^F)$.

We observe that since $a_{ik} \succcurlyeq \overline{0}$ for all $i,k \in \mathbb{N}$ and $x_k^-(\lambda) \leq x_k^+(\lambda)$ for all $\lambda \in [0,1]$, we have,

$a_{ik}^-(\lambda)x_k^-(\lambda) \leq a_{ik}^-(\lambda)x_k^+(\lambda)$ and $a_{ik}^+(\lambda)x_k^-(\lambda) \leq a_{ik}^+(\lambda)x_k^+(\lambda)$,
which implies,

$(a_{ik}x_k)^-(\lambda) = \min\{a_{ik}^-(\lambda)x_k^-(\lambda), a_{ik}^-(\lambda)x_k^+(\lambda), a_{ik}^+(\lambda)x_k^-(\lambda), a_{ik}^+(\lambda)x_k^+(\lambda)\} = \min\{a_{ik}^-(\lambda)x_k^-(\lambda), a_{ik}^+(\lambda)x_k^-(\lambda)\}.$

Similarly,

$(a_{ik}x_k)^+(\lambda) = \max\{a_{ik}^-(\lambda)x_k^-(\lambda), a_{ik}^-(\lambda)x_k^+(\lambda), a_{ik}^+(\lambda)x_k^-(\lambda), a_{ik}^+(\lambda)x_k^+(\lambda)\} = \max\{a_{ik}^-(\lambda)x_k^+(\lambda), a_{ik}^+(\lambda)x_k^+(\lambda)\}.$

For the converse part, let us consider that $A \in (\ell_1^F : \ell_p^F)$, so that

$$\sum_i(D(A_i(x),\overline{0}))^p < \infty,$$

on ℓ_1^F where $A_i(x) = \sum_k a_{ik}x_k$.

To show $\sup_k \sum_i(D(a_{ik},\overline{0}))^p < \infty \ \ (1 \leq p < \infty)$, it is sufficient to show that

$$\sup_k \sum_{i=1}^{\infty} \sup_{\lambda\in[0,1]} |a_{ik}^-(\lambda)|^p < \infty \tag{10}$$

and

$$\sup_k \sum_{i=1}^{\infty} \sup_{\lambda\in[0,1]} |a_{ik}^+(\lambda)|^p < \infty, \tag{11}$$

for all $k \in \mathbb{N}$.

Since $\sum_k a_{ik}x_k$ converges for each i whenever $x = (x_k) \in \ell_1^F$, we have, $\sum_k(a_{ik}x_k)^-(\lambda)$ and $\sum_k(a_{ik}x_k)^+(\lambda)$ converges uniformly in $\lambda \in [0,1]$ for each i whenever $x_k^-(\lambda) \in \ell_1^-$ and $x_k^+(\lambda) \in \ell_1^+$ for all $k \in \mathbb{N}$, where

$\ell_1^- = \{x^-(\lambda) : x = [x]_\lambda = [(x_k)]_\lambda = [x_k^-(\lambda), x_k)^+(\lambda)] \in \ell_1^F \textit{ for all } \lambda \in [0,1]\},$

and

$\ell_1^+ = \{x^+(\lambda) : x = [x]_\lambda = [(x_k)]_\lambda = [x_k^-(\lambda), x_k)^+(\lambda)] \in \ell_1^F \, for all \, \lambda \in [0,1]\}$, corresponding to each $x = (x_k) \in \ell_1^F$. It is easy to see that $\ell_1^- \subseteq \ell_1$ and $\ell_1^+ \subseteq \ell_1$.

Using Banach-Steinhaus theorem, we get,

$$\sup_k |a_{ik}^-(\lambda)| < \infty,$$

and

$$\sup_k |a_{ik}^+(\lambda)| < \infty,$$

for all $\lambda \in [0,1]$ and for all each i.

Each member of ℓ_1 can be regarded as a member of ℓ_1^F and our desired condition (11) is based on crisp terms (λ-level sets), let us define a function g_n on ℓ_1 as follows:

If $x^+(\lambda) \in \ell_1^+$,

$$\begin{aligned} g_n(x^+(\lambda)) &= (\sum_{i=1}^n |\sum_k \max\{a_{ik}^-(\lambda)x_k^+(\lambda), a_{ik}^+(\lambda)x_k^+(\lambda)\}|^p)^{1/p} \\ &= (\sum_{i=1}^n |\sum_k (a_{ik}x_k)^+(\lambda)|^p)^{1/p} \\ &= (\sum_{i=1}^n |(A_i(x))^+(\lambda)|)^{1/p}, \end{aligned}$$

for all $\lambda \in [0,1]$ and for all $k \in \mathbb{N}$.

If $x \in \ell_1 \setminus \ell_1^+$ then $x^+(\lambda) = x^-(\lambda) = x$ for all $\lambda \in [0,1]$, we have $g_n(x) = (\sum_{i=1}^n |\sum_k a_{ik}x_k|^p)^{1/p}$.

Thus, each g_n is a seminorm on ℓ_1. Also, each $(A_i(x))^+(\lambda)$ is a bounded linear functional on ℓ_1^+. It is easy to see that g_n is bounded on ℓ_1, and in particular on ℓ_1^+. So we get a sequence (g_n) of continuous seminorms on ℓ_1 such that,

$$g_n(x^+(\lambda)) = (\sum_{i=1}^n |(A_i(x))^+(\lambda)|^p)^{1/p} \le (\sum_{i=1}^\infty |(A_i(x))^+(\lambda)|^p)^{1/p} < \infty,$$

for each $x^+(\lambda) \in \ell_1$. It follows from the Banach-Steinhaus theorem that there exists a constant $G < \infty$ such that for all $\lambda \in [0,1]$,

$$(\sum_{i=1}^\infty |(A_i(x))^+(\lambda)|)^p)^{1/p} < G\|x^+(\lambda)\|,$$

on ℓ_1 for all $\lambda \in [0,1]$.

Now letting $x = (x_k)$ defined by

$$\begin{aligned} x = (x_k) &:= \overline{1}, \quad i = k, \\ &= \overline{0}, \quad i \neq k, \end{aligned}$$

for all $i \in \mathbb{N}$ and considering $(a_{ik}x_k)^+(\lambda) = \max\{a^-_{ik}(\lambda)x^+_k(\lambda), a^+_{ik}(\lambda)x^+_k(\lambda)\} = a^+_{ik}(\lambda)x^+_k(\lambda)$, we get for all $\lambda \in [0,1]$,

$$(A_i(x))^+(\lambda) = \sum_k (a_{ik}x_k)^+(\lambda) = \sum_k a^+_{ik}(\lambda)x^+_k(\lambda) = a^+_{ik}(\lambda),$$

for each fixed $k \in \mathbb{N}$.

Thus,

$$(\sum_{i=1}^{\infty} |a^+_{ik}(\lambda)|)^p)^{1/p} \leq G\|x^+(\lambda)\| < \infty,$$

for each fixed $k \in \mathbb{N}$ and for all $\lambda \in [0,1]$. Which implies that,

$$\sup_k \sum_{i=1}^{\infty} \sup_{\lambda\in[0,1]} |a^+_{ik}(\lambda)|^p < \infty.$$

Similarly, considering $(a_{ik}x_k)^+(\lambda) = \max\{a^-_{ik}(\lambda)x^+_k(\lambda), a^+_{ik}(\lambda)x^+_k(\lambda)\} = a^-_{ik}(\lambda)x^+_k(\lambda)$, we have the following condition (10),

$$\sup_k \sum_{i=1}^{\infty} \sup_{\lambda\in[0,1]} |a^-_{ik}(\lambda)|^p < \infty.$$

Consequently, we get that,

$$\sup_k \sum_i (D(a_{ik}, \overline{0}))^p < \infty, \quad (1 \leq p < \infty).$$

This step completes the proof. □

Theorem 2.6. *Let $A = (a_{nk})$ be a two dimensional matrix of non-negative fuzzy numbers with $a_{nk} \succcurlyeq \overline{0}$ for all $n,k \in \mathbb{N}$. Then $A \in (\ell_1^F : \ell_1^F; P)$ if and only if the conditions*

$$\sup_k \sum_n D(a_{nk}, \overline{0}) < \infty, \tag{12}$$

$$\sum_n a_{nk} = \overline{1}, \tag{13}$$

for all $k \in \mathbb{N}$.

Proof. Let us suppose the conditions (12)-(13) hold and let $x = (x_k) \in \ell_1^F$. Since (12) holds, putting $p = 1$ in Theorem 2.5, we get $A \in (\ell_1^F : \ell_1^F)$. Also, since the condition (13) holds, considering $(a_{nk}x_k)^+(\lambda) = \max\{a^-_{nk}(\lambda)x^+_k(\lambda), a^+_{nk}(\lambda)x^+_k(\lambda)\} = a^+_{nk}(\lambda)x^+_k(\lambda)$, we get for all $\lambda \in [0,1]$,

$$\begin{aligned}
\sum_n (A_n(x))^+(\lambda) &= \sum_n \sum_k (a_{nk}x_k)^+(\lambda) \\
&= \sum_n \sum_k a^+_{nk}(\lambda)x^+_k(\lambda) \\
&= \sum_k x^+_k(\lambda) \sum_n a^+_{nk}(\lambda) \\
&= \sum_k x^+_k(\lambda).
\end{aligned}$$

Similarly, taking $(A_n(x))^-(\lambda) = \sum_k (a_{nk}x_k)^-(\lambda)$ for all $n \in \mathbb{N}$ and for all $\lambda \in [0,1]$, we get, $\sum_n (A_n(x))^-(\lambda) = \sum_k x^-_k(\lambda)$.

From the above we can see that $A \in (\ell_1^F : \ell_1^F; P)$.

For the converse part, suppose that $A \in (\ell_1^F : \ell_1^F, P)$. Then obviously (12) holds.

Taking $(a_{nk}x_k)^+(\lambda) = \max\{a_{nk}^-(\lambda)x_k^+(\lambda), a_{nk}^+(\lambda)x_k^+(\lambda)\} = a_{nk}^-(\lambda)x_k^+(\lambda)$, we get,

$\sum_n (A_n(x))^+(\lambda) = \sum_n \sum_k (a_{nk}x_k)^+(\lambda) = \sum_n \sum_k a_{nk}^-(\lambda)x_k^+(\lambda) = \sum_k x_k^+(\lambda)$.
for all $\lambda \in [0, 1]$.

Again considering

$(a_{nk}x_k)^+(\lambda) = \max\{a_{nk}^-(\lambda)x_k^+(\lambda), a_{nk}^+(\lambda)x_k^+(\lambda)\} = a_{nk}^+(\lambda)x_k^+(\lambda)$, we get,

$\sum_n (A_n(x))^+(\lambda) = \sum_n \sum_k (a_{nk}x_k)^+(\lambda) = \sum_n \sum_k a_{nk}^+(\lambda)x_k^+(\lambda) = \sum_k x_k^+(\lambda)$.
for all $\lambda \in [0, 1]$.

Now letting $x = (x_k)$ defined by

$$\begin{aligned} x = (x_k) &:= \overline{1}, \quad k = r, \\ &= \overline{0}, \quad k \neq r, \end{aligned}$$

for all $k \in \mathbb{N}$, we get,

$$|\sum_n a_{nr}^-(\lambda) - 1| = 0,$$

and

$$|\sum_n a_{nr}^+(\lambda) - 1| = 0,$$

for all $\lambda \in [0, 1]$. Which implies that the condition (13) holds as r is arbitrary.

This step completes the proof. □

Acknowledgement

The authors are very much grateful to the anonymous referees for their constructive comments and suggestions.

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