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Stochastic Volterra Integro-differential Equations Driven by a Fractional Brownian Motion with Delayed Impulses

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Abstract. In this paper, the problem of existence of mild solutions for a stochastic Volterra integro-differential equation with delayed impulses and driven by a fractional Brownian motion (Hurst parameter $H \in (\frac{1}{2}, 1)$) is investigated. Here, we assume that the delayed impulses are linear and impulsive transients depend on not only their current but also historical states of the system. Utilizing the fixed point theorem combine with fractional power of operators and the semi-group theory, sufficient conditions that guarantee the existence and uniqueness of mild solutions for such equation are obtained. Finally, an example is presented to demonstrate the effectiveness of the proposed results.

1. Introduction

The existence, uniqueness, and asymptotic properties of solutions of non-stochastic Volterra integro-differential equations have been significantly studied by many investigators [1-6], to mention a few. However, in many applications, due to the complex random nature of the situation, the phenomenon should be studied more realistically considered in a stochastic framework, resulting in a stochastic integro-differential equation[7-9], and the references therein. In 2015, the author(in [10]) studied the fractional order Volterra integro-differential equation in terms of variational iteration method, the fractional derivative was described in the Caputo sense. In 2016, the author (in [11]) studied the density of solutions to stochastic Volterra integro-differential equations with multi-fractional noises.

In reality, the external disturbances laws of many things changes and developments have dependence in different extent at different point time. In this case, using standard Brownian motions to describe random disturbance has gradually displayed limitation, instead it should be fractional Brownian motions(fBms in short for the remainder of this paper). FBm plays a central role in modelling and analysis of many complex phenomena in applications when the systems are subject to 'rough' external forcing. A FBm with Hurst

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parameter $H \in (0,1)$ is a centralized Gaussian process, when $H = \frac{1}{2}$, it reduces to the standard Brownian motion. That is to say, fBms are generalization of Brownian motions. However, fBms behave different significantly form standard Brownian motions. Specially, Brownian motions are Markov processes, Ito processes and Martingale processes but fBms neither semi-martingales nor Markov processes. Thus, the Ito theorem and $\mathcal{L}$ operator can't be used to deal with stochastic Volterra integro-differential equations driven by fBms, and should be finding new methods to study this kind of equations. In 2015, in [12], the author studied the stochastic Volterra integro-differential equations driven by fBms in a Hilbert space by fixed point theorems, the existence uniqueness conditions of mild solutions were provided respectively under the case of Lipschitz impulses and bounded impulses.

When modeling dynamical systems which are subjected to abrupt state changes at certain moments of time, impulsive system is the most potential candidate. Impulsive systems have found important applications in various fields, such as control systems with communication constraints, etc. In recent years, Ito stochastic differential equations with impulses had studied, such as [13,14], and then, impulsive stochastic differential equations driven by fBms had studied in [12,15,16]. However, in literatures [12-16], the impulse only depended on the current states of the system. What we are interested in that impulsive transients depend on not only their current but also historical states of the system[17-18]. These impulses we called delayed impulses, and differential equations with delayed impulses arising in such applications as automatic control, secure communication and population dynamics. For instance, in communication security systems based on impulsive synchronization, there exist transmission and sampling delays during the information transmission process, where the sampling delay created from sampling, the impulses at some discrete instances causes the impulsive transients depended on their historical states, we refer [19,20] and references therein.

What we are interested in the delayed impulsive stochastic Volterra integro-differential equation driven by fBms, the impulses are delayed impulses, the fBms are fractional Brownian motions with Hurst parameter $H \in (\frac{1}{2}, 1)$. Because fBms neither semi-martingales nor Markov processes, the traditional tools of Ito stochastic analysis cannot be applied effectively for these equations. In addition, impulse may influences not only qualitative properties but also existence and uniqueness of the solution. We are inspired by the work of Nguyen Tien Dung [12], the author who obtained the existence and uniqueness of mild solutions, the impulses only depended on current states but not depended on historical states of the system. So, our goal is to find the existence and uniqueness conditions of mild solutions for the delayed impulsive stochastic Volterra integro-differential equation driven by a fBm in this paper.

The rest of this paper is organized as follows. In Section 2, we introduce some notations, concepts, and basic results about fBms. In Section 3, existence and uniqueness conditions of mild solutions are established. In Section 4, an illustrative example is given. At last, the conclusion is given in Section 5.

2. Preliminaries

In this paper, we consider the following delayed impulsive stochastic Volterra integro-differential equation driven by fBms in a Hilbert space of the form:

$$\begin{cases} \mathrm{d}x(t) = [Ax(t) + F(t, x_t, \int_0^t K(t,s)x(s)\mathrm{d}s)]\mathrm{d}t + G(t)\mathrm{d}W^H(t), t \in [t_0, T], t \neq t_k \\ \Delta x(t_k) := x(t_k^+) - x(t_k^-) = d_k x(t_k^- - \delta), k = 1, 2, ..., m \\ x(t_0 + \theta) = \phi(\theta), \theta \in (-\infty, 0]. \end{cases} \tag{1}$$

Where A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators in a Hilbert space X. $W^H(t)$ is a fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$ on a real and separable Hilbert space Y.The Volterra kernel $K(t,s)$ is non-negative continuous for $0 \leqslant s \leqslant t$. The impulsive moments satisfy $0 \leq t_0 < t_1 < t_2 < ... < t_m = T$. $x(t_k^+)$ and $x(t_k^-)$ are respectively the right and left limits of $x(t)$ at $t = t_k$. $\Delta x(t_k)$ represent the jump in the state $x(t)$ at t_k. Denote $\overline{\tau} = \max\{t_k - t_{k-1}\}$ and $\underline{\tau} = \min\{t_k - t_{k-1}\}$, δ is delay constant satisfying $0 \leq \delta < \underline{\tau}$. d_k are positive numbers. $\mathfrak{B}_{t_0}$ and $\mathfrak{B}_T$ are abstract phase spaces and defined by $\mathfrak{B}_{t_0} = \{\phi : (-\infty, t_0] \to X$:for any $t^* > 0$, the second moments of $\phi(t)$ is bounded and measurable function on $[-t^*, t_0]$ with $\phi(0) = 0\}$, and $\mathfrak{B}_T = \{x : (-\infty, T] \to X : x(t)$ is continuous everywhere except a finite number of point t_k at which $x(t_k)^+, x(t_k)^-$ exist and $x_{t_0} = \phi \in \mathfrak{B}_{t_0}$, $k = 1, 2, ..., m\}$. For $\phi(t) \in \mathfrak{B}_{t_0}$, $\|\phi\|_{\mathfrak{B}_{t_0}} = \sup_{s\in(-\infty,t_0]}\|\phi(s)\| < +\infty$. The history x_t is defined by $x_t(\theta) = x(t + \theta), \theta \leq t_0$. $T > 0$ be arbitrary fixed horizon. $F : [t_0, T] \times \mathfrak{B}_T \times X \to X$, $G : [t_0, T] \to \mathcal{L}_2^0(Y, X)$, where $\mathcal{L}_2^0(Y, X)$ is the space of all Q-Hilbert-Schmidt operator $\psi : Y \to X$.

Have been introduced the equation, then, we recall fBms conceptions, Wiener integrals with respect to fBms and lemmas, which will be needed throughout the whole of this paper.

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space and $T > 0$ be an arbitrary fixed horizon, $H \in (\frac{1}{2}, 1)$ is a constant, $\{W^H(t), t \geqslant 0\}$ is called one-dimensional fBm with Hurst parameter H, if the $W^H(0) = \mathbb{E}W^H(t)$ for any $T > t > 0$ and the covariance function satisfy:

$$R_H(t, s) = \mathbb{E}[W^H(t)W^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), 0 < s, t < T.$$

Where $\mathbb{E}$ refers to the mathematical expectation of probability P.

It is known that $W^H(t)$ with $H \in (\frac{1}{2}, 1)$ has the following Wiener integral representation:

$$W^H(t) = \int_0^t K_H(t, s)\mathrm{d}W(s).$$

Where $W(t) = \{W(t) : t \in [t_0, T]\}$ is a standard Wiener process, and $K_H(t, s)$ is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \mathrm{d}u, t > s.$$

With $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H,H-\frac{1}{2})}}$ and $B(,)$ denotes the Beta function. Let $K_H(t, s) = 0$ for $t \leq s$. It is not difficult to see that

$$\frac{\partial K_H(t, s)}{\partial t} = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t - s)^{H-\frac{1}{2}}.$$

For the deterministic function $\varphi \in ([t_0, T])$, it is known from [21] that the fractional Wiener integral with respect to $W^H(t)$ can be defined by

$$\int_0^T \varphi(t)\mathrm{d}W^H(t) = \int_0^T (K_H^*\varphi)(t)\mathrm{d}W(t).$$

Where $(K_H^*\varphi)(s) = \int_s^t \varphi(t)\frac{\partial K_H(t,s)}{\partial t}\mathrm{d}t$.

Let X and Y be real, separable Hilbert space, $\mathcal{L}(Y, X)$ denote the space of all bounded linear operators from Y to X. Let $Q \in \mathcal{L}(Y, X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n$, where $\lambda_n, n = 1, 2, ...$ are nonnegative real numbers, $\{e_n, n = 1, 2, ..\}$ is a complete orthonormal basis in Y. Define a $Y-$ valued Gaussian process as

$$W^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n W_n^H(t).$$

Where $W_n^H(t)$ are real, independent fBms. It has the covariance:

$$\mathbb{E}\langle W^H(t), x\rangle\langle W^H(s), y\rangle = R(t, s)\langle Q(x), y\rangle.$$

For all $x, y \in Y$ and $t, s \in [t_0, T]$. The $\mathcal{L}_2^0(Y, X)$ which mentioned before is the space of all Q-Hilbert-Schmidt operator $\psi : Y \to X$, equipped with the following norm and inner product

$$\left\|\psi\right\|_{\mathcal{L}_2^0(Y,X)} = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n}\psi e_n\right\|^2,$$

$$\langle\phi,\psi\rangle_{\mathcal{L}_2^0(Y,X)}=\sum_{n=1}^{\infty}\langle\phi e_n,\psi e_n\rangle.$$

From the above, it is easy to see that the space $\mathcal{L}_2^0(Y,X)$ is a separable Hilbert space. Then from [12,25], the fractional Wiener integral of the function $\psi:[t_0,T]\to\mathcal{L}_2^0(Y,X)$ with respect to fBm is defined by

$$\int_0^t\psi(s)\mathrm{d}W^H(s)=\sum_{n=1}^{\infty}\int_0^t\sqrt{\lambda_n}\psi(s)e_n\mathrm{d}W_n^H(s)=\sum_{n=1}^{\infty}\int_0^t\sqrt{\lambda_n}K_H^*(\psi e_n)(s)\mathrm{d}W_n(s).$$

Lemma 1 [12]. If $\psi:[t_0,T]\to\mathcal{L}_2^0(Y,X)$ satisfies $\int_{t_0}^T\left\|\psi(s)\right\|^2_{\mathcal{L}_2^0(Y,X)}\mathrm{d}s<+\infty$, the $\int_{t_0}^t\psi(s)dW^H(s)$ is well defined as a X-valued random variable and the following inequality is established

$$\mathbb{E}\left\|\int_{t_0}^t\psi(s)\mathrm{d}W^H(s)\right\|^2\leq 2H(t-t_0)^{2H-1}\int_{t_0}^t\left\|\psi(s)\right\|^2_{\mathcal{L}_2^0(Y,X)}\mathrm{d}s.$$

We end this section by giving the definition of mild solution of the Eq. (1), for simplicity, we can assume that $x(0)=\phi(0)=0$.

Definition 1. A $X-$valued stochastic process $\{x(t),t\in(-\infty,T]\}$ is called a mild solution of the Eq.(1) if $x_0=\phi\in\mathfrak{B}_{t_0}$ with $\phi(0)=0$ and the following conditions hold,

(i) for each $t\in[t_0,T]$, x_t is a $\mathfrak{B}_T-$ valued function and $x(.)$ is continuous on $[t_0,t_1]$ and each interval $(t_k,t_{k+1}],k=1,2,...,m$.

(ii) for each k, the limits $x(t_k^+),x(t_k^-)$ exist and $x(t_k^-)=x(t_k)$ and $\Delta x(t_k):=x(t_k^+)-x(t_k^-)=d_kx(t_k-\delta)$

(iii) for each $t\in(-\infty,T]$, we have

$$x(t)=\begin{cases}\phi(t),\quad t\in(-\infty,t_0]\\ \int_{t_0}^tS(t-s)F(s,x_s,\int_0^sK(s,u)x(u)\mathrm{d}u)\mathrm{d}s+\int_{t_0}^tS(t-s)G(s)\mathrm{d}W^H(s),\quad t\in[t_0,t_1]\\ S(t-t_k)[x(t_k^-)+d_kx(t_k-\delta)]+\int_{t_k}^tS(t-s)G(s)\mathrm{d}W^H(s)+\int_{t_k}^tS(t-s)F(s,x_s,\int_0^sK(s,u)x(u)\mathrm{d}u)\mathrm{d}s,\\ \qquad t\in(t_k,t_{k+1}],k=1,2,...,m.\end{cases}\tag{2}$$

3. Existence and Uniqueness of Mild Solution

In this section, we investigate the existence and uniqueness conditions of mild solution for the Eq.(1) by means of the fixed point theory. In order to attain the results, we assume that the following assumptions hold.

(H_1) For the strongly continuous linear operator semigroup $\{S(t)\}_{t\geq0}$, we assume it is exponentially stable, that is to say, there exist a constant $M>0$ and a real number $r>0$ such that $\|S(t)\|\leq Me^{-rt}$, $t\geq0$.

(H_2) For any $T>t_0$, the function G satisfies

$$\int_{t_0}^Te^{rs}\|G(s)\|^2_{\mathcal{L}_2^0(Y,X)}\mathrm{d}s<\infty,\text{for some }r>0.$$

(H_3) The mapping F satisfies the following conditions, for any $\Phi,\psi\in\mathfrak{B}^0_{TD}$, $x,y\in X_D$, there exist constants $L_1>0,L_2>0$, such that

$$\left\|F(t,x,\Phi)-F(t,y,\psi)\right\|^2\leq L_1\left\|\Phi-\psi\right\|^2+L_2\left\|x-y\right\|^2.$$

Where $\mathfrak{B}^0_{TD}\subset\mathfrak{B}_{TD}$ is defined by $\mathfrak{B}^0_{TD}=\{x\in\mathfrak{B}_{TD},\mathbb{E}\|x\|^2\leq(m+1)(N+\frac{M^2C(L_1+aL_2)}{r^2})\}$, $X_D\subset X$ is defined by $X_D=\{\psi\in X,\mathbb{E}\|\psi\|^2\leq a(m+1)(N+\frac{M^2C(L_1+aL_2)}{r^2})\}$, with $C=\sup_{\theta\in(-\infty,t_0]}\mathbb{E}\|x(t_0+\theta)\|^2$, N is a constant which depends on the upper bound of $\int_0^\infty e^{rs}\|G(s)\|^2_{\mathcal{L}_2^0(Y,X)}\mathrm{d}s$. Note that (H_2) and Lemma1, it is not difficult to get $\mathbb{E}\|x\|^2\leq(m+1)(N+\frac{M^2C(L_1+aL_2)}{r^2})$.

Remark 2.1 The H_3 of this paper is to show that the function F satisfy inequality $\left\|F(t,x,\Phi)-F(t,y,\psi)\right\|^2 \leq L_1\left\|\Phi-\psi\right\|^2 + L_2\left\|x-y\right\|^2$ for $\Phi,\psi \in \mathfrak{B}^0_{TD}$, $x,y \in X_D$, not for $\Phi,\psi \in \mathfrak{B}_{TD}$, $x,y \in X$. The results under this hypothetical condition should be less conservative than the results is proved under the global Lipschitz condition of F.

Before giving the results, let's take the following formula first,

$$\begin{aligned}\Xi =&C_k^1\sum_{i_1=1}^{k} d_{i_1}^2 e^{-2r(k-i_1)\underline{\tau}} + C_k^2\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}^2 d_{i_2}^2 e^{-2r((k-i_1)\underline{\tau}-\delta)} + C_k^3\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}^2 d_{i_2}^2 d_{i_3}^2 e^{-2r((k-i_1)\underline{\tau}-2\delta)} + ...\\ &+ C_k^k\sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\cdots\sum_{i_k>i_{k-1}}^{k} d_{i_1}^2 d_{i_2}^2 ... d_{i_k}^2 e^{-2r((k-i_1)\underline{\tau}-(k-1)\delta)}.\end{aligned}$$

Where $C_k^j = \begin{pmatrix} j \\ k \end{pmatrix}, j = 1,2,..,k.$

Remark 2.2 In the above formula, when $d_{i_j} = 0, j = 1,2,...,k$, then $\Xi = 0$. It's easy to see each term $2r(k-i_1)\underline{\tau} \geq 0, i_1 = 1,2,...,k$, $2r((k-i_1)\underline{\tau}-\delta) > 0, i_1 = 1,2,...,k-1$, ..., $2r((k-i_1)\underline{\tau}-(k-1)\delta) > 0, i_1 = 1$, when $d_{i_j} = \frac{1}{k^2}$, then

$$\begin{aligned}\Xi &\leq C_k^1\sum_{i_1=1}^{k}\frac{1}{k^4} + C_k^2\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k}\frac{1}{k^8} + C_k^3\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k}\frac{1}{12} + ... + C_k^k\sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\cdots\sum_{i_k>i_{k-1}}^{k}\frac{1}{k^{4j}}\\ &= \sum_{j=1}^{k}(C_k^j\frac{1}{k^{2j}})^2\\ &\leq (\sum_{j=1}^{k} C_k^j\frac{1}{k^{2j}})^2\\ &= ((1+\frac{1}{k^2})^k - 1)^2.\end{aligned}$$

It is not hard to see that $((1+\frac{1}{k^2})^k-1)^2 \in (0,1]$. Now, we can state the existence and uniqueness results for the Eq.(1)

Theorem 3.1 Suppose that (H_1)–(H_3) hold. Then, for all $t \in (-\infty, T]$, the mild solution to the Eq.(1) exists uniquely, provided that

$$\max_{k=1,2,...,m}\Big\{(k+1)\frac{M^2}{r^2}\Big(L_1+L_2a\Big)\times\Big((1-e^{-r(k+1)\overline{\tau}})^2+\Xi\Big)\Big\} < 1. \tag{3}$$

Proof. Firstly, let us introduce the set $\mathfrak{B}_{TD}$ is the Banach space of all stochastic processes $x(t)$ from $(-\infty, T]$ into X, equipped with the supremum norm $\|\xi\|^2_{\mathfrak{B}_{TD}} = \sup_{s\in(-\infty,T]}\mathbb{E}\|\xi(s)\|^2$, $\xi \in \mathfrak{B}_{TD}$. The closed subset $\bar{\mathfrak{B}}_{TD}$ satisfying $x(t) \in \mathfrak{B}_{TD}$, $x(t) = \phi(t), t \in (-\infty, t_0]$ and the (i) and (ii) in Definition1, provided with the same norm $\|.\|_{\mathfrak{B}_{TD}}$. We define an operator $J : \bar{\mathfrak{B}}_{TD} \to \bar{\mathfrak{B}}_{TD}$ by $(Jx)(t) = \phi(t), t \in (-\infty, t_0]$ and

$$(Jx)(t) = \begin{cases}\int_0^t S(t-s)F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s + \int_0^t S(t-s)G(s)\mathrm{d}W^H(s), t\in[t_0,t_1]\\ S(t-t_k)[x(t_k^-)+d_kx(t_k-\delta)] + \int_{t_k}^t S(t-s)G(s)\mathrm{d}W^H(s) + \int_{t_k}^t S(t-s)F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s,\\ \qquad t\in(t_k,t_{k+1}], k=1,2,...,m.\end{cases} \tag{4}$$

In order to give the existence and uniqueness conditions of the mild solution of the Eq.(1), it is enough to show that the operator J has a unique fixed point. So we use the Banach fixed point theorem. From above, we can see that J maps $\bar{\mathfrak{B}}_T$ into itself. Then we need to show that J is a contraction mapping.

Let $x, x^* \in \bar{\mathfrak{B}}_{TD}$, then for all $t \in [t_0, t_1]$ we have,

$$\begin{aligned}
&\mathbb{E}\,\|(Jx)(t) - (Jx^*)(t)\|^2 \\
&= \mathbb{E}\left\| \int_{t_0}^{t} S(t-s)F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s - \int_{t_0}^{t} S(t-s)F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\mathrm{d}s \right\|^2 \\
&= \mathbb{E}\left\| \int_{t0}^{t} S(t-s)\Big(F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u) - F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \right\|^2. \qquad (5)
\end{aligned}$$

For $t \in (t_1, t_2]$, we have,

$$\begin{aligned}
&\mathbb{E}\,\|(Jx)(t) - (Jx^*)(t)\|^2 \\
&= \mathbb{E}\Big\| S(t-t_1)\Big(x(t_1^-) - x^*(t_1^-)\Big) + S(t-t_1)d_1\Big(x(t_1-\delta) - x^*(t_1-\delta)\Big) \\
&\quad + \int_{t_1}^{t} S(t-s)F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s - \int_{t_1}^{t} S(t-s)F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\mathrm{d}s \Big\|^2.
\end{aligned}$$

In terms of (2), then we have, $t \in (t_1, t_2]$,

$$\begin{aligned}
&\mathbb{E}\,\|(Jx)(t) - (Jx^*)(t)\|^2 \\
&= \mathbb{E}\Big\| S(t-t_1)\Big(\int_{t_0}^{t_1} S(t_1-s)F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s - \int_{t0}^{t_1} S(t_1-s)F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\mathrm{d}s\Big) \\
&\quad + S(t-t_1)d_1\Big(\int_{t_0}^{t_1-\delta} S(t_1-\delta-s)F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s \\
&\quad - \int_{t_0}^{t_1-\delta} S(t_1-\delta-s)F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\mathrm{d}s\Big) \\
&\quad + \int_{t_1}^{t} S(t-s)F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s - \int_{t_1}^{t} S(t-s)F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\mathrm{d}s \Big\|^2 \\
&= \mathbb{E}\Big\| \int_{t_0}^{t} S(t-s)\Big(F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u) - F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + d_1 \int_{t_0}^{t_1-\delta} S(t-\delta-s)\Big(F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u) - F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \Big\|^2. \qquad (6)
\end{aligned}$$

For $t \in (t_2, t_3]$, we have,

$$\begin{aligned}
&\mathbb{E}\,\|(Jx)(t) - (Jx^*)(t)\|^2 \\
&= \mathbb{E}\Big\| S(t-t_2)\Big(x(t_2^-) - x^*(t_2^-)\Big) + S(t-t_2)d_2\Big(x(t_2-\delta) - x^*(t_2-\delta)\Big) \\
&\quad + \int_{t_2}^{t} S(t-s)F(s, x_s, \int_0^s K(s,u)x(u)\mathrm{d}u)\mathrm{d}s - \int_{t_2}^{t} S(t-s)F(s, x_s^*, \int_0^s K(s,u)x^*(u)\mathrm{d}u)\mathrm{d}s \Big\|^2. \qquad (7)
\end{aligned}$$

In terms of (2) and (7), when $t \in (t_2, t_3]$, we get,

$$
\begin{aligned}
&\mathbb{E}\,\|(Jx)(t)-(Jx^*)(t)\|^2\\
&=\mathbb{E}\Big\|S(t-t_2)\Big(S(t_2-t_1)\big(x(t_1^-)-x^*(t_1^-)\big)+S(t_2-t_1)d_1\big(x(t_1-\delta)-x^*(t_1-\delta)\big)\\
&\quad+\int_{t_1}^{t_2}S(t_2-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u\Big)\mathrm{d}s\Big)\\
&\quad+S(t-t_2)d_2\Big(S(t_2-\delta-t_1)\big(x(t_1^-)-x^*(t_1^-)\big)+S(t_2-\delta-t_1)d_1\big(x(t_1-\delta)-x^*(t_1-\delta)\big)\Big)\\
&\quad+\int_{t_1}^{t_2-\delta}S(t_2-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u\Big)\mathrm{d}s\Big)\\
&\quad+\int_{t_2}^{t}S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\Big\|^2.
\end{aligned}
$$

Computing the above formula, when $t \in (t_2, t_3]$, we have

$$
\begin{aligned}
&\mathbb{E}\,\|(Jx)(t)-(Jx^*)(t)\|^2\\
&=\mathbb{E}\Big\|\int_{t_0}^{t_1}S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+d_1\int_{t_0}^{t_1-\delta}S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+d_2\int_{t_0}^{t_1}S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+d_1d_2\int_{t_0}^{t_1-\delta}S(t-2\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+d_2\int_{t_1}^{t_2-\delta}S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s.\\
&\quad+\int_{t_2}^{t}S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\Big\|^2.
\end{aligned}\tag{8}
$$

Computing (8), we can get,

$$
\begin{aligned}
&\mathbb{E}\,\|(Jx)(t)-(Jx^*)(t)\|^2\\
&=\mathbb{E}\Big\|\int_{t_0}^{t}S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+d_1\int_{t_0}^{t_1-\delta}S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+d_1d_2\int_{t_0}^{t_1-\delta}S(t-2\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\\
&\quad+d_2\int_{t_0}^{t_2-\delta}S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\Big\|^2.
\end{aligned}\tag{9}
$$

Similarly, when $t \in (t_3, t_4]$, we get,

$$
\begin{aligned}
&\mathbb{E}\,\|(Jx)(t) - (Jx^*)(t)\|^2 \\
&= \mathbb{E}\Big\|\int_{t_0}^{t} S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + \sum_{i_1=1}^{3} d_{i_1}\int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + \sum_{i_1=1}^{2}\sum_{i_2>i_1}^{3} d_{i_1}d_{i_2}\int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + d_1d_2d_3\int_{t_0}^{t_1-\delta} S(t-3\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\Big\|^2. \qquad (10)
\end{aligned}
$$

It is easy to see, when $t \in (t_k, t_{k+1}], k = 1, 2, ..., m$, we get,

$$
\begin{aligned}
&\mathbb{E}\,\|(Jx)(t) - (Jx^*)(t)\|^2 \\
&= \mathbb{E}\Big\|\int_{t_0}^{t} S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + \sum_{i_1=1}^{k} d_{i_1}\int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + \sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}d_{i_2}\int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + \sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}d_{i_2}d_{i_3}\int_{t_0}^{t_{i_1}-\delta} S(t-3\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s \\
&\quad + \ldots + \sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\ldots\sum_{i_k>i_{k-1}}^{k} d_{i_1}d_{i_2}\ldots d_{i_k}\int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s) \\
&\qquad \times\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\Big\|^2. \qquad (11)
\end{aligned}
$$

Next, we estimate (11),

$$
\begin{aligned}
&\mathbb{E}\,\|(Jx)(t) - (Jx^*)(t)\|^2 \\
&\leq (k+1)\Bigg\{\mathbb{E}\Big\|\int_{t_0}^{t} S(t-s)\Big(F(s,x_s,\int_{t_0}^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\Big\|^2 \\
&\quad + \mathbb{E}\Big\|\sum_{i_1=1}^{k} d_{i_1}\int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\Big\|^2.
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\left\|\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}d_{i_2}\int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2\\
&+\mathbb{E}\left\|\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}d_{i_2}d_{i_3}\int_{t_0}^{t_{i_1}-\delta} S(t-3\delta-s)\right.\\
&\quad\left.\times\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2\\
&+\ldots\\
&+\mathbb{E}\left\|\sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\ldots\sum_{i_k>i_{k-1}}^{k} d_{i_1}d_{i_2}\ldots d_{i_k}\int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s)\right.\\
&\quad\left.\times\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2\Bigg\}\\
&=(k+1)\sum_{j=0}^{k} Q_j\ . \qquad (12)
\end{aligned}
$$

Denote $a=\Big(\sup_{t\in[0,T]}\int_0^t K(t,s)ds\Big)^2$, by Holder inequality, $(H_1)-(H_3)$, we have,

$$
\begin{aligned}
Q_0&=\mathbb{E}\left\|\int_{t_0}^{t} S(t-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2\\
&\le\mathbb{E}\Big\{\int_{t_0}^{t}\|S(t-s)\|\left\|F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\right\|\Big\}^2\\
&\le\int_{t_0}^{t} M^2e^{-r(t-s)}\mathrm{d}s\int_{t_0}^{t} e^{-r(t-s)}\mathbb{E}\left\|F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\right\|^2\mathrm{d}s \qquad (13)\\
&\le\frac{M^2}{r}(1-e^{-r(t-t_0)})\int_{t_0}^{t} e^{-r(t-s)}\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2\,\mathrm{d}s\\
&\le\frac{M^2}{r^2}(1-e^{-r(k+1)\overline{\tau}})^2\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2 .
\end{aligned}
$$

$$
\begin{aligned}
Q_1&=\mathbb{E}\left\|\sum_{i_1=1}^{k} d_{i_1}\int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2\\
&\le k\sum_{i_1=1}^{k} d_{i_1}^2\mathbb{E}\left\|\int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2\\
&\le k\sum_{i_1=1}^{k} d_{i_1}^2M^2\int_{t_0}^{t_{i_1}-\delta} e^{-r(t-\delta-s)}\mathrm{d}s\int_{t_0}^{t_{i_1}-\delta} e^{-r(t-\delta-s)}\\
&\quad\times\mathbb{E}\left\|F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\right\|^2\mathrm{d}s \qquad (14)\\
&\le\frac{M^2}{r^2}k\sum_{i_1=1}^{k} d_{i_1}^2e^{-2r(t-t_{i_1})}(1-e^{-r(t_{i_1}-\delta-t_0)})^2\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2\\
&\le\frac{M^2}{r^2}k\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2\sum_{i_1=1}^{k} d_{i_1}^2e^{-2r(t-t_{i_1})}.
\end{aligned}
$$

$$
\begin{aligned}
Q_2 &= \mathbb{E}\left\|\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}d_{i_2}\int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\right\|^2 \\
&\le C_k^2\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}^2d_{i_2}^2\mathbb{E}\left\|\int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2 \\
&\le C_k^2\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}^2d_{i_2}^2M^2\int_{t_0}^{t_{i_1}-\delta} e^{-r(t-2\delta-s)}\mathrm{d}s\int_{t_0}^{t_{i_1}-\delta} e^{-r(t-2\delta-s)} \\
&\quad\times\mathbb{E}\left\|F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\right\|^2\mathrm{d}s \\
&\le \frac{M^2}{r^2}C_k^2\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}^2d_{i_2}^2e^{-2r(t-t_{i_1}-\delta)}(1-e^{-r(t_{i_1}-\delta-t_0)})^2\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2 \\
&\le \frac{M^2}{r^2}C_k^2\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}^2d_{i_2}^2e^{-2r(t-t_{i_1}-\delta)}.
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
Q_3 &= \mathbb{E}\left\|\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}d_{i_2}d_{i_3}\int_{t_0}^{t_{i_1}-\delta} S(t-3\delta-s)\right. \\
&\quad\left.\times\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\right\|^2 \\
&\le C_k^3\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}^2d_{i_2}^2d_{i_3}^2\mathbb{E}\left\|\int_{t_0}^{t_{i_1}-\delta} S(t-3\delta-s)\right. \\
&\quad\left.\times\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\mathrm{d}s\right\|^2 \\
&\le M^2C_k^3\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}^2d_{i_2}^2d_{i_3}^2\int_{t_0}^{t_{i_1}-\delta} e^{-r(t-3\delta-s)}\mathrm{d}s\int_{t_0}^{t_{i_1}-\delta} e^{-r(t-3\delta-s)} \\
&\quad\times\mathbb{E}\left\|F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\right\|^2\mathrm{d}s \\
&\le \frac{M^2}{r^2}C_k^3\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}^2d_{i_2}^2d_{i_3}^2e^{-2r(t-t_{i_1}-2\delta)}(1-e^{-r(t_{i_1}-\delta-t_0)})^2\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2 \\
&\le \frac{M^2}{r^2}C_k^3\Big(L_1+L_2a\Big)\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}^2d_{i_2}^2d_{i_3}^2e^{-2r(t-t_{i_1}-2\delta)}\sup_{s\in(-\infty,T]}\mathbb{E}\,\|x(s)-x^*(s)\|^2.
\end{aligned}
\tag{16}
$$

The rest what can be done in the same manner, we can get,

$$
\begin{aligned}
Q_k &= \mathbb{E}\left\|\sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\dots\sum_{i_k>i_{k-1}}^{k} d_{i_1}d_{i_2}\dots d_{i_k}\int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s)\right. \\
&\quad\left.\times\Big(F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u)-F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\Big)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le M^2 C_k^k \sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\cdots\sum_{i_k>i_{k-1}}^{k} d_{i_1}^2 d_{i_2}^2 \ldots d_{i_k}^2 \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-k\delta-s)}\mathrm{d}s \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-k\delta-s)} \\
&\quad\times \mathbb{E}\left\|F(s,x_s,\int_0^s K(s,u)x(u)\mathrm{d}u) - F(s,x_s^*,\int_0^s K(s,u)x^*(u)\mathrm{d}u)\right\|^2 \mathrm{d}s \\
&\le \frac{M^2}{r^2} C_k^k \sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\cdots\sum_{i_k>i_{k-1}}^{k} d_{i_1}^2 d_{i_2}^2 \ldots d_{i_k}^2 e^{-2r(t-t_{i_1}-(k-1)\delta)}(1-e^{-r(t_{i_1}-\delta-t_0)})^2\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\|x(s)-x^*(s)\|^2 \\
&\le \frac{M^2}{r^2} C_k^k\Big(L_1+L_2a\Big) \sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\cdots\sum_{i_k>i_{k-1}}^{k} d_{i_1}^2 d_{i_2}^2 \ldots d_{i_k}^2 e^{-2r(t-t_{i_1}-(k-1)\delta)} \sup_{s\in(-\infty,T]}\mathbb{E}\|x(s)-x^*(s)\|^2 .
\end{aligned}
\tag{17}
$$

Substituting (13)-(17) into (12), we obtain the following inequality, for $t\in[t_0,t_1]$ and $t\in(t_k,t_{k+1}]$, $k=1,2,...,m$

$$
\begin{aligned}
&\mathbb{E}\|(Jx)(t)-(Jx^*)(t)\|^2 \\
&\le (k+1)\frac{M^2}{r^2}\Big(L_1+L_2a\Big)\sup_{s\in(-\infty,T]}\mathbb{E}\|x(s)-x^*(s)\|^2 \times \Big\{(1-e^{-r(k+1)\overline{\tau}})^2 + C_k^1\sum_{i_1=1}^{k} d_{i_1}^2 e^{-2r(k-i_1)\underline{\tau}} \\
&\quad + C_k^2 \sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}^2 d_{i_2}^2 e^{-2r((k-i_1)\underline{\tau}-\delta)} + C_k^3 \sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}^2 d_{i_2}^2 d_{i_3}^2 e^{-2r((k-i_1)\underline{\tau}-2\delta)} \\
&\quad + \ldots + C_k^k \sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\cdots\sum_{i_k>i_{k-1}}^{k} d_{i_1}^2 d_{i_2}^2\ldots d_{i_k}^2 e^{-2r((k-i_1)\underline{\tau}-(k-1)\delta)}\Big\}.
\end{aligned}
\tag{18}
$$

Combing (18) with (3), implies that J is a contraction map and hence it has a unique fixed point on the interval $[t_0,T]$, which is a mild solution of the Eq.(1). Thus, the theorem is proved.

Remark 3.1 When the impulses only depends on current state, not the historical state of the systems that is $\delta=0$, the Eq. (1) becomes the following form,

$$
\begin{cases}
\mathrm{d}x(t) = [Ax(t) + F(t,x_t,\int_0^t K(t,s)x(s)\mathrm{d}s]\mathrm{d}t + G(t)\mathrm{d}W^H(t), t\in[t_0,T], t\ne t_k \\
\Delta x(t_k) = d_k x(t_k^-), k=1,2,...m \\
x(t) = \phi(t), t\in(-\infty,t_0].
\end{cases}
\tag{19}
$$

Where the operators A, F and $G(t)$ are defined the same as before. By Theorem 3.1, we can get the following results.

Corollary 3.2 Suppose that (H_1)-(H_3) and the following conditions hold, then the mild solution of Eq. (19) exists a unique solution.

$$
\max_{k=1,2,\ldots,m}\Big\{(k+1)\frac{M^2}{r^2}\Big(L_1+L_2a\Big)\times\Big((1-e^{-r(k+1)\overline{\tau}})^2+\Xi'\Big)\Big\}<1.
\tag{20}
$$

Where

$$
\begin{aligned}
\Xi' = &C_k^1\sum_{i_1=1}^{k} d_{i_1}^2 e^{-2r(k-i_1)\underline{\tau}} + C_k^2\sum_{i_1=1}^{k-1}\sum_{i_2>i_1}^{k} d_{i_1}^2 d_{i_2}^2 e^{-2r((k-i_1)\underline{\tau})} + C_k^3\sum_{i_1=1}^{k-2}\sum_{i_2>i_1}^{k-1}\sum_{i_3>i_2}^{k} d_{i_1}^2 d_{i_2}^2 d_{i_3}^2 e^{-2r((k-i_1)\underline{\tau})} + \ldots \\
&+ C_k^k\sum_{i_1=1}^{1}\sum_{i_2>i_1}^{2}\cdots\sum_{i_k>i_{k-1}}^{k} d_{i_1}^2 d_{i_2}^2\ldots d_{i_k}^2 e^{-2r((k-i_1)\underline{\tau})}.
\end{aligned}
$$

In terms of Remark 2.2, we can easy to see that the condition (20) is viable. For example, when $d_k=\frac{1}{k^2}$, we can prove $\Xi'\in(0,1]$. Then

$$
\max_{k=1,2,\ldots,m}\Big\{(k+1)\frac{M^2}{r^2}\Big(L_1+L_2a\Big)\Big((1-e^{-r(k+1)\overline{\tau}})^2+\Xi'\Big)\Big\}\le(m+1)\frac{2M^2(L_1+aL_2)}{r^2}.
$$

We only need $L_1 + aL_2 < \frac{r^2}{2M^2(m+1)}$, then the Eq.(19) has a unique solution.

Remark 3.3 In literature [12], the author studied the stochastic Volterra equation driven by fBm with impulses, which is similar to the Eq.(19), described as

$$\begin{cases} dx(t) = [Ax(t) + F(t, x_t, \int_0^t K(t,s)x(s)ds]dt + G(t)dW^H(t), t \in [0,T], t \neq t_k \\ \Delta x(t_k) = I_k(x(t_k^-)), k = 1, 2, ...m \\ x(t) = \phi(t), t \in (-\infty, 0]. \end{cases} \tag{21}$$

And the I_k satisfied Lipschitz conditions $\|I_k(x) - I_k(y)\|^2 \leq \rho_k \|x - y\|^2$, and the other is the same as the Eq.(19) when $t_0 = 0$.

Remark 3.4 In [12], the author provided the existence and unique conditions of solution for the Eq.(21) as follows: exists $M > 0$ such that $\|S(t)\| \leq M$, the function F satisfies that there exist $L_1, L_2 > 0$ such that $\mathbb{E}\|F(t,\psi,x) - F(t,\phi,y)\|^2 \leq L_1\|\psi - \phi\|^2 + L_2\mathbb{E}\|x - y\|^2$, for any $t \in [0,T]$, $\psi, \phi \in \mathfrak{B}_h$, $x, y \in L^2(\Omega, X)$, and the (H_4) in [12] is the same (H_2) in this paper, and the following inequality:

$$\max_{k=1,2,...,m} \left(3M^2(1 + \rho_k + T^2(L_1a^2 + L_2K^*))\right) < 1. \tag{22}$$

Where $T > t_0 = 0$ is arbitrary. So, from the Eq.(22), the condition is very difficult to arrive due to T in the Eq.(22). In addition to this, when $M = 1$, it is impossible to the Eq.(22), no mater what value of L_1, L_2, ρ_k. However, by Corollary 3.2 in this paper, we can choose $M = 1$ and $d_k = \frac{1}{k^2}$, then the Eq.(1) has a unique solution under the condition $L_1 + aL_2 < \frac{r^2}{2(m+1)}$. It is not difficult to see that our results are better than Theorem 3.1 literature[12].

Remark 3.5 When the impulses disappear, that is $d_k = 0, k = 1, 2, ..., m$, the Eq.(1) reduces to the following stochastic Volterra equation:

$$\begin{cases} dx(t) = [Ax(t) + F(t, x_t, \int_0^t K(t,s)x(s)ds]dt + G(t)dW^H(t), t \in [t_0, T]. \\ x(t) = \phi(t), t \in (-\infty, t_0] \end{cases} \tag{23}$$

Where the operators A, F and $G(t)$ are defined the same as before. Here $\mathfrak{B}_T = \{x : (-\infty, T] \to X : x(t)$ is continuous$\}$, and endowed with the supremum norm $\|\phi\|_{\mathfrak{B}_T} = \sup_{s\in(-\infty,0]}\|\phi(s)\|$, for $\phi(t) \in \mathfrak{B}_T$. By using the same technique in Theorem 3.1, we can easily deduce the following corollary.

Corollary 3.6 Suppose that (H_1)-(H_3) and $L_1 + L_2a < \frac{r^2}{M^2}$ hold, then the mild solution of the Eq.(23) exists a unique solution.

Remark 3.7 When $I_k(x(t_k^-)) = 0, k = 1, 2, ..., m$, the Eq.(21) reduce to Eq.(23) when $t_0 = 0$. Then $\rho_k = 0$ and the (22) turn to be $3M^2(1 + T^2(L_1a^2 + L_2K^*)) < 1$. The same thing is the condition $3M^2(1 + T^2(L_1a^2 + L_2K^*)) < 1$ is very difficult to achieve. But Corollary 3.6 in this paper, the condition is easy to achieve.

Remark 3.8 Theorem 3.1 shows that the delayed impulsive differential equation driven by fBms has unique mild solution when the impulses frequency and their amplitude must be suitably related to the growth rate of function F. Corollary 3.6 shows that the stochastic Volterra differential equation driven by fBms without impulses has unique solution when the strongly continuous linear operator semigroup $S(t)$ exponential decay rate should be suitably related to the growth rate of function F.

Remark 3.9 In this paper, we only considered the additive noise only. In the future, we will further study existence, uniqueness and qualitative properties of mild solutions for stochastic differential equations driven by multiplicative noise, for example when the term $G(t)dW^H(t)$ is replaced by $G(t, X(t))dW^H(t)$ term, it may be one of our interesting directions of the future work.

4. Example

In this section, we provide an example to illustrate the obtained results. Let us consider the delayed impulsive stochastic Volterra equation driven by a fBm as follows,

$$\begin{cases} \frac{\partial Z(t,x)}{\partial t} = \frac{\partial^2 Z(t,x)}{\partial x^2} + \sigma_1(t)x(t-\sin t) + \sigma_2(t)\int_0^t K(t,s)x(s)\mathrm{d}s + e^{-rt}\mathrm{d}W^H(t), t\in[0,T], t\neq t_k \\ \Delta Z(t_k,x) := \frac{\rho}{k^2}Z(t_k^- - \delta), t = t_k, k = 1,2,...m \\ Z(t,0) = Z(t,\pi) = 0, t\in[0,T] \\ Z(t,x) = \phi(t,x), t\in(-\infty,0]. \end{cases} \tag{24}$$

Where $\sigma_1(t)$ and $\sigma_2(t)$ are bounded functions of t, $\rho > 0$ is constant. Let $X = L^2[0,\pi]$ and $Y = L^2[0,\pi]$, the operator $A : X \to X$ by $Ax = x''$ with domain $\mathfrak{D}(A) = \{x \in X, x, x'$ are absolutely continuous $x'' \in X, x(0) = x(\pi) = 0\}$. Then, $Ax = \sum_{n=1}^{\infty} n^2\langle x, x_n\rangle x_n, x \in \mathfrak{D}(A)$, where $x_n(t) = \sqrt{\frac{2}{\pi}}\sin(nt), n = 1,2,...$ is the orthogonal set of eigenvectors in A. It is easy to know that , A is the infinitesimal generator of an analytic semi-group $\{S(t)\}_{t\geq 0}$ in X. Furthermore, we have $S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, x_n\rangle x_n$ for all $x \in X, t > 0$. We known that $\|S(t)\| \leq e^{-\pi^2 t}$. For defining the operator $Q : Y \to X$, we can elect a sequence $\{\lambda_n\}_{n\geq 1} \subset R^+$ and set $Q\omega_n = \lambda_n\omega_n$, where ω_n is a complete orthonormal basis in Y. Also, assuming that $tr(Q) = \sum_{n=1}^{\infty}\left(\lambda_n\right)^{\frac{1}{2}} < \infty$. Now we denote the process $W^H(t)$ by $W^H(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\omega_n W_n^H(t)$, where $H \in (\frac{1}{2}, 1)$ and $W_n^H(t)$ is a sequence of two-sided one-dimensional fBms mutually independent.

Now, we verify the conditions of Theorem 3.1. Since $\|S(t)\| \leq e^{-\pi^2 t}$, we can choose $M = 1$ and $r = \pi^2$ in (H_1). The function F as follows,

$$F(t, x_t, \int_0^t K(t,s)x(s)\mathrm{d}s) = \frac{\sigma_1(t)}{\sqrt{2\pi}}x(t-\sin t) + \frac{\sigma_2(t)}{\sqrt{2\pi}}\int_0^t K(t,s)x(s)\mathrm{d}s.$$

So, we can choose $L_1 = \frac{\|\sigma_1(t)\|^2}{\pi^2}$ and $L_2 = \frac{\|\sigma_2(t)\|^2}{\pi^2}$ in (H_3). $G(t) = e^{-rt}$, then (H_2) holds. For the convenience of calculation, let $K(t,s) = e^{-s}$, $\tau^* = t_{k+1} - t_k$, $\delta = \frac{\tau^*}{2}$, then $a = 1, \underline{\tau} = \overline{\tau} = \tau^*$ and if the following inequality

$$\max_{k=1,2,...,m}\Big\{\frac{k+1}{\pi^4}\frac{\|\sigma_1(t)\|^2 + \|\sigma_2(t)\|^2}{\pi^2}\Big((1 - e^{-\pi^2(k+1)\tau^*})^2 + \left(C_k^1\right)^2\frac{\rho^2}{k^4} + \left(C_k^2\right)^2\frac{\rho^4}{k^8}e^{-\pi^2\tau^*} + ... + \left(C_k^k\right)^2\frac{\rho^k}{k^{4k}}e^{-\pi^2(k-1)\tau^*}\Big)\Big\} < 1 \tag{25}$$

holds, then all the assumptions in Theorem 3.1 are satisfied. By Theorem 3.1, the Eq.(24) exists unique mild solution. In term of Remark 2.2, it is easy to known that if the following formula holds,

$$\sigma_1 + \sigma_2 < \frac{\pi^6}{2(m+1)}.$$

Then the (25) holds. Where $\sigma_1 = \|\sigma_1(t)\|^2$ and $\sigma_2 = \|\sigma_2(t)\|^2$.

5. Conclusion

In this article, we obtain the existence and uniqueness conditions of mild solutions for a class of delayed impulsive stochastic Volterra equations driven by a fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. In addition, an example is given to show the effectiveness of the obtained theoretical results. It is noteworthy that the impulse delay in this article only affect one single time interval $(0 \leq \delta < \underline{\tau})$. In fact, however, the same ideology can be also applied for the impulse delay with two or more intervals. And the impulses can also be described as $\Delta x(t_k) = d_k x(t_k^- - \delta_k)$. In order to discuss conveniently, we assume the $\delta_k = \delta$ and $(0 \leq \delta < \underline{\tau})$.

We conclude this paper with an open question: As we stated in the Section 1, the properties and theories of stochastic differential equations driven by fBms are in the first stage of studying and few literatures study the qualitative properties. Moreover, basically all the works are base on Hurst parameter $H \in (\frac{1}{2}, 1)$ of fBms, one problem is that how to investigate the existence and uniqueness and stability behaviour of mild solutions for impulsive stochastic differential equations driven by fBms with Hurst parameter $H \in (0, \frac{1}{2})$.

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