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The Multiple Composition of the Left and Right Fractional Riemann-Liouville Integrals – Analytical and Numerical Calculations

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Abstract. New fractional integral operators of order $\alpha \in \mathbb{R}_+$ are introduced. These operators are defined as the composition of the left and right (or the right and left) Riemann-Liouville fractional order integrals. Some of their properties are studied. Analytical results of fractional integrals of several functions are presented. For a numerical calculation of fractional order integrals, two numerical procedures are given. In the final part of this paper, examples of numerical evaluations of these operators of three different functions are shown in plots and the comparison of the numerical accuracy was analyzed in tables.

1. Introduction

Fractional calculus is as old as the traditional calculus proposed independently by Newton and Leibniz. The theory of operators of non-integer order was initiated in 1695. Since then, several scientists studied this question, among them Euler, Abel, Fourier, Liouville, Riemann, Grünwald, Letnikov, and many others.

Initially, the progress of fractional calculus was quite slow. However, after 1900 the theory of operators of non-integer order has developed much faster. Many definitions of integral and differential operators are proposed e.g. left and right Riemann-Liouville derivative and integral, left and right Caputo derivative, left and right Grünwald-Letnikov derivative, Hadamard integral, Weyl integral, left and right Chen integral, etc. For a review of definitions of fractional order operators that appear in mathematics, physics, and engineering, we refer the reader to books [1, 11, 12, 20] and papers [13, 21].

Recently, a subtopic of fractional calculus has been growing in importance - it is fractional variational calculus [2, 14–18, 23]. In such an approach we get differential and integral equations containing the left and right fractional operators, simultaneously. Solutions of this type of equations contain a composition of the left and right fractional Riemann-Liouville integrals [2, 7, 14]. Such a new type of fractional integral operators has other properties than well-known operators like the left and right Riemann-Liouville integrals.

The study of properties of these new fractional integral operators it seems to be an interesting issue. We present a selected properties, which could be useful in finding a solution of the Euler–Lagrange equations, in the following part of the paper.

On the other hand, the ability to calculate this type of integrals is very important to get a graphical interpretation of solutions of fractional variational differential equations. The analytical evaluations of

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new fractional integrals for any function are difficult to achieve. In some cases, we can express them through special functions but it also causes difficulties with calculation of the function values. Therefore, numerical methods are a useful tool to obtain an approximation of integrals with different types of kernel [8–10, 19, 24, 27].

In the book [14], Klimek presented an original approach to obtain the solution for the Euler–Lagrange equation where the composition of the left and right fractional integrals was used. Next, these types of integral operators appeared when Blaszczyk and Ciesielski transformed a type of the Euler–Lagrange equation into an integral form in [3, 4, 6]. The new notation of this type of the fractional operator was presented for the first time in the paper [5]. All these above mentioned facts were inspiration for us to deeply analyse this issue. Here, we study several properties and give two numerical procedures for the numerical calculation of new fractional integral operators.

2. Preliminaries

Let us introduce new fractional integral operators, namely $\mathcal{I}^{\alpha,n}_{a^+,b^-}$ and $\mathcal{I}^{\alpha,n}_{b^-,a^+}$.

Definition 2.1. *We define the fractional integral operators $\mathcal{I}^{\alpha,n}_{a^+,b^-}$ and $\mathcal{I}^{\alpha,n}_{b^-,a^+}$ of order $\alpha > 0$, for $n \in \mathbb{N} \cup \{0\}$, on the finite interval $t \in [a,b]$, for function $f \in L_1(a,b)$, as follows*

$$\mathcal{I}^{\alpha,n}_{a^+,b^-} f(t) := \left(I^{\alpha}_{a^+} I^{\alpha}_{b^-}\right)^n f(t) = \underbrace{\left(I^{\alpha}_{a^+} I^{\alpha}_{b^-} \;\; I^{\alpha}_{a^+} I^{\alpha}_{b^-} \;\; \ldots\right)}_{n\text{ times}} f(t), \tag{1}$$

$$\mathcal{I}^{\alpha,n}_{b^-,a^+} f(t) := \left(I^{\alpha}_{b^-} I^{\alpha}_{a^+}\right)^n f(t) = \underbrace{\left(I^{\alpha}_{b^-} I^{\alpha}_{a^+} \;\; I^{\alpha}_{b^-} I^{\alpha}_{a^+} \;\; \ldots\right)}_{n\text{ times}} f(t). \tag{2}$$

Definition 2.2. *(Left and right Riemann-Liouville fractional integrals). The left and right Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order $\alpha > 0$ are defined by [11, 12, 20, 22, 25]*

$$I^{\alpha}_{a^+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad \text{for } t > a, \tag{3}$$

$$I^{\alpha}_{b^-} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} d\tau \quad \text{for } t < b, \tag{4}$$

respectively. Here $\Gamma(\alpha)$ denotes the Euler's gamma function.

Definitions (1) and (2) for $n = 1$ can also be written in the following way

$$\mathcal{I}^{\alpha,1}_{a^+,b^-} f(t) = \frac{1}{\Gamma^2(\alpha)} \int_a^t \frac{1}{(t-\tau)^{1-\alpha}} \int_\tau^b \frac{f(\xi)}{(\xi-\tau)^{1-\alpha}} \, d\xi \, d\tau, \tag{5}$$

$$\mathcal{I}^{\alpha,1}_{b^-,a^+} f(t) = \frac{1}{\Gamma^2(\alpha)} \int_t^b \frac{1}{(\tau-t)^{1-\alpha}} \int_a^\tau \frac{f(\xi)}{(\tau-\xi)^{1-\alpha}} \, d\xi \, d\tau, \tag{6}$$

and for $\alpha = 1$ are simplified to the forms

$$\mathcal{I}^{1,1}_{a^+,b^-} f(t) = \int_a^t \int_\tau^b f(\xi) \, d\xi \, d\tau, \quad \mathcal{I}^{1,1}_{b^-,a^+} f(t) = \int_t^b \int_a^\tau f(\xi) \, d\xi \, d\tau. \tag{7}$$

3. Properties

Now, we introduce some properties of fractional integral operators (1) and (2). Let us begin with linearity of such operators.

Property 3.1. *(Linearity) Let c_1 and c_2 be constants and $f, g \in L_1(a, b)$, then*

$$\mathcal{I}^{\alpha,n}_{a^+,b^-}\left(c_1 f(t) + c_2 g(t)\right) = c_1 \mathcal{I}^{\alpha,n}_{a^+,b^-} f(t) + c_2 \mathcal{I}^{\alpha,n}_{a^+,b^-} g(t), \tag{8}$$

$$\mathcal{I}^{\alpha,n}_{b^-,a^+}\left(c_1 f(t) + c_2 g(t)\right) = c_1 \mathcal{I}^{\alpha,n}_{b^-,a^+} f(t) + c_2 \mathcal{I}^{\alpha,n}_{b^-,a^+} g(t). \tag{9}$$

Proof. It follows from linearity of the fractional Riemann-Liouville operators.

Next, we introduce very useful recurrence formulas for fractional integral operators

Property 3.2. *(Recurrence formula for integrals)*

$$\mathcal{I}^{\alpha,n}_{a^+,b^-} f(t) = \begin{cases} f(t), & \text{if } n = 0 \\ \mathcal{I}^{\alpha,1}_{a^+,b^-} \mathcal{I}^{\alpha,n-1}_{a^+,b^-} f(t), & \text{if } n > 0 \end{cases}, \tag{10}$$

$$\mathcal{I}^{\alpha,n}_{b^-,a^+} f(t) = \begin{cases} f(t), & \text{if } n = 0 \\ \mathcal{I}^{\alpha,1}_{b^-,a^+} \mathcal{I}^{\alpha,n-1}_{b^-,a^+} f(t), & \text{if } n > 0 \end{cases}. \tag{11}$$

Proof. It follows directly from definitions (1) and (2).

Property 3.3. *(Symmetry) Let $\alpha > 0$ and Q is the reflection operator on the interval $[a, b]$, then for $f(t) \in L_1(a, b)$ the following formulas are valid*

$$Q\mathcal{I}^{\alpha,n}_{a^+,b^-} f(t) = \mathcal{I}^{\alpha,n}_{b^-,a^+} f(a + b - t), \tag{12}$$

$$Q\mathcal{I}^{\alpha,n}_{b^-,a^+} f(t) = \mathcal{I}^{\alpha,n}_{a^+,b^-} f(a + b - t). \tag{13}$$

Proof. Let us start from the following relation (using the following properties [25]: $QI^{\alpha}_{a^+} = I^{\alpha}_{b^-}Q$ and $QI^{\alpha}_{b^-} = I^{\alpha}_{a^+}Q$)

$$Q\mathcal{I}^{\alpha,1}_{a^+,b^-} f(t) = QI^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t) = I^{\alpha}_{b^-} QI^{\alpha}_{b^-} f(t) = I^{\alpha}_{b^-} I^{\alpha}_{a^+} Qf(t) = \mathcal{I}^{\alpha,1}_{b^-,a^+} Qf(t). \tag{14}$$

Next, by using the above relation and recurrency (see Eq. (10)) we can write

$$Q\mathcal{I}^{\alpha,n}_{a^+,b^-} f(t) = Q\mathcal{I}^{\alpha,1}_{a^+,b^-} \mathcal{I}^{\alpha,n-1}_{a^+,b^-} f(t) = \mathcal{I}^{\alpha,1}_{b^-,a^+} Q\mathcal{I}^{\alpha,n-1}_{a^+,b^-} f(t) = \mathcal{I}^{\alpha,2}_{b^-,a^+} Q\mathcal{I}^{\alpha,n-2}_{a^+,b^-} f(t) = ... = \mathcal{I}^{\alpha,n}_{b^-,a^+} Qf(t). \tag{15}$$

The operator Q for function $f(t)$ on the interval $[a, b]$ has property $Q f(t) = f(a + b - t)$. By using such property and putting it into Eq. (15) we obtain formula (12). In a similar way one can prove formula (13).

Property 3.4. *(Fractional integration by parts) Let $n \in \mathbb{N} \cup \{0\}$, $f(t)$ and $g(t)$ are the arbitrary integrable functions, then the following rules occur*

$$\int_a^b f(t) \cdot \mathcal{I}^{\alpha,n}_{a^+,b^-} g(t)\, dt = \int_a^b \mathcal{I}^{\alpha,n}_{a^+,b^-} f(t) \cdot g(t)\, dt, \tag{16}$$

$$\int_a^b f(t) \cdot \mathcal{I}^{\alpha,n}_{b^-,a^+} g(t)\, dt = \int_a^b \mathcal{I}^{\alpha,n}_{b^-,a^+} f(t) \cdot g(t)\, dt. \tag{17}$$

Proof. We are reminded of the rule for fractional integration by parts (proved in Samko et al [25])

$$\int_a^b f(t) \cdot I_{a^+}^{\alpha} g(t)\, dt = \int_a^b I_{b^-}^{\alpha} f(t) \cdot g(t)\, dt. \tag{18}$$

First, we show the proof of the first relation (16) for the case of $n = 1$. We obtain

$$\int_a^b f(t) \cdot \mathcal{I}_{a^+,b^-}^{\alpha,1} g(t)\, dt = \int_a^b f(t) \cdot I_{a^+}^{\alpha} I_{b^-}^{\alpha} g(t)\, dt = \int_a^b I_{b^-}^{\alpha} f(t) \cdot I_{b^-}^{\alpha} g(t)\, dt = \int_a^b I_{a^+}^{\alpha} I_{b^-}^{\alpha} f(t) \cdot g(t)\, dt =$$
$$= \int_a^b \mathcal{I}_{a^+,b^-}^{\alpha,1} f(t) \cdot g(t)\, dt. \tag{19}$$

By using the recurrence formula (see: Property 3.2) we have

$$\int_a^b f(t) \cdot \mathcal{I}_{a^+,b^-}^{\alpha,n} g(t)\, dt = \int_a^b f(t) \cdot \mathcal{I}_{a^+,b^-}^{\alpha,1} \mathcal{I}_{a^+,b^-}^{\alpha,n-1} g(t)\, dt = \int_a^b \mathcal{I}_{a^+,b^-}^{\alpha,1} f(t) \cdot \mathcal{I}_{a^+,b^-}^{\alpha,n-1} g(t)\, dt = \dots =$$
$$= \int_a^b \mathcal{I}_{a^+,b^-}^{\alpha,n-1} f(t) \cdot \mathcal{I}_{a^+,b^-}^{\alpha,1} g(t)\, dt = \int_a^b \mathcal{I}_{a^+,b^-}^{\alpha,n} f(t) \cdot g(t)\, dt. \tag{20}$$

The proof of (17) should be treated in a similar manner.

In particular, the following relation is satisfied

$$\int_a^b \mathcal{I}_{a^+,b^-}^{\alpha,n-n_1} f(t) \cdot \mathcal{I}_{a^+,b^-}^{\alpha,n_1} g(t)\, dt = \int_a^b \mathcal{I}_{a^+,b^-}^{\alpha,n-n_2} f(t) \cdot \mathcal{I}_{a^+,b^-}^{\alpha,n_2} g(t)\, dt, \tag{21}$$

where $n_1 \in \{0, ..., n\}$ and $n_2 \in \{0, ..., n\}$.

Property 3.5. *(Composition fractional integrals with m-th integer order integral I^m) Let $m \in \mathbb{N} \cup \{0\}$, then the following relations take place*

$$I^m \mathcal{I}_{a^+,b^-}^{\alpha,n} f(t) = \mathcal{I}_{a^+,b^-}^{\alpha,n} I^m f(t), \quad I^m \mathcal{I}_{b^-,a^+}^{\alpha,n} f(t) = \mathcal{I}_{b^-,a^+}^{\alpha,n} I^m f(t). \tag{22}$$

Property 3.6. *(Identity operator) For $\alpha = 0$ and $n \geq 0$ we define the Identity operator I^0 as*

$$\mathcal{I}_{a^+,b^-}^{0,n} := I^0, \quad \mathcal{I}_{b^-,a^+}^{0,n} := I^0 \tag{23}$$

and we obtain respectively

$$\mathcal{I}_{a^+,b^-}^{0,n} f(t) = I^0 f(t) = f(t), \quad \mathcal{I}_{b^-,a^+}^{0,n} f(t) = I^0 f(t) = f(t). \tag{24}$$

4. Certain Examples of Integrals

The fractional integrals (1) and (2) for $n = 1$ of the constant function $f(t) = C$ are given by

$$\begin{aligned} \mathcal{I}^{\alpha,1}_{a^+,b^-} C &= C\frac{(t-a)^{\alpha}(b-a)^{\alpha}}{\Gamma^2(\alpha+1)}\, {}_2F_1\left(1,-\alpha;\alpha+1;\frac{t-a}{b-a}\right) \\ &= C\frac{(t-a)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(-\alpha)} G^{1,2}_{2,2}\left(-\frac{t-a}{b-a}\,\middle|\, \begin{matrix} 0, 1+\alpha \\ 0, -\alpha \end{matrix}\right), \end{aligned} \tag{25}$$

$$\begin{aligned} \mathcal{I}^{\alpha,1}_{b^-,a^+} C &= C\frac{(b-a)^{\alpha}(b-t)^{\alpha}}{\Gamma^2(\alpha+1)}\, {}_2F_1\left(1,-\alpha;\alpha+1;\frac{b-t}{b-a}\right) \\ &= C\frac{(b-a)^{\alpha}(b-t)^{\alpha}}{\Gamma(\alpha+1)\Gamma(-\alpha)} G^{1,2}_{2,2}\left(-\frac{b-t}{b-a}\,\middle|\, \begin{matrix} 0, 1+\alpha \\ 0, -\alpha \end{matrix}\right), \end{aligned} \tag{26}$$

where ${}_2F_1$ and $G^{1,2}_{2,2}$ are special functions (hypergeometric function and Meijer G-function, respectively) [11, 25, 26].

The fractional integrals of functions $f(t) = (b-t)^{-\alpha}$ and $f(t) = (t-a)^{-\alpha}$ for $0 < \alpha < 1$ and $n = 1$ have the forms

$$\mathcal{I}^{\alpha,1}_{a^+,b^-}(b-t)^{-\alpha} = \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}(t-a)^{\alpha}, \tag{27}$$

$$\mathcal{I}^{\alpha,1}_{b^-,a^+}(t-a)^{-\alpha} = \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}(b-t)^{\alpha}. \tag{28}$$

For a more general form of functions $f(t) = (b-t)^{k-\alpha}$ and $f(t) = (t-a)^{k-\alpha}$, $k \in \mathbb{N} \cup \{0\}$ and $k > \alpha - 1$, for $n = 1$ we obtained

$$\begin{aligned} \mathcal{I}^{\alpha,1}_{a^+,b^-}(b-t)^{k-\alpha} &= \frac{\Gamma(k-\alpha+1)}{\Gamma(k+1)}\frac{(b-a)^k}{\Gamma(\alpha+1)}(t-a)^{\alpha}\, {}_2F_1\left(-k,1;\alpha+1;\frac{t-a}{b-a}\right) \\ &= (t-a)^{\alpha}(b-a)^k\Gamma(k-\alpha+1)\sum_{j=0}^{k}\frac{(-1)^j}{\Gamma(k-j+1)\Gamma(\alpha+j+1)}\left(\frac{t-a}{b-a}\right)^j, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathcal{I}^{\alpha,1}_{b^-,a^+}(t-a)^{k-\alpha} &= \frac{\Gamma(k-\alpha+1)}{\Gamma(k+1)}\frac{(b-a)^k}{\Gamma(\alpha+1)}(b-t)^{\alpha}\, {}_2F_1\left(-k,1;\alpha+1;\frac{b-t}{b-a}\right) \\ &= (b-t)^{\alpha}(b-a)^k\Gamma(k-\alpha+1)\sum_{j=0}^{k}\frac{(-1)^j}{\Gamma(k-j+1)\Gamma(\alpha+j+1)}\left(\frac{b-t}{b-a}\right)^j. \end{aligned} \tag{30}$$

The next example, for function $f(t) = t^{\alpha}$, is based on the results presented in [14]

$$\mathcal{I}^{\alpha,1}_{0^+,b^-} t^{\alpha} = b^{3\alpha} G^{2\,1}_{3\,3}\left[\frac{t}{b}\,\middle|\, \begin{matrix} \alpha, 2\alpha, 3\alpha+1 \\ 3\alpha, \alpha, 0 \end{matrix}\right]. \tag{31}$$

In the case of $\alpha = 1$ and $n = 1$ for functions $f(t) = 1$, $f(t) = t$, $f(t) = t^n$ and $f(t) = \sin(t)$ we get

$$\mathcal{I}^{1,1}_{a^+,b^-} 1 = \frac{a^2}{2} - ab + bt - \frac{t^2}{2}, \tag{32}$$

$$\mathcal{I}^{1,1}_{b^-,a^+} 1 = \frac{b^2}{2} - ab + at - \frac{t^2}{2}, \tag{33}$$

$$\mathcal{I}^{1,1}_{a^+,b^-} t = \frac{a^3}{6} - \frac{ab^2}{2} + \frac{b^2 t}{2} - \frac{t^3}{6}, \tag{34}$$

$$\mathcal{I}^{1,1}_{b^-,a^+} t = \frac{b^3}{6} - \frac{a^2 b}{2} + \frac{a^2 t}{2} - \frac{t^3}{6}, \tag{35}$$

$$\mathcal{I}^{1,1}_{a^+,b^-} t^n = \frac{a^{n+2}}{(n+1)(n+2)} - \frac{ab^{n+1}}{n+1} + \frac{b^{n+1} t}{n+1} - \frac{t^{n+2}}{(n+1)(n+2)}, \tag{36}$$

$$\mathcal{I}^{1,1}_{b^-,a^+} t^n = \frac{b^{n+2}}{(n+1)(n+2)} - \frac{a^{n+1} b}{n+1} + \frac{a^{n+1} t}{n+1} - \frac{t^{n+2}}{(n+1)(n+2)}, \tag{37}$$

$$\mathcal{I}^{1,1}_{a^+,b^-} \sin(t) = \sin(t) + (a-t)\cos(b) - \sin(a), \tag{38}$$

$$\mathcal{I}^{1,1}_{b^-,a^+} \sin(t) = \sin(t) + (b-t)\cos(a) - \sin(b). \tag{39}$$

On the basis of relations (36) and (37) for $a = 0$ and $b = 1$ (calculated by using the recurrence) for $\alpha = 1$ and $n = 1, ..., 5$ we obtain

$$\mathcal{I}^{1,1}_{0^+,1^-} t = \frac{t}{2} - \frac{t^3}{6}, \tag{40}$$

$$\mathcal{I}^{1,2}_{0^+,1^-} t = \frac{5t}{24} - \frac{t^3}{12} + \frac{t^5}{120}, \tag{41}$$

$$\mathcal{I}^{1,3}_{0^+,1^-} t = \frac{61t}{720} - \frac{5t^3}{144} + \frac{t^5}{240} - \frac{t^7}{5040}, \tag{42}$$

$$\mathcal{I}^{1,4}_{0^+,1^-} t = \frac{277t}{8064} - \frac{61t^3}{4320} + \frac{t^5}{576} - \frac{t^7}{10080} + \frac{t^9}{362880}, \tag{43}$$

$$\mathcal{I}^{1,5}_{0^+,1^-} t = \frac{50521t}{3628800} - \frac{277t^3}{48384} + \frac{61t^5}{86400} - \frac{t^7}{24192} + \frac{t^9}{725760} - \frac{t^{11}}{39916800}. \tag{44}$$

The above integrals for $\alpha = 1$ allow us to verify the validity of the numerical methods by comparing the numerical values and analytical values.

5. Numerical Evaluation of Fractional Integrals

The definitions of both fractional integral operators (1) and (2) seem to be a very complicated. The mathematical software known to us allows one to determine only the values of particular cases of functions. But the analytical evaluations of fractional integrals for any function are rather impossible. Also, the closed-form solution for the fractional integrals (expressed i.e. through the Meijer's G function or any hypergeometric function – see previous Chapter) causes difficulties with respect to the calculation of the function values with the plot visualization. For this reason the numerical methods are a useful tool. In this section we propose a numerical method that allows us to obtain a useful approximation for fractional integrals of analytic functions.

Taking into account the important property of the recurrence relations (10) and (11), the complicated problem is reduced to the interchangeable calculations of the left and right Riemann–Liouville fractional integrals and they are repeated in a loop. We make the following decompositions

$$\begin{aligned} &\varphi_{(0)}(t) := f(t), \\ &\mathcal{I}^{\alpha,n-k+1}_{a^+,b^-} \varphi_{(k-1)}(t) := \begin{cases} \mathcal{I}^{\alpha,n-k}_{a^+,b^-} \varphi_{(k)}(t), & \text{if } n > k-1, \\ \varphi_{(k-1)}(t), & \text{if } n = k-1, \end{cases} \quad \text{for } k = 1, ..., n, \end{aligned} \tag{45}$$

where

$$\begin{aligned} &\varphi_{(k)}(t) := \mathcal{I}^{\alpha,1}_{a^+,b^-}\varphi_{(k-1)}(t) = I^{\alpha}_{a^+}I^{\alpha}_{b^-}\varphi_{(k-1)}(t) = I^{\alpha}_{a^+}\phi_{(k)}(t)\,, \\ &\phi_{(k)}(t) := I^{\alpha}_{b^-}\varphi_{(k-1)}(t)\,, \end{aligned} \tag{46}$$

and for the second integral operator

$$\begin{aligned} &\varphi_{(0)}(t) := f(t)\,, \\ &\mathcal{I}^{\alpha,n-k+1}_{b^-,a^+}\varphi_{(k-1)}(t) := \begin{cases} \mathcal{I}^{\alpha,n-k}_{b^-,a^+}\varphi_{(k)}(t)\,, & \text{if } n > k-1, \\ \varphi_{(k-1)}(t)\,, & \text{if } n = k-1, \end{cases} \quad \text{for } k = 1,...,n, \end{aligned} \tag{47}$$

where

$$\begin{aligned} &\varphi_{(k)}(t) := \mathcal{I}^{\alpha,1}_{b^-,a^+}\varphi_{(k-1)}(t) = I^{\alpha}_{b^-}I^{\alpha}_{a^+}\varphi_{(k-1)}(t) = I^{\alpha}_{b^-}\phi_{(k)}(t)\,, \\ &\phi_{(k)}(t) := I^{\alpha}_{a^+}\varphi_{(k-1)}(t)\,. \end{aligned} \tag{48}$$

The functions $\varphi_{(k)}(t)$, for $k = 0,...,n$ and $\phi_{(k)}(t)$, for $k = 1,...,n$ are auxiliary functions. We can especially use only two functions $\varphi(t)$ and $\phi(t)$ if they are interchangeably used in successive iterations in the above recurrences. The substitution $\varphi_{(k=0)}(t) := f(t)$ allows us to use any simplification of notations and simultaneously it is the initial step in iterations.

In the numerical approach, the continuous time interval $t \in [a,b]$ is replaced by a grid constructed by discrete points. Here, we use the homogeneous grid of nodes: $t_i = a + i\,\Delta t$, $i = 0,...,N$, with the constant time step $\Delta t = (b-a)/N$, where $N+1$ is a number of nodes. Now, we present the scheme for the numerical evaluation of both integral operators of fractional order (3) and (4) obtained by the method which is based on the direct discretization of integral operators using the trapezoidal rule of integration [20]. We introduce the function $\psi(t)$ ($\equiv \varphi(t)$ or $\phi(t)$) and the values of the function $\psi(t)$ at the node t_i we denote by $\psi_i = \psi(t_i)$. We will determine function values at the nodes t_i. In our early papers [4, 5] we have determined the discrete forms for the approximation of the fractional integral operators in the following forms

$$I^{\alpha}_{a^+}\psi(t)\Big|_{t=t_i} \approx \sum_{j=0}^{i} u^{(\alpha)}_{i,j}\psi_j, \tag{49}$$

$$I^{\alpha}_{b^-}\psi(t)\Big|_{t=t_i} \approx \sum_{j=i}^{N} v^{(\alpha)}_{i,j}\psi_j, \tag{50}$$

where coefficients $u^{(\alpha)}_{i,j}$ and $v^{(\alpha)}_{i,j}$ are as follows

$$u^{(\alpha)}_{i,j} = \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)}\begin{cases} 0 & \text{if } i = 0 \wedge j = 0 \\ (i-1)^{\alpha+1} - i^{\alpha+1} + i^{\alpha}(\alpha+1) & \text{if } i > 0 \wedge j = 0 \\ (i-j+1)^{\alpha+1} - 2(i-j)^{\alpha+1} + (i-j-1)^{\alpha+1} & \text{if } i > 0 \wedge 0 < j < i \\ 1 & \text{if } i > 0 \wedge j = i \end{cases}, \tag{51}$$

$$v^{(\alpha)}_{i,j} = \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)}\begin{cases} 0 & \text{if } i = N \wedge j = N \\ (N-i-1)^{\alpha+1} - (N-i)^{\alpha+1} + (N-i)^{\alpha}(\alpha+1) & \text{if } i < N \wedge j = N \\ (j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1} & \text{if } i < N \wedge i < j < N \\ 1 & \text{if } i < N \wedge j = i \end{cases}. \tag{52}$$

One can notice that in order to compute the values of operators (49) and (50) at every node t_i, we need to use values of functions at many nodes of the domain.

From a computational point of view, in order to find the discrete values of the function $\varphi(t)$ at nodes t_i, $i = 0,...,N$ as results of the action of the operators $\mathcal{I}^{\alpha,n}_{a^+,b^-}f(t)$ and $\mathcal{I}^{\alpha,n}_{b^-,a^+}f(t)$ on the given analytic function

$f(t)$, we propose two procedures CALC_$\mathcal{I}_{RL}$ and CALC_$\mathcal{I}_{LR}$, respectively – see the listings of Algorithm 1.

Algorithm 1 Numerical evaluations of $\varphi(t) = \mathcal{I}^{\alpha,n}_{a^+,b^-} f(t)$ (procedure CALC_$\mathcal{I}_{RL}$) and $\varphi(t) = \mathcal{I}^{\alpha,n}_{b^-,a^+} f(t)$ (procedure CALC_$\mathcal{I}_{LR}$)

1: **procedure** CALC_$\mathcal{I}_{RL}(f(t),\alpha,n,a,b,\varphi,N)$
2:     **integer** n,N,i,k;
3:     **real** $\alpha,a,b,\Delta t$;
4:     **real array** $(\varphi)_{0:N},(\phi)_{0:N}$;
5:     $\Delta t \leftarrow (b-a)/N$
6:     **for** $i \leftarrow 0$ **to** N **do**
7:         $\varphi_i \leftarrow f(a + i\,\Delta t)$
8:     **end for**
9:     **for** $k \leftarrow 1$ **to** n **do**
10:         **for** $i \leftarrow 0$ **to** N **do**
11:             $\phi_i \leftarrow \sum_{j=i}^{N} v^{(\alpha)}_{i,j} \varphi_j$
12:         **end for**
13:         **for** $i \leftarrow 0$ **to** N **do**
14:             $\varphi_i \leftarrow \sum_{j=0}^{i} u^{(\alpha)}_{i,j} \phi_j$
15:         **end for**
16:     **end for**
17:     **return** φ
18: **end procedure**

1: **procedure** CALC_$\mathcal{I}_{LR}(f(t),\alpha,n,a,b,\varphi,N)$
2:     **integer** n,N,i,k;
3:     **real** $\alpha,a,b,\Delta t$;
4:     **real array** $(\varphi)_{0:N},(\phi)_{0:N}$;
5:     $\Delta t \leftarrow (b-a)/N$
6:     **for** $i \leftarrow 0$ **to** N **do**
7:         $\varphi_i \leftarrow f(a + i\,\Delta t)$
8:     **end for**
9:     **for** $k \leftarrow 1$ **to** n **do**
10:         **for** $i \leftarrow 0$ **to** N **do**
11:             $\phi_i \leftarrow \sum_{j=0}^{i} u^{(\alpha)}_{i,j} \varphi_j$
12:         **end for**
13:         **for** $i \leftarrow 0$ **to** N **do**
14:             $\varphi_i \leftarrow \sum_{j=i}^{N} v^{(\alpha)}_{i,j} \phi_j$
15:         **end for**
16:     **end for**
17:     **return** φ
18: **end procedure**

The procedures CALC_$\mathcal{I}_{RL}$ and CALC_$\mathcal{I}_{LR}$ are defined with seven arguments. These arguments have the same meaning as described before. As mentioned earlier, we use 2 auxiliary arrays φ and ψ of $N+1$ elements, which store the discrete values at the nodes t_i: $\varphi_i \equiv \varphi(t_i)$ and $\phi_i \equiv \phi(t_i)$. In lines 6–7, we set the initial values in array φ – it corresponds to the case for $k = 0$. In each procedure there is the main loop (lines 9–16) with two internal loops, in which the discrete values at each node are calculated for the particular left and right fractional order integrals. Finally, the array φ stores the discrete values being the approximation of the fractional operator and this array is returned back in procedures.

The running time of both proposed procedures in Algorithm 1 is estimated as $O(n \cdot N^2)$ and the memory usage is equal to $O(N)$.

6. Example of Computations

In Figures 1–3 we present plots of the numerical values of $\mathcal{I}^{\alpha,n}_{a^+,b^-} f(t)$ for $a = 0$, $b = 1$, $n \in \{1, 2, 5\}$, $\alpha \in \{0.25, 0.5, 0.75, 1, 1.5, 2\}$ and for three different functions: $f(t) = 1$ (Fig. 1), $f(t) = t^\alpha$ (Fig. 2), and $f(t) = \sin(\pi t)$ (Fig. 3). Additionally, in one of the plots in each figure, the maximal values of $\mathcal{I}^{\alpha,n}_{0^+,1^-} f(t)$ on the time interval $t \in [0, 1]$ for the range $n = 0, ..., 100$ and different parameters α are shown. The calculations are performed on the basis of the procedure CALC_$\mathcal{I}_{RL}$ for $N = 1000$ presented in Algorithm 1.

If we make the comparison between the obtained numerical results with the early given analytical solutions of fractional operators for the selected cases (especially when parameter α is an integer number), we can conclude that the numerical solutions seem to be quite acceptable.

In order to verify the accuracy of the presented numerical algorithms, we considered two cases of the evaluation of fractional integral $\mathcal{I}^{\alpha,1}_{a^+,b^-} f(t)$. We have chosen two functions $f(t) = 1$ and $f(t) = t^\alpha$. In both cases we assumed $\alpha = 0.8$ and $n = 1$. The analytical solutions are given in Equations (25) and (31), respectively.

The analytical values of the special functions occurring in both equations have been obtained by using Mathematica software. In Tables 1 and 2 we show the values of analytical solutions at the selected values of argument t_i. Additionally, we presented the corresponding numerical values calculated for the different numbers of grid nodes $N \in \{10, 100, 1000, 10000\}$ as the differences between analytical and numerical values. One can note that the differences decrease with decreasing the time step.

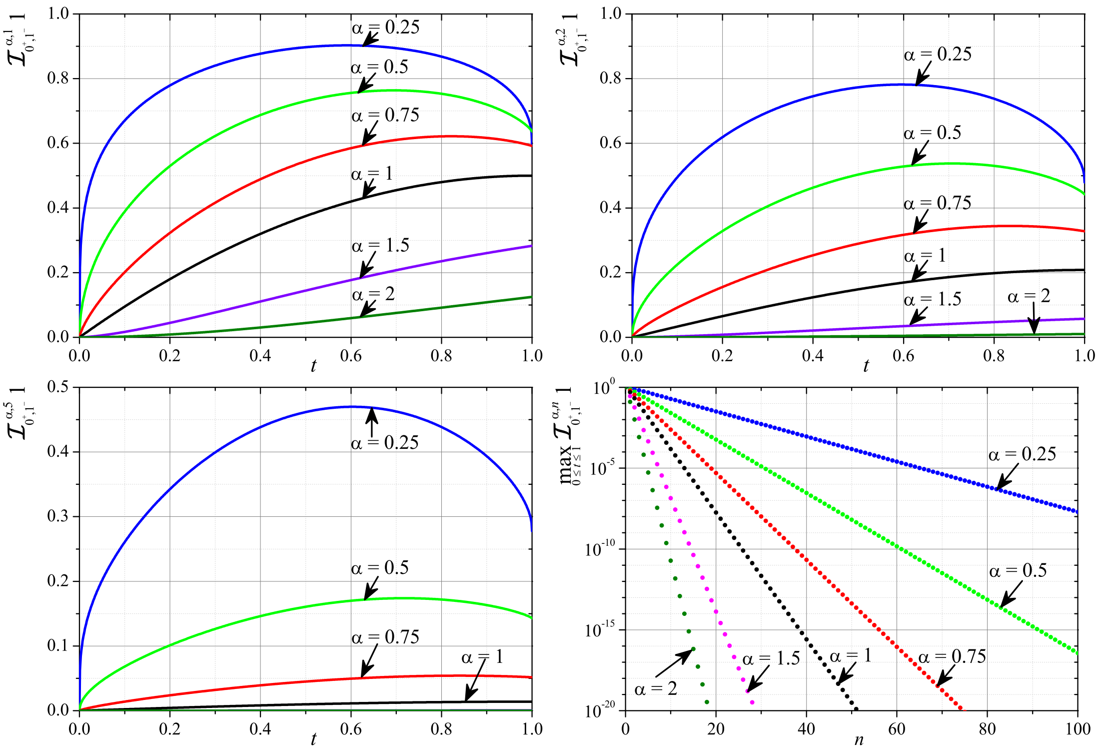

Figure 1: Numerical evaluation of $\mathcal{I}^{\alpha,n}_{0^+,1^-}1$ and maximal values of $\mathcal{I}^{\alpha,n}_{0^+,1^-}1$ for different values of α

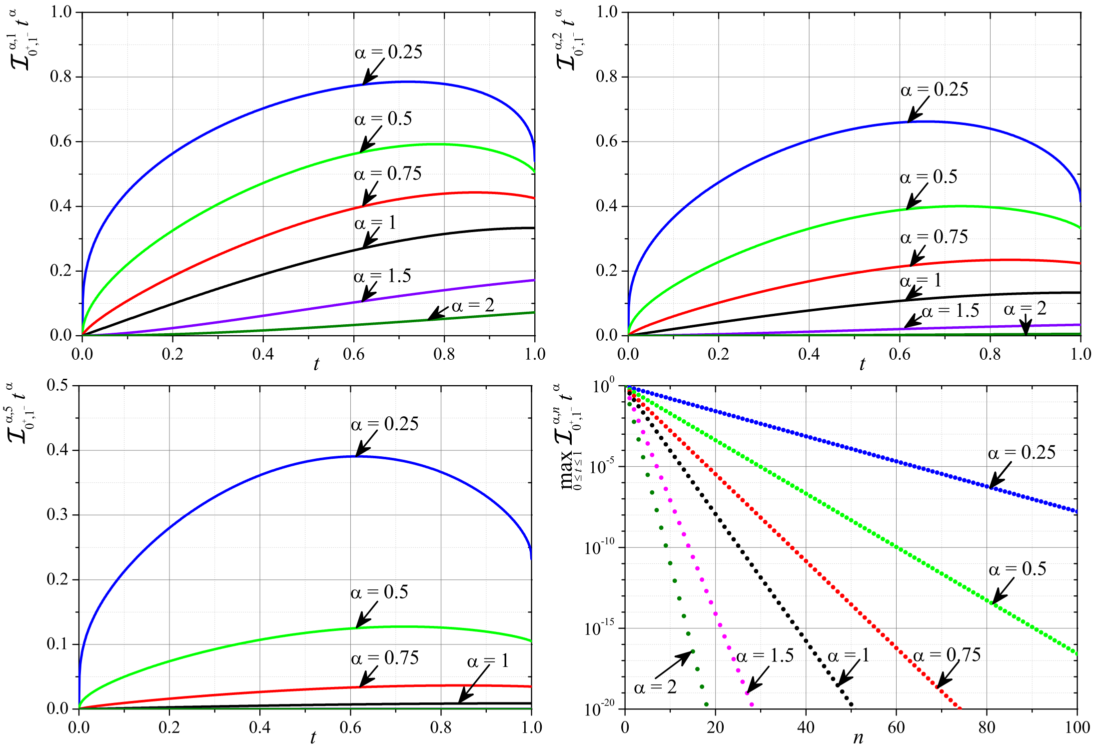

Figure 2: Numerical evaluation of $\mathcal{I}^{\alpha,n}_{0^+,1^-}t^{\alpha}$ and maximal values of $\mathcal{I}^{\alpha,n}_{0^+,1^-}t^{\alpha}$ for different values of α

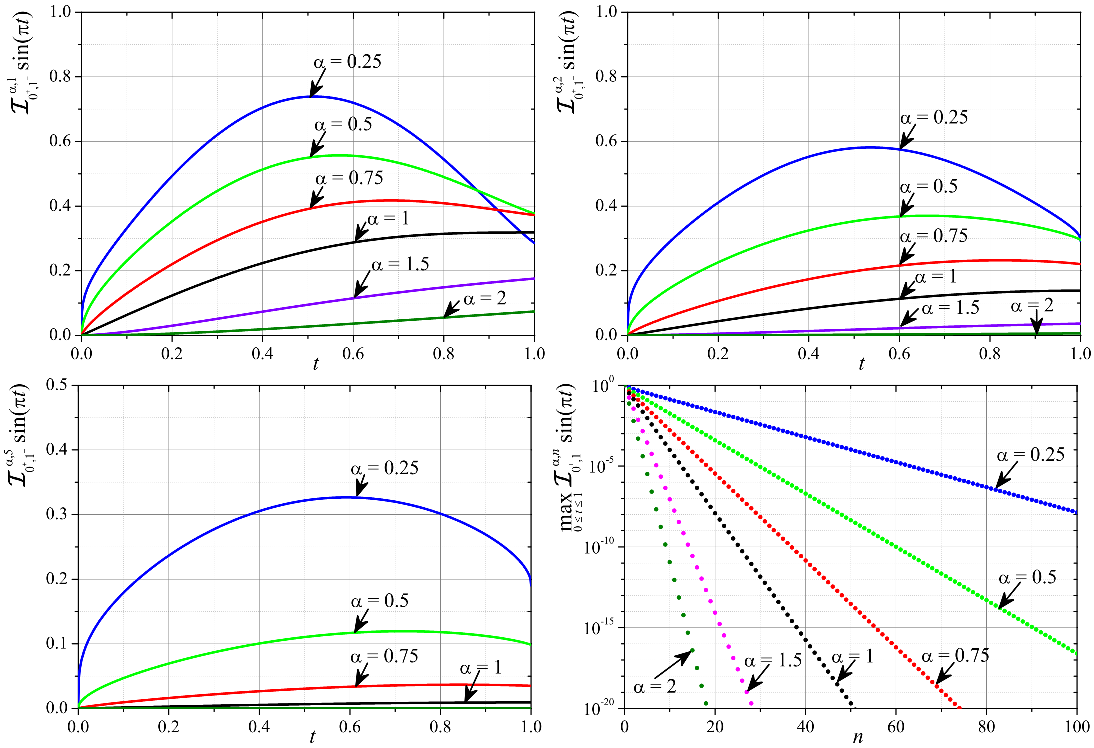

Figure 3: Numerical evaluation of $\mathcal{I}^{\alpha,n}_{0^+,1^-}\sin(\pi t)$ and maximal values of $\mathcal{I}^{\alpha,n}_{0^+,1^-}\sin(\pi t)$ for different values of α

Table 1: Comparison of analytical and numerical results for $F(t) = \mathcal{I}^{\alpha,1}_{1^+,5^-}1$ where $\alpha = 0.8$

t_i	$F_{analytical}(t_i)$	$F_{analytical}(t_i) - F_{numerical}(t_i)$			
		$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
1.0	0.0	0.0	0.0	0.0	0.0
1.4	1.603782641276	$2.272\cdot10^{-4}$	$2.380\cdot10^{-6}$	$2.397\cdot10^{-8}$	$2.400\cdot10^{-10}$
1.8	2.659414587381	$4.360\cdot10^{-4}$	$4.492\cdot10^{-6}$	$4.512\cdot10^{-8}$	$4.515\cdot10^{-10}$
2.2	3.491362960862	$6.622\cdot10^{-4}$	$6.783\cdot10^{-6}$	$6.806\cdot10^{-8}$	$6.810\cdot10^{-10}$
2.6	4.154917792624	$9.229\cdot10^{-4}$	$9.428\cdot10^{-6}$	$9.456\cdot10^{-8}$	$9.460\cdot10^{-10}$
3.0	4.673908578891	$1.239\cdot10^{-3}$	$1.264\cdot10^{-5}$	$1.268\cdot10^{-7}$	$1.268\cdot10^{-9}$
3.4	5.060354802376	$1.644\cdot10^{-3}$	$1.679\cdot10^{-5}$	$1.683\cdot10^{-7}$	$1.684\cdot10^{-9}$
3.8	5.319816544103	$2.206\cdot10^{-3}$	$2.257\cdot10^{-5}$	$2.264\cdot10^{-7}$	$2.265\cdot10^{-9}$
4.2	5.452808068017	$3.087\cdot10^{-3}$	$3.178\cdot10^{-5}$	$3.188\cdot10^{-7}$	$3.190\cdot10^{-9}$
4.6	5.453459496831	$4.866\cdot10^{-3}$	$5.111\cdot10^{-5}$	$5.137\cdot10^{-7}$	$5.141\cdot10^{-9}$
5.0	5.296740032259	$1.899\cdot10^{-2}$	$5.197\cdot10^{-4}$	$1.348\cdot10^{-5}$	$3.429\cdot10^{-7}$

Table 2: Comparison of analytical and numerical results for $F(t) = \mathcal{I}^{\alpha,1}_{0^+,5^-} t^\alpha$ where $\alpha = 0.8$

t_i	$F_{analytical}(t_i)$	$F_{analytical}(t_i) - F_{numerical}(t_i)$			
		$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
0.0	0.0	0.0	0.0	0.0	0.0
0.5	4.431319693990	$2.149{\cdot}10^{-2}$	$2.159{\cdot}10^{-4}$	$2.172{\cdot}10^{-6}$	$2.176{\cdot}10^{-8}$
1.0	7.765185719592	$2.898{\cdot}10^{-2}$	$2.934{\cdot}10^{-4}$	$2.947{\cdot}10^{-6}$	$2.950{\cdot}10^{-8}$
1.5	10.699870501652	$3.504{\cdot}10^{-2}$	$3.547{\cdot}10^{-4}$	$3.559{\cdot}10^{-6}$	$3.562{\cdot}10^{-8}$
2.0	13.288578942473	$4.049{\cdot}10^{-2}$	$4.097{\cdot}10^{-4}$	$4.110{\cdot}10^{-6}$	$4.113{\cdot}10^{-8}$
2.5	15.519176458783	$4.579{\cdot}10^{-2}$	$4.632{\cdot}10^{-4}$	$4.644{\cdot}10^{-6}$	$4.647{\cdot}10^{-8}$
3.0	17.355754017428	$5.130{\cdot}10^{-2}$	$5.189{\cdot}10^{-4}$	$5.202{\cdot}10^{-6}$	$5.205{\cdot}10^{-8}$
3.5	18.747377186550	$5.750{\cdot}10^{-2}$	$5.819{\cdot}10^{-4}$	$5.833{\cdot}10^{-6}$	$5.836{\cdot}10^{-8}$
4.0	19.626840258994	$6.539{\cdot}10^{-2}$	$6.631{\cdot}10^{-4}$	$6.648{\cdot}10^{-6}$	$6.651{\cdot}10^{-8}$
4.5	19.898439868907	$7.822{\cdot}10^{-2}$	$7.999{\cdot}10^{-4}$	$8.024{\cdot}10^{-6}$	$8.029{\cdot}10^{-8}$
5.0	19.365920767018	$1.564{\cdot}10^{-1}$	$3.285{\cdot}10^{-3}$	$7.578{\cdot}10^{-5}$	$1.836{\cdot}10^{-6}$

7. Conclusions

In this paper we studied fractional integral operators which are defined as the multiply composition of the left and right (or the right and left) Riemann-Liouville fractional integrals. Certain properties of these operators were shown, like linearity. symmetry or integrating by parts. Analytical results of fractional integrals of a constant and power functions are presented. Two numerical procedures are proposed for numerical evaluation of fractional order integrals. Examples of numerical evaluations of these operators of three different functions are also shown. The presented new type of fractional integral operators over a finite interval of integration can be extended to an infinite interval of integration, meaning that one can consider the following forms $\mathcal{I}^{\alpha,n}_{-\infty,\infty}$ and $\mathcal{I}^{\alpha,n}_{\infty,-\infty}$. Such a type seems to be an interesting issue for consideration i.e. of field potential (as in the case of the Riesz or Weyl fractional integrals). We think that our proposition of new integral operators and the obtained results will be an open topic for discussion and future research-works. We invite researchers to study this topic, especially to extend our analytical and numerical results presented in this paper, to search for the application of these operators in various fields of science.

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