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On Coupled Hadamard Type Sequential Fractional Differential Equations with Variable Coefficients and Nonlocal Integral Boundary Conditions

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Abstract. In this paper, we develop the existence criteria for the solutions of a system of Hadamard type sequential fractional differential equations with variable coefficients and nonlocal integral boundary conditions. The main results rely on the standard tools of fixed-point theory. An illustrative example is also discussed.

1. Introduction

We investigate the sufficient criteria ensuring the existence as well as existence and uniqueness of solutions for a coupled system of Hadamard type sequential fractional differential equations with variable coefficients and nonlocal integral boundary conditions given by

$$\begin{cases} (D^q + k_1(t)D^{q-1})u(t) = f(t, u(t), v(t), D^{\alpha}v(t)),\ 1 < q \leqslant 2, 0 < \alpha < 1 \\ (D^p + k_2(t)D^{p-1})v(t) = g(t, u(t), D^{\delta}u(t), v(t)), 1 < p \leqslant 2,\ 0 < \delta < 1, \\ u(1) = 0,\ u(e) = I^{\gamma}v(\eta) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma-1}\frac{v(s)}{s}ds,\ \gamma > 0,\ 1 < \eta < e, \\ v(1) = 0,\ v(e) = I^{\beta}u(\zeta) = \dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\beta-1}\frac{u(s)}{s}ds, \beta > 0,\ 1 < \zeta < e, \end{cases} \tag{1}$$

where $D^{(.)}$ and $I^{(.)}$ respectively denote the Hadamard fractional derivative and Hadamard fractional integral (to be defined later), $k_1(t), k_2(t)$ are increasing continuous functions on $[1, e]$ and $f, g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ are

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appropriately chosen functions.

Fractional differential equations appear in the mathematical modelling of many real world problems due to the nonlocal nature of fractional derivative, which can take care of the past history of the involved phenomenon. The literature on the topic ranging from its theoretical to applications aspects, is now much enriched, for instance, see the texts ([16]-[17],[19]) and a series of papers ([2],[6]-[8],[10],[13],[21],[22],[26],[28]-[30]). However, much of the work on fractional differential equations either involves Riemann-Liouville or Caputo type fractional derivative. We find in the classical text [20] that Hadamard in 1892 [12] suggested a concept of fractional integro-differentiation in terms of the fractional power of the type $(t\frac{d}{dt})^q$ in contrast to its Riemann-Liouville counterpart of the form $(\frac{d}{dt})^q$. This kind of derivative introduced by Hadamard contains logarithmic function of arbitrary exponent in the kernel of the integral appearing in its definition. One can notice that Hadamard's construction is invariant in relation to dilation and is well suited to the problems containing half axes. In [5], it was shown that the Lamb-Bateman integral equation could be expressed in terms of Hadamard fractional derivatives of order 1/2. In [9], a modified Lamb-Bateman equation, based on Hadamard derivatives and fractional Hyper-Bessel-type operators, was presented. For some recent work on Hadamard type fractional differential equations and inclusions, we refer the reader to the papers ([3],[16],[18],[25],[27]) and references cited therein. Coupled systems of fractional order differential equations have also been investigated by many authors as such systems appear naturally in many real world situations, for example, see ([1],[4],[14],[23],[24]) and the references cited therein.

We emphasize that the present work is motivated by a recent article [4], where the authors considered the problem (1) with constants coefficients k_1 and k_2 instead of $k_1(t)$ and $k_2(t)$. With the introduction of variable coefficients, the problem (1) turns out to be more interesting and practical. However, it needs careful analysis due to its complexity caused by the presence of variable coefficients.

2. Basic Concepts and an Important Lemma

Let us first recall some important definitions [16].

Definition 2.1. *The Hadamard fractional integral of order q for a function $g \in L^p[y,x]$, $0 \le y \le t \le x \le \infty$, is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_y^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0.$$

Definition 2.2. *Let $[y,x] \subset \mathbb{R}$, $\delta = t\frac{d}{dt}$ and $AC_\delta^n[y,x] = \{g : [y,x] \to \mathbb{R} : \delta^{n-1}[g(t)] \in AC[y,x]\}$. The Hadamard derivative of fractional order q for a function $g \in AC_\delta^n[y,x]$ is defined as*

$$D^q g(t) = \delta^n (I^{n-q})(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Recall that the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral in the space $L^p[y,x], 0 < y < x < \infty, 1 \le p \le \infty$, that is, $D^q I^q f(t) = f(t)$ (Theorem 4.8, [15]).

Lemma 2.3. *Let $h_1, h_2 \in C[1,e]$, then the solution of the linear system of fractional differential equations:*

$$\begin{aligned} (D^q + k_1(t)D^{q-1})u(t) &= h_1(t), \\ (D^p + k_2(t)D^{p-1})v(t) &= h_2(t), \end{aligned} \tag{2}$$

supplemented with the boundary conditions of (1) is given by

$$\begin{aligned}
u(t) &= \frac{1}{\Delta}\Big(e^{-\mu_1(t)}\int_1^t \frac{e^{\mu_1(s)}}{s}(\log s)^{q-2}ds\Big)\Big\{-A_2\Big[\frac{e^{-\mu_2(e)}}{\Gamma(p-1)}\int_1^e \frac{e^{\mu_2(s)}}{s}\Big(\int_1^s\Big(\log\frac{s}{\tau}\Big)^{p-2}\frac{h_2(\tau)}{\tau}d\tau\Big)ds\\
&- \frac{1}{\Gamma(q-1)\Gamma(\beta)}\int_1^\zeta\Big(\log\frac{\zeta}{s}\Big)^{\beta-1}\frac{e^{-\mu_1(s)}}{s}\Big(\int_1^s\frac{e^{\mu_1(\tau)}}{\tau}\Big(\int_1^\tau\Big(\log\frac{\tau}{m}\Big)^{q-2}\frac{h_1(m)}{m}dm\Big)d\tau\Big)ds\Big]\\
&- B_2\Big[\frac{1}{\Gamma(p-1)\Gamma(\gamma)}\int_1^\eta\Big(\log\frac{\eta}{s}\Big)^{\gamma-1}\frac{e^{-\mu_2(s)}}{s}\Big(\int_1^s\frac{e^{\mu_2(\tau)}}{\tau}\Big(\int_1^\tau\Big(\log\frac{\tau}{m}\Big)^{p-2}\frac{h_2(m)}{m}dm\Big)d\tau\Big)ds\\
&- \frac{e^{-\mu_1(e)}}{\Gamma(q-1)}\int_1^e\frac{e^{\mu_1(s)}}{s}\Big(\int_1^s\Big(\log\frac{s}{\tau}\Big)^{q-2}\frac{h_1(\tau)}{\tau}d\tau\Big)ds\Big]\Big\}\\
&+ \frac{e^{-\mu_1(t)}}{\Gamma(q-1)}\int_1^t\frac{e^{\mu_1(s)}}{s}\Big(\int_1^s\Big(\log\frac{s}{\tau}\Big)^{q-2}\frac{h_1(\tau)}{\tau}d\tau\Big)ds, \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
v(t) &= \frac{1}{\Delta}\Big(e^{-\mu_2(t)}\int_1^t \frac{e^{\mu_2(s)}}{s}(\log s)^{p-2}ds\Big)\Big\{-B_1\Big[\frac{e^{-\mu_2(e)}}{\Gamma(q-1)}\int_1^e \frac{e^{\mu_2(s)}}{s}\Big(\int_1^s\Big(\log\frac{s}{\tau}\Big)^{q-2}\frac{h_1(\tau)}{\tau}d\tau\Big)ds\\
&- \frac{1}{\Gamma(p-1)\Gamma(\gamma)}\int_1^\eta\Big(\log\frac{\eta}{s}\Big)^{\gamma-1}\frac{e^{-\mu_2(s)}}{s}\Big(\int_1^s\frac{e^{\mu_2(\tau)}}{\tau}\Big(\int_1^\tau\Big(\log\frac{\tau}{m}\Big)^{p-2}\frac{h_2(m)}{m}dm\Big)d\tau\Big)ds\Big]\\
&- A_1\Big[\frac{1}{\Gamma(q-1)\Gamma(\beta)}\int_1^\zeta\Big(\log\frac{\zeta}{s}\Big)^{\beta-1}\frac{e^{-\mu_1(s)}}{s}\Big(\int_1^s\frac{e^{\mu_1(\tau)}}{\tau}\Big(\int_1^\tau\Big(\log\frac{\tau}{m}\Big)^{q-2}\frac{h_1(m)}{m}dm\Big)d\tau\Big)ds\\
&- \frac{e^{-\mu_2(e)}}{\Gamma(p-1)}\int_1^e\frac{e^{\mu_2(s)}}{s}\Big(\int_1^s\Big(\log\frac{s}{\tau}\Big)^{p-2}\frac{h_2(\tau)}{\tau}d\tau\Big)ds\Big]\Big\}\\
&+ \frac{e^{-\mu_2(t)}}{\Gamma(p-1)}\int_1^t\frac{e^{\mu_2(s)}}{s}\Big(\int_1^s\Big(\log\frac{s}{\tau}\Big)^{p-2}\frac{h_2(\tau)}{\tau}d\tau\Big)ds, \qquad (4)
\end{aligned}$$

where

$$\Delta = B_1A_2 - A_1B_2 \neq 0, \qquad (5)$$

$$A_1 = e^{-\mu_1(e)}\int_1^e \frac{e^{\mu_1(s)}}{s}(\log s)^{q-2}ds, \quad A_2 = \frac{-1}{\Gamma(\gamma)}\int_1^\eta\Big(\log\frac{\eta}{s}\Big)^{\gamma-1}\frac{e^{-\mu_2(s)}}{s}\left(\int_1^s\frac{e^{\mu_2(\tau)}}{\tau}(\log\tau)^{p-2}d\tau\right)ds, \qquad (6)$$

$$B_1 = \frac{-1}{\Gamma(\beta)}\int_1^\zeta\Big(\log\frac{\zeta}{s}\Big)^{\beta-1}\frac{e^{-\mu_1(s)}}{s}\left(\int_1^s\frac{e^{\mu_1(\tau)}}{\tau}(\log\tau)^{q-2}d\tau\right)ds, \quad B_2 = e^{-\mu_2(e)}\int_1^e\frac{e^{\mu_2(s)}}{s}(\log s)^{p-2}ds, \qquad (7)$$

$\mu_i(t) = \int_1^t \frac{k_i(\tau)}{\tau}d\tau$ *with* $\mu_i(t) \neq 0,\ \forall t \in [1, e]$.

Proof. As argued in [16], the general solution of the system (2) can be written as

$$u(t) = a_0e^{-\mu_1(t)} + a_1e^{-\mu_1(t)}\int_1^t\frac{e^{\mu_1(s)}}{s}(\log s)^{q-2}ds + e^{-\mu_1(t)}\int_1^t\frac{e^{\mu_1(s)}}{s}I^{q-1}h_1(s)ds, \qquad (8)$$

$$v(t) = b_0e^{-\mu_2(t)} + b_1e^{-\mu_2(t)}\int_1^t\frac{e^{\mu_2(s)}}{s}(\log s)^{p-2}ds + e^{-\mu_2(t)}\int_1^t\frac{e^{\mu_2(s)}}{s}I^{p-1}h_2(s)ds, \qquad (9)$$

where $a_i,\ b_i, (i = 0, 1)$ are unknown arbitrary constants. Using the boundary conditions given by (1) in (8) and (9), we obtain

$$a_0 = 0,\ b_0 = 0,\ a_1 = \frac{(A_2\mathcal{J}_2 - B_2\mathcal{J}_1)}{\Delta},\ b_1 = \frac{(B_1\mathcal{J}_1 - A_1\mathcal{J}_2)}{\Delta}, \qquad (10)$$

where $\Delta, A_i, B_i\ (i = 1, 2)$ are respectively given by (5) (6), (7), and

$$\begin{aligned}\mathcal{J}_1 &= \frac{1}{\Gamma(\gamma)\Gamma(p-1)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma-1}\frac{e^{-\mu_2(s)}}{s}\left(\int_1^s \frac{e^{\mu_2(\tau)}}{\tau}\left(\int_1^\tau \left(\log\frac{\tau}{m}\right)^{p-2}\frac{h_2(m)}{m}dm\right)d\tau\right)ds\\ &- \frac{e^{-\mu_1(e)}}{\Gamma(q-1)}\int_1^e \frac{e^{\mu_1(s)}}{s}\left(\int_1^s \left(\log\frac{s}{\tau}\right)^{q-2}\frac{h_1(\tau)}{\tau}d\tau\right)ds, \qquad (11)\\ \mathcal{J}_2 &= \frac{1}{\Gamma(\beta)\Gamma(q-1)}\int_1^\zeta \left(\log\frac{\zeta}{s}\right)^{\beta-1}\frac{e^{-\mu_1(s)}}{s}\left(\int_1^s \frac{e^{\mu_1(\tau)}}{\tau}\left(\int_1^\tau \left(\log\frac{\tau}{m}\right)^{q-2}\frac{h_1(m)}{m}dm\right)d\tau\right)ds\\ &- \frac{e^{-\mu_2(e)}}{\Gamma(p-1)}\int_1^e \frac{e^{\mu_2(s)}}{s}\left(\int_1^s \left(\log\frac{s}{\tau}\right)^{p-2}\frac{h_2(\tau)}{\tau}d\tau\right)ds. \qquad (12)\end{aligned}$$

Substituting the values (10) in (8) and (9), we obtain the solution (3)-(4). The converse follows by direct computation. This completes the proof.

3. Existence and Uniqueness Results

Let $X = \left\{x : x \in C([1, e], \mathbb{R}) \text{ and } D^\vartheta x \in C([1, e], \mathbb{R})\right\}$ denote the space equipped with the norm $\|x\|_X = \|x\| + \|D^\vartheta x\| = \sup_{t\in[1,e]} |x(t)| + \sup_{t\in[1,e]} |D^\vartheta x(t)|$, where $\vartheta = \delta, \alpha$. Clearly the product space $(X \times X, \|.\|_{X\times X})$ is a Banach space endowed with the norm $\|(x_1, x_2)\|_{X\times X} = \|x_1\|_X + \|x_2\|_X$ for $(x_1, x_2) \in X \times X$.

Using Lemma 2.3, we introduce an operator $T : X \times X \to X \times X$ as follows:

$$T(u, v)(t) := (T_1(u, v)(t), T_2(u, v)(t)), \qquad (13)$$

where

$$\begin{aligned}T_1(u,v)(t) &= \frac{1}{\Delta}\Big(e^{-\mu_1(t)}\int_1^t \frac{e^{\mu_1(s)}}{s}(\log s)^{q-2}ds\Big)\Big\{-A_2\, e^{-\mu_2(e)}\int_1^e \frac{e^{\mu_2(s)}}{s}\Big(I^{p-1}g(s,u(s),D^\delta u(s),v(s))\Big)ds\\ &+ \frac{A_2}{\Gamma(\beta)}\int_1^\zeta \Big(\log\frac{\zeta}{s}\Big)^{\beta-1}\frac{e^{-\mu_1(s)}}{s}\Big(\int_1^s \frac{e^{\mu_1(\tau)}}{\tau}\Big(I^{q-1}f(\tau,u(\tau),v(\tau),D^\alpha v(\tau))\Big)d\tau\Big)ds\\ &- \frac{B_2}{\Gamma(\gamma)}\int_1^\eta \Big(\log\frac{\eta}{s}\Big)^{\gamma-1}\frac{e^{-\mu_2(s)}}{s}\Big(\int_1^s \frac{e^{\mu_2(\tau)}}{\tau}\Big(I^{p-1}g(\tau,u(\tau),D^\delta u(\tau),v(\tau))\Big)d\tau\Big)ds\\ &+ B_2\, e^{-\mu_1(e)}\int_1^e \frac{e^{\mu_1(s)}}{s}\Big(I^{q-1}f(s,u(s),v(s),D^\alpha v(s))\Big)ds\Big\}\\ &+ e^{-\mu_1(t)}\int_1^t \frac{e^{\mu_1(s)}}{s}\Big(I^{q-1}f(s,u(s),v(s),D^\alpha v(s))\Big)ds, \qquad (14)\end{aligned}$$

$$\begin{aligned}T_2(u,v)(t) &= \frac{1}{\Delta}\Big(e^{-\mu_2(t)}\int_1^t \frac{e^{\mu_2(s)}}{s}(\log s)^{p-2}ds\Big)\Big\{-B_1\, e^{-\mu_2(e)}\int_1^e \frac{e^{\mu_2(s)}}{s}\Big(I^{q-1}f(s,u(s),v(s),D^\alpha v(s))\Big)ds\\ &+ \frac{B_1}{\Gamma(\gamma)}\int_1^\eta \Big(\log\frac{\eta}{s}\Big)^{\gamma-1}\frac{e^{-\mu_2(s)}}{s}\Big(\int_1^s \frac{e^{\mu_2(\tau)}}{\tau}\Big(I^{p-1}g(\tau,u(\tau),D^\delta u(\tau),v(\tau))\Big)d\tau\Big)ds\\ &- \frac{A_1}{\Gamma(\beta)}\int_1^\zeta \Big(\log\frac{\zeta}{s}\Big)^{\beta-1}\frac{e^{-\mu_1(s)}}{s}\Big(\int_1^s \frac{e^{\mu_1(\tau)}}{\tau}\Big(I^{q-1}f(\tau,u(\tau),v(\tau),D^\alpha(\tau))\Big)d\tau\Big)ds\\ &+ A_1\, e^{-\mu_2(e)}\int_1^e \frac{e^{\mu_2(s)}}{s}\Big(I^{p-1}g(s,u(s),D^\delta(s),v(s))\Big)ds\Big\}\\ &+ e^{-\mu_2(t)}\int_1^t \frac{e^{\mu_2(s)}}{s}I^{p-1}g(s,u(s),D^\delta(s),v(s))ds. \qquad (15)\end{aligned}$$

Next we enlist the assumptions that we need in the sequel.

(H_1) Let $f, g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous functions and there exist real constants $\mu_j, \lambda_j \geq 0$ $(j = 1, 2, 3)$ and $\mu_0 > 0$, $\lambda_0 > 0$ such that $|f(t, x_1, x_2, x_3)| \leq \mu_0 + \mu_1|x_1| + \mu_2|x_2| + \mu_3|x_3|$, and $|g(t, x_1, x_2, x_3)| \leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2| + \lambda_3|x_3|, \forall x_j \in \mathbb{R},\ j = 1, 2, 3.$

(H_2) There exist positive constants $l,\ l_1$ such that

$$\begin{aligned} |f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| &\leq l\,[|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|]\,, \\ |g(t, u_1, u_2, u_3) - g(t, v_1, v_2, v_3)| &\leq l_1\,[|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|]\,, \forall t \in [1, e],\ u_j,\ v_j \in \mathbb{R}. \end{aligned}$$

For computational convenience, we set

$$\rho_1 = \frac{1}{|\Delta|(q-1)}\Big[\frac{|A_2|\big(\log\zeta\big)^{q+\beta}}{\Gamma(q+1)\Gamma(\beta+1)} + \frac{|B_2|}{\Gamma(q+1)} + \frac{|\Delta|}{q\Gamma(q-1)}\Big], \tag{16}$$

$$\rho_2 = \frac{1}{|\Delta|(q-1)}\Big[\frac{|A_2|}{\Gamma(p+1)} + \frac{|B_2|\big(\log\eta\big)^{p+\gamma}}{\Gamma(\gamma+1)\Gamma(p+1)}\Big], \tag{17}$$

$$\rho_3 = \frac{1}{|\Delta|(p-1)}\Big[\frac{|B_1|}{\Gamma(q+1)} + \frac{|A_1|\big(\log\zeta\big)^{q+\beta}}{\Gamma(q+1)\Gamma(\beta+1)}\Big], \tag{18}$$

$$\rho_4 = \frac{1}{|\Delta|(p-1)}\Big[\frac{|B_1|\big(\log\eta\big)^{p+\gamma}}{\Gamma(\gamma+1)\Gamma(p+1)} + \frac{|A_1|}{\Gamma(p+1)} + \frac{|\Delta|}{p\Gamma(p-1)}\Big], \tag{19}$$

$$\varpi_1 = \Big(1 + \frac{1}{\Gamma(1-\delta)}\Big)\Big(\mu_0\rho_1 + \lambda_0\rho_2\Big) + \Big(1 + \frac{1}{\Gamma(1-\alpha)}\Big)\Big(\mu_0\rho_3 + \lambda_0\rho_4\Big), \tag{20}$$

$$\varpi_2 = \Big(\mu_1\rho_1 + \max\{\lambda_1, \lambda_2\}\rho_2\Big)\Big(1 + \frac{1}{\Gamma(1-\delta)}\Big) + \Big(\mu_1\rho_3 + \max\{\lambda_1, \lambda_2\}\rho_4\Big)\Big(1 + \frac{1}{\Gamma(1-\alpha)}\Big), \tag{21}$$

$$\varpi_3 = \Big(\max\{\mu_2, \mu_3\}\rho_1 + \lambda_3\rho_2\Big)\Big(1 + \frac{1}{\Gamma(1-\delta)}\Big) + \Big(\max\{\mu_2, \mu_3\}\rho_3 + \lambda_3\rho_4\Big)\Big(1 + \frac{1}{\Gamma(1-\alpha)}\Big). \tag{22}$$

Now the platform is set to present our main work. The first result dealing with the existence of solutions for the given problem is based on Leray-Schauder alternative.

Lemma 3.1. *(Leray-Schauder alternative [11]) Let $F : E \to E$ be a completely continuous operator. Let $\varepsilon(F) = \{x \in E : x = \lambda F(x)$ for some $0 < \lambda < 1\}$. Then either the set $\varepsilon(F)$ is unbounded or F has at least one fixed point.*

Theorem 3.2. *Assume that (H_1) holds. In addition it is assumed that max$\{\varpi_2,\ \varpi_3\} < 1$, where ϖ_2 and ϖ_3 are given by (21) and (22) respectively. Then the boundary value problem (1) has at least one solution $[1, e]$.*

Proof. In the first step, we show that the operator $T :\ X \times X \to X \times X$ is completely continuous. By continuity of the functions f and g, it follows that the operators T_1 and T_2 are continuous. In consequence, the operator T is continuous. Next we show that the operator T is uniformly bounded. For that, let $\Omega \subset X \times X$ be a bounded set. Then there exist positive constants L_1 and L_2 such that $|f(t, u(t), v(t), D^\alpha v(t))| \leq L_1,\ |g(t, u(t), D^\delta u(t), v(t))| \leq L_2,\ \forall (u, v) \in \Omega$. Then, for any $(u, v) \in \Omega$ together with the notations (16) and (17), we obtain $\|T_1(u, v)\| \leq L_1\rho_1 + L_2\rho_2, \|D^\delta T_1(u, v)\| \leq \frac{1}{\Gamma(1-\delta)}\left(L_1\rho_1 + L_2\rho_2\right)$, which lead to

$$\|T_1(u, v)\|_X = \|T_1(u, v)\| + \|D^\delta T_1(u, v)\| \leq \Big(1 + \frac{1}{\Gamma(1-\delta)}\Big)\left(L_1\rho_1 + L_2\rho_2\right). \tag{23}$$

In a similar manner, we find that

$$\|T_2(u, v)\|_X = \|T_2(u, v)\| + \|D^\alpha T_2(u, v)\| \leq \Big(1 + \frac{1}{\Gamma(2-\alpha)}\Big)\left(L_1\rho_3 + L_2\rho_4\right), \tag{24}$$

where ρ_3 and ρ_4 are respectively given by (18), (19). From the inequalities (23) and (24), we deduce that T_1 and T_2 are uniformly bounded, which implies that the operator T is uniformly bounded.
Next, we show that T is equicontinuous. For $1 < t_1 < t_2 < e$, we have

$$|T_1(u,v)(t_2) - T_1(u,v)(t_1)| \to 0, \qquad |D^\delta T_1(u,v)(t_2) - D^\delta T_1(u,v)(t_1)| \to 0,$$
$$|T_2(u,v)(t_2) - T_2(u,v)(t_1)| \to 0, \qquad |D^\alpha T_2(u,v)(t_2) - D^\alpha T_2(u,v)(t_1)| \to 0,$$

independent of (u,v) as $t_2 \to t_1$. Thus the operators T_1 and T_2 are equicontinuous, which implies that the operator T is equicontinuous. Therefore, by Arzela-Ascoli's theorem, we deduce that the operator T is compact (completely continuous).
Finally, it will be established that the set $\varepsilon(T) = \{(u,v) \in X \times X : (u,v) = \lambda T(u,v)\,;\ 0 \le \lambda \le 1\}$ is bounded. Let $(u,v) \in \varepsilon(T)$. Then $(u,v) = \lambda T(u,v)$. For any $t \in [1,e]$, we have $u(t) = \lambda T_1(u,v)(t),\ v(t) = \lambda T_2(u,v)(t)$. Thus

$$|u(t)| = \lambda |T_1(u,v)(t)| \le |T_1(u,v)(t)|, \quad (|\lambda| \le 1).$$

Using (H_1) in (14) and the notations (16) and (17), we find that

$$\|u\| \le (\mu_0 + \mu_1\|u\|_X + \max\{\mu_2,\mu_3\}\|v\|_X)\rho_1 + (\lambda_0 + \max\{\lambda_1,\lambda_2\}\|u\|_X + \lambda_3\|v\|_X)\rho_2,$$
$$\|D^\delta u\| \le \frac{1}{\Gamma(1-\delta)}\Big\{(\mu_0 + \mu_1\|u\|_X + \max\{\mu_2,\mu_3\}\|v\|_X)\rho_1 + (\lambda_0 + \max\{\lambda_1,\lambda_2\}\|u\|_X + \lambda_3\|v\|_X)\rho_2\Big\}.$$

Consequently, we have

$$\begin{aligned}\|u\|_X &= \|u\| + \|D^\delta u\| \le \Big(1 + \frac{1}{\Gamma(1-\delta)}\Big)\Big\{(\mu_0 + \mu_1\|u\|_X + \max\{\mu_2,\mu_3\}\|v\|_X)\rho_1 \\ &\quad + (\lambda_0 + \max\{\lambda_1,\lambda_2\}\|u\|_X + \lambda_3\|v\|_X)\rho_2\Big\}.\end{aligned} \tag{25}$$

Likewise, we can obtain

$$\begin{aligned}\|v\|_X &\le \Big(1 + \frac{1}{\Gamma(1-\alpha)}\Big)\Big\{(\mu_0 + \mu_1\|u\|_X + \max\{\mu_2,\mu_3\}\|v\|_X)\rho_3 \\ &\quad + (\lambda_0 + \max\{\lambda_1,\lambda_2\}\|u\|_X + \lambda_3\|v\|_X)\rho_4\Big\}.\end{aligned} \tag{26}$$

Using the notations (20)-(22), it follows from (25) and (26) that

$$\|u\|_X + \|v\|_X \le \varpi_1 + \max\{\varpi_2,\varpi_3\}\|(u,v)\|_{X\times X}, \tag{27}$$

which, together with $\|(u,v)\|_{X\times X} = \|u\|_X + \|v\|_X$, yields

$$\|(u,v)\|_{X\times X} \le \frac{\varpi_1}{1 - \max\{\varpi_2,\varpi_3\}}.$$

This shows that $\varepsilon(T)$ is bounded. Thus, Leary-Schauder Lemma applies and that the operator T has at least one fixed point. Consequently, the boundary value problem (1) has at least one solution on $[1,e]$. This completes the proof. □

Our next result deals with the uniqueness of solutions for the problem (1), which relies on Banach's contraction mapping principle. For computational convenience, we introduce the notations:

$$\begin{aligned}&\Lambda = \ell\rho_1 + \ell_1\rho_2,\ \bar{M} = r_1\rho_1 + r_2\rho_2,\ \Lambda' = \ell\rho_3 + \ell_1\rho_4,\ \bar{N} = r_1\rho_3 + r_2\rho_4, \\ &r_1 = \sup_{t\in[1,e]} f(t,0,0,0) < \infty,\ r_2 = \sup_{t\in[1,e]} g(t,0,0,0) < \infty.\end{aligned} \tag{28}$$

Theorem 3.3. *Assume that* (H_2) *holds. Then the boundary value problem (1) has a unique solution on* $[1,e]$, *provided that*

$$\Lambda + \frac{\Lambda}{\Gamma(1-\delta)} < \frac{1}{2} \quad \text{and} \quad \Lambda' + \frac{\Lambda'}{\Gamma(1-\alpha)} < \frac{1}{2}, \tag{29}$$

where Λ *and and* Λ' *are given by (28).*

Proof. Let us fix

$$r \geq \max\left\{ \frac{\bar{M} + \frac{\bar{M}}{\Gamma(1-\delta)}}{\frac{1}{2} - (\Lambda + \frac{\Lambda}{\Gamma(1-\delta)})}, \frac{\bar{N} + \frac{\bar{N}}{\Gamma(1-\alpha)}}{\frac{1}{2} - (\Lambda' + \frac{\Lambda'}{\Gamma(1-\alpha)})} \right\},$$

where $\Lambda,\ \Lambda'$, and $\bar{M},\ \bar{N}$ are given by (28). Then we show that $TB_r \subset B_r$, where

$$B_r = \{(u,v) \in X \times X : \|(u,v)\|_{X\times X} \leq r\}.$$

For $(u,v) \in B_r$, we have

$$\begin{aligned} |f(t,u(t),v(t),D^\alpha v(t))| &\leq |f(t,u(t),v(t),D^\alpha v(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\ &\leq \ell[|u(t)| + |v(t)| + |D^\alpha v(t)|] + r_1 \\ &\leqslant \ell[\|u\|_X + \|v\|_X] + r_1 \leqslant \ell\|(u,v)\|_{X\times X} + r_1 \leqslant \ell r + r_1. \end{aligned}$$

Similarly, we have $|g(t,u(t),D^\delta u(t),v(t))| \leq \ell_1 r + r_2$. In consequence, we get

$$|T_1(u,v)(t)| \leqslant (\ell r + r_1)\rho_1 + (\ell_1 r + r_2)\rho_2 = (\ell\rho_1 + \ell_1\rho_2)r + r_1\rho_1 + r_2\rho_2 \leqslant \Lambda r + \bar{M},$$

and

$$|D^\delta T_1(u,v)(t)| \leqslant \frac{1}{\Gamma(1-\delta)}\Big(t\frac{d}{dt}\Big)\int_1^t \Big(\log\frac{t}{s}\Big)^{-\delta}\frac{|T_1(u,v)(s)|}{s}ds \leqslant \frac{\Lambda r + \bar{M}}{\Gamma(1-\delta)}.$$

Therefore,

$$\|T_1(u,v)\|_X = \|T_1(u,v)\| + \|D^\delta T_1(u,v)\| \leqslant \Big(1 + \frac{1}{\Gamma(1-\delta)}\Big)\Big(\Lambda r + \bar{M}\Big) \leq \frac{r}{2}. \tag{30}$$

In similar manner, we can obtain $|T_2(u,v)(t)| \leq \Lambda' r + \bar{N}$ and $|D^\alpha T_2(u,v)(t)| \leqslant \frac{\Lambda' r + \bar{N}}{\Gamma(1-\alpha)}$. Consequently, we get

$$\|T_2(u,v)\|_X = \|T_2(u,v)\| + \|D^\alpha T_2(u,v)\| \leqslant \Big(1 + \frac{1}{\Gamma(1-\alpha)}\Big)\Big(\Lambda' r + \bar{N}\Big) \leq \frac{r}{2}. \tag{31}$$

Thus, from (30) and (31), we deduce that $\|T(u,v)\|_{X\times X} = \|T_1(u,v)\|_X + \|T_2(u,v)\|_X \leqslant r$, which implies $TB_r \subset B_r$.

Now we prove that the operator T is a contraction. For $u_i, v_i \in B_r,\ i = 1,2$ and for each $t \in [1,e]$, it follows by the condition (H_2) that

$$|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq \Lambda\,[\|u_1 - u_2\|_X + \|v_1 - v_2\|_X]$$

and

$$|D^\delta T_1(u_1,v_1)(t) - D^\delta T_1(u_2,v_2)(t)| \leq \frac{\Lambda}{\Gamma(1-\delta)}[\|u_1 - u_2\|_X + \|v_1 - v_2\|_X],$$

which imply that

$$\|T_1(u_1,v_1) - T_1(u_2,v_2)\|_X \leq \left(\Lambda + \frac{\Lambda}{\Gamma(1-\delta)}\right)(\|u_1 - u_2\|_X + \|v_1 - v_2\|_X). \tag{32}$$

Similarly, we can find that

$$\|T_2(u_1,v_1) - T_2(u_2,v_2)\|_X \leq \left(\Lambda' + \frac{\Lambda'}{\Gamma(1-\alpha)}\right][\|u_1 - u_2\|_X + \|v_1 - v_2\|_X). \tag{33}$$

From (32) and (33), we deduce that

$$\begin{aligned} \|T(u_1,v_1) - T(u_2,v_2)\|_{X\times X} &= \|T_1(u_1,v_1) - T_1(u_2,v_2)\|_X + \|T_2(u_1,v_1) - T_2(u_2,v_2)\|_X \\ &\leq \left(\Lambda + \Lambda' + \frac{\Lambda}{\Gamma(1-\delta)} + \frac{\Lambda'}{\Gamma(1-\alpha)}\right)[\|u_1 - u_2\|_X + \|v_1 - v_2\|_X], \end{aligned}$$

which, in view of the condition (29), shows that T is a contraction. Hence, by Banach's fixed point theorem, the operator T has a unique fixed point which corresponds to a unique solution of problem (1) on $[1,e]$. This completes the proof.

3.1. Example

Consider the following coupled system of fractional differential equations

$$\begin{aligned}(D^{\frac{7}{4}}+5tD^{\frac{3}{4}})x(t) &= e^{-4t}+\frac{1}{8\sqrt{3+t^2}}\left(\frac{|x(t)|}{1+|x(t)|}+\sin(y(t))+\tan^{-1}(D^{\frac{3}{4}}y(t))\right), \quad t\in[1,e]\\ (D^{\frac{5}{3}}+2t^2D^{\frac{2}{3}})y(t) &= \frac{5}{25+t^2}+\frac{e^{1-t}}{\sqrt{80+t^2}}\left(\sin(x(t))+\frac{|D^{\frac{1}{4}}x(t)|}{1+|D^{\frac{1}{4}}x(t)|}+y(t)\right),\end{aligned} \tag{34}$$

supplemented with nonlocal coupled integral boundary conditions:

$$\begin{aligned}u(1) &= 0,\ u(e)=I^{3/2}v(9/4)\\ v(1) &= 0,\ v(e)=I^{1/2}u(7/4).\end{aligned} \tag{35}$$

Here, $k_1(t)=5t$, $k_2(t)=2t^2$, $q=7/4$, $p=5/3$, $\alpha=3/4$, $\delta=1/4$, $\zeta=7/4$, $\eta=9/4$, $\gamma=3/2, \beta=1/2$, $f(t,x(t),y(t),D^\alpha y(t))=e^{-4t}+\frac{1}{8\sqrt{3+t^2}}\left(\frac{|x(t)|}{1+|x(t)|}+\sin(y(t))+\tan^{-1}(D^{\frac{3}{4}}y(t))\right)$ and $g(t,x(t),D^\delta x(t),y(t))=\frac{5}{25+t^2}+\frac{e^{1-t}}{\sqrt{80+t^2}}\sqrt{81+t^2}\left(\sin(x(t))+\frac{|D^{\frac{1}{4}}x(t)|}{1+|D^{\frac{1}{4}}x(t)|}+y(t)\right)$. From the inequalities:

$$\begin{aligned}&|f(t,x_1(t),y_1(t),D^\alpha y_1(t))-f(t,x_2(t),y_2(t),D^\alpha y_2(t))|\\ &\qquad\leq\frac{1}{16}\left(|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|+|D^{\frac{3}{4}}y_1(t)-D^{\frac{3}{4}}y_2(t)|\right),\\ &|g(t,x_1(t),D^\delta x_1(t),y_1(t))-g(t,x_2(t),D^\delta x_2(t),y_2(t))\\ &\qquad\leq\frac{1}{9}\left(|x_1(t)-x_2(t)|+|D^{\frac{1}{4}}x_1(t)-D^{\frac{1}{4}}x_2(t)|+|y_1(t)-y_2(t)|\right),\end{aligned}$$

we get $l=\frac{1}{16}$ and $l_1=\frac{1}{9}$. Using the given data, we find that $A_1\leq 4/3, |A_2|\leq 0.71656$, $|B_1|\leq 0.72821$, $B_2\leq 3/2$, $|\Delta|\leq 1.4782$, $\rho_1\leq 1.6825$, $\rho_2\leq 0.77794$, $\Lambda\leq 0.19159$, $\rho_3\leq 0.71655, \rho_4\leq 1.7542$, $\Lambda'\leq 0.23970$. Further $\Lambda(1+1/\Gamma(3/4))\leq 0.34794<0.5$, $\Lambda'(1+1/\Gamma(1/4))\leq 0.30581<0.5$. Thus all the conditions of Theorem 3.3 are satisfied. In consequence, by the conclusion of Theorem 3.3, there exists a unique solution for the problem (34)-(35) on $[1,e]$.

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