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Hermite-Hadamard and Simpson-like Type Inequalities for Differentiable p-quasi-convex Functions

İmdat İşcan[a], Sercan Turhan[b], Selahattin Maden[c]

[a]*Giresun University, Department of Mathematics, Faculty of Arts and Sciences, 28100, Giresun, Turkey.*
[b]*Giresun University, Dereli Vocational School, 28100, Giresun, Turkey.*
[c]*Ordu University, Department of Mathematics, Faculty of Arts and Sciences, 52000, Ordu, Turkey.*

Abstract. In this paper, we give a new concept which is a generalization of the concepts quasi-convexity and harmonically quasi-convexity and establish a new identity. A consequence of the identity is that we obtain some new general inequalities containing all of the Hermite-Hadamard and Simpson-like type for functions whose derivatives in absolute value at certain power are p-quasi-convex. Some applications to special means of real numbers are also given.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

Following inequality is well known in the literature as Simpson inequality:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4 .$$

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Email addresses: imdati@yahoo.com, imdat.iscan@giresun.edu.tr (İmdat İşcan), sercanturhan28@gmail.com, sercan.turhan@giresun.edu.tr (Sercan Turhan), maden55@mynet.com (Selahattin Maden)

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(\alpha x + (1-\alpha)y) \leq \sup\{f(x), f(y)\},$$

for any $x, y \in [a, b]$ and $\alpha \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [1]).

For some results which generalize, improve and extend the inequalities(1) related to quasi-convex functions we refer the reader to see [1–4, 6, 10, 11, 15] and plenty of references therein.

In [5], the author gave the definition of harmonically convex function as follow and established Hermite-Hadamard's inequality for harmonically convex functions.

Definition 1.2. *Let $I \subseteq \mathbb{R}\setminus\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

In [15], Zhang et al. defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

Definition 1.3. *A function $f : I \subseteq (0, \infty) \to [0, \infty)$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \sup\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

We would like to point out that any harmonically convex function on $I \subseteq (0, \infty)$ is a harmonically quasi-convex function, but not conversely. For example, the function

$$f(x) = \begin{cases} 1, & x \in (0, 1]; \\ (x-2)^2, & x \in [1, 4]. \end{cases}$$

is harmonically quasi-convex on $(0, 4]$, but it is not harmonically convex on $(0, 4]$.

In [9], Zhang and Wan gave definition of p-convex function as follow:

Definition 1.4. *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be a p-convex function or belongs to the class $PC(I)$, if*

$$f\left(\left[\alpha x^p + (1-\alpha)y^p\right]^{1/p}\right) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in I$ and $\alpha \in [0, 1]$.

Remark 1.5 ([9]). *An interval I is said to be a p-convex set if $\left[\alpha x^p + (1-\alpha)y^p\right]^{1/p} \in I$ for all $x, y \in I$ and $\alpha \in [0, 1]$, where $p = 2k + 1$ or $p = n/m$, $n = 2r + 1$, $m = 2t + 1$ and $k, r, t \in \mathbb{N}$.*

Remark 1.6 ([7]). *If $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}\setminus\{0\}$, then $\left[\alpha x^p + (1-\alpha)y^p\right]^{1/p} \in I$ for all $x, y \in I$ and $\alpha \in [0, 1]$.*

According to Remark 1.6, we can give a different version of the definition of p-convex function as follow:

Definition 1.7. *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if*

$$f\left(\left[\alpha x^p + (1-\alpha)y^p\right]^{1/p}\right) \leq \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in I$ and $\alpha \in [0,1]$.

According to Definition 1.7, It can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

In [16, Theorem 5], if we take $I \subset (0,\infty)$, $p \in \mathbb{R}\backslash\{0\}$ and $h(t) = t$, then we have the following Theorem.

Theorem 1.8. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then we have*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

For some results related to p-convex functions and its generalizations, we refer the reader to see [7–9, 12–14, 16].

2. Main Results

Definition 2.1. *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be p-quasi-convex, if*

$$f\left(\left[tx^p + (1-t)y^p\right]^{1/p}\right) \leq \max\{f(x), f(y)\} \tag{4}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (4) is reversed, then f is said to be p-quasi-concave.

It can be easily seen that for $r = 1$ and $r = -1$, p-quasi convexity reduces to ordinary quasi convexity and harmonically quasi convexity of functions defined on $I \subset (0,\infty)$, respectively. Morever every p-convex function is a p-quasi-convex function.

Example 2.2. *Let $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^p, p \in \mathbb{R}\backslash\{0\}$, and $g : (0,\infty) \to \mathbb{R}$, $g(x) = c$, $c \in \mathbb{R}$, then f and g are p-quasi-convex functions.*

Proposition 2.3. *Let $I \subset (0,\infty)$ be a real interval, $p \in \mathbb{R}\backslash\{0\}$ and $f : I \to \mathbb{R}$ is a function, then ;*

1. *if $p \leq 1$ and f is quasi-convex and nondecreasing function then f is p-quasi-convex.*
2. *if $p \geq 1$ and f is p-quasi-convex and nondecreasing function then f is quasi-convex.*
3. *if $p \leq 1$ and f is p-quasi-concave and nondecreasing function then f is quasi-concave.*
4. *if $p \geq 1$ and f is quasi-concave and nondecreasing function then f is p-quasi-concave.*
5. *if $p \geq 1$ and f is quasi-convex and nonincreasing function then f is p-quasi-convex.*
6. *if $p \leq 1$ and f is p-quasi-convex and nonincreasing function then f is quasi-convex.*
7. *if $p \geq 1$ and f is p-quasi-concave and nonincreasing function then f is quasi-concave.*
8. *if $p \leq 1$ and f is quasi-concave and nonincreasing function then f is p-quasi-concave.*

Proof. Since $g(x) = x^p$, $p \in (-\infty,0) \cup [1,\infty)$, is a convex function on $(0,\infty)$ and $g(x) = x^p$, $p \in (0,1]$, is a concave function on $(0,\infty)$,the proof is obvious from the following power mean inequalities

$$\left[tx^p + (1-t)y^p\right]^{1/p} \geq tx + (1-t)y,\ p \geq 1,$$

and

$$\left[tx^p + (1-t)y^p\right]^{1/p} \leq tx + (1-t)y,\ p \leq 1.$$

□

The following proposition is obvious.

Proposition 2.4. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ and if we consider the function $g : [a^p, b^p] \to \mathbb{R}$, defined by $g(t) = f\left(t^{1/p}\right)$, $p \neq 0$, then f is p-quasi-convex on $[a,b]$ if and only if g is quasi-convex on $[a^p, b^p]$.*

In order to prove our main results we need the following lemma:

Lemma 2.5. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I$ with $a < b$. If $f' \in L[a,b]$ and $p \in \mathbb{R}\setminus\{0\}$, then for $\lambda \in [0,1]$ we have the equality*

$$(1-\lambda) f\left(M_p\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx = \frac{b^p - a^p}{2p}\left[\int_0^{1/2} \frac{\lambda - 2t}{M_{t,p}^{p-1}} f'\left(M_{t,p}\right)dt + \int_{1/2}^1 \frac{2-\lambda-2t}{M_{t,p}^{p-1}} f'\left(M_{t,p}\right)dt\right],$$

where $M_{t,p} = M_{t,p}(a,b) = \left[ta^p + (1-t)b^p\right]^{1/p}$ and $M_{1/2,p} = M_p$.

Proof. It suffices to note that

$$\begin{aligned} I_1 &= \frac{b^p - a^p}{p}\int_0^{1/2} \frac{\lambda - 2t}{M_{t,p}^{p-1}} f'\left(M_{t,p}\right)dt \\ &= (2t-\lambda) f\left(M_{t,p}\right)\Big|_0^{1/2} - 2\int_0^{1/2} f\left(M_{t,p}\right)dt \\ &= (1-\lambda) f\left(M_p\right) + \lambda f(b) - 2\int_0^{1/2} f\left(M_{t,p}\right)dt. \end{aligned}$$

Setting $x^p = ta^p + (1-t)b^p$ and $px^{p-1}dx = (a^p - b^p)\,dt$, which gives

$$I_1 = (1-\lambda) f\left(M_p\right) + \lambda f(b) - \frac{2p}{b^p - a^p}\int_{M_p}^b \frac{f(x)}{x^{1-p}}dx.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \frac{b^p - a^p}{p}\int_{1/2}^1 \frac{2-\lambda-2t}{M_{t,p}^{p-1}} f'\left(M_{t,p}\right)dt \\ &= \lambda f(a) + (1-\lambda) f\left(M_p\right) - \frac{2p}{b^p - a^p}\int_a^{M_p} \frac{f(x)}{x^{1-p}}dx. \end{aligned}$$

Thus,

$$\frac{I_1 + I_2}{2} = (1-\lambda) f\left(M_p\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx$$

which is required. □

Theorem 2.6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$, and $f' \in L[a,b]$. If $|f'|^q$ is p-quasi-convex on $[a,b]$ for $q \geq 1$ and $p \in \mathbb{R}\backslash\{0\}$ then we have the following inequality for $\lambda \in [0,1]$*

$$\left|(1-\lambda) f\left(M_p(a,b)\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \tag{5}$$
$$\leq \frac{b^p-a^p}{2p}\left(C(\lambda;p;a,b)+C(\lambda;p;b,a)\right)\left(\max\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}\right)^{1/q}$$

where for $p=-1$

$$C(\lambda;-1;u,\vartheta) = \frac{1}{(\vartheta^{-1}-u^{-1})^2}\left[-2\ln\left(\frac{\vartheta M_{-1}(u,\vartheta)}{M^2_{\frac{\lambda}{2},-1}(u,\vartheta)}\right) + \left(\vartheta^{-1}+M^{-1}_{\lambda,-1}(u,\vartheta)\right)\left(\vartheta + M_{-1}(u,\vartheta) - 2M_{\frac{\lambda}{2},-1}(u,\vartheta)\right)\right],$$

and for $p \in \mathbb{R}\backslash\{-1,0\}$

$$\begin{aligned} C(\lambda;p;u,\vartheta) \quad = \quad & \frac{p}{(p+1)(\vartheta^p-u^p)^2}\left[2\left(\vartheta^{p+1} + M_p^{p+1}(u,\vartheta) - 2M^{p+1}_{\frac{\lambda}{2},p}(u,\vartheta)\right)\right. \\ & \left. -(p+1)\left(\vartheta^p + M^p_{\lambda,p}(u,\vartheta)\right)\left(\vartheta + M_p(u,\vartheta) - 2M_{\frac{\lambda}{2},p}(u,\vartheta)\right)\right],\ u,\vartheta>0, \end{aligned}$$

and $M_{t,p}(a,b) = [ta^p+(1-t)b^p]^{1/p}$ and $M_{1/2,p}(a,b) = M_p(a,b)$.

Proof. From Lemma 2.5 and using the power mean integral inequality, we have

$$\begin{aligned} & \left|(1-\lambda) f\left(M_p(a,b)\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\ & \leq \frac{b^p-a^p}{2p}\left[\int_0^{1/2}\frac{|\lambda-2t|}{M^{p-1}_{t,p}(a,b)}\left|f'\left(M_{t,p}(a,b)\right)\right|dt + \int_{1/2}^1 \frac{|2-\lambda-2t|}{M^{p-1}_{t,p}(a,b)}\left|f'\left(M_{t,p}(a,b)\right)\right|dt\right] \\ & \leq \frac{b^p-a^p}{2p}\left\{\left(\int_0^{1/2}\frac{|\lambda-2t|}{M^{p-1}_{t,p}(a,b)}dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\frac{|\lambda-2t|}{M^{p-1}_{t,p}(a,b)}\left|f'\left(M_{t,p}(a,b)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_{1/2}^1\frac{|2-\lambda-2t|}{M^{p-1}_{t,p}(a,b)}dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1\frac{|2-\lambda-2t|}{M^{p-1}_{t,p}(a,b)}\left|M_{t,p}(a,b)\right|^q dt\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Hence, by p-quasi convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned} & \left|(1-\lambda) f\left(M_p(a,b)\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \leq \frac{b^p-a^p}{2p} \\ & \times\left\{\left(\int_0^{1/2}\frac{|\lambda-2t|}{M^{p-1}_{t,p}(a,b)}dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\frac{|\lambda-2t|\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}}{M^{p-1}_{t,p}(a,b)}dt\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_{1/2}^1\frac{|2-\lambda-2t|}{M^{p-1}_{t,p}(a,b)}dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1\frac{|2-\lambda-2t|\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}}{M^{p-1}_{t,p}(a,b)}dt\right)^{\frac{1}{q}}\right\} \end{aligned}$$

$$\leq \frac{b^p - a^p}{2p} \left(C(\lambda; p; a, b) + C(\lambda; p; b, a) \right) \left(\max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{1/q}$$

It is easily check that

$$\int_0^{1/2} \frac{|\lambda - 2t|}{M_{t,p}^{p-1}(a,b)} dt = C(\lambda; p; a, b)$$

and

$$\int_{1/2}^{1} \frac{|2 - \lambda - 2t|}{M_{t,p}^{p-1}(a,b)} dt = C(\lambda; p; b, a).$$

This concludes the proof. □

In Theorem 2.6, if we take $p = 1$, then we obtain the following result for quasi-convex functions.

Corollary 2.7. *Under the assumptions Theorem 2.6 with $p = 1$, we have*

$$\left| (1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \left(\frac{f(a)+f(b)}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\, dx \right|$$

$$\leq \frac{b-a}{2} \left(C(\lambda; 1; a, b) + C(\lambda; 1; b, a) \right) \left(\max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{1/q}.$$

In Theorem 2.6, if we take $p = -1$, then we obtain the following result for harmonically quasi-convex functions.

Corollary 2.8. *Under the assumptions Theorem 2.6 with $p = -1$, we have*

$$\left| (1-\lambda) f\left(\frac{2ab}{a+b}\right) + \lambda \left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right|$$

$$\leq \frac{b-a}{2ab} \left(C(\lambda; -1; a, b) + C(\lambda; -1; b, a) \right) \left(\max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{1/q}.$$

Corollary 2.9. *Under the assumptions Theorem 2.6 with $q = 1$, we have*

$$\left| (1-\lambda) f\left(M_p(a,b)\right) + \lambda \left(\frac{f(a)+f(b)}{2}\right) - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right|$$

$$\leq \frac{b^p - a^p}{2p} \left(C(\lambda; p; a, b) + C(\lambda; p; b, a) \right) \left(\max \left\{ \left| f'(a) \right|, \left| f'(b) \right| \right\} \right). \tag{6}$$

Corollary 2.10. *Under the assumptions Theorem 2.6 with $\lambda = 0$, we have*

$$\left| f\left(M_p(a,b)\right) - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \leq \frac{b^p - a^p}{2p} \left(C(0; p; a, b) + C(0; p; b, a) \right) \left(\max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{1/q}$$

where for $p=-1$

$$C(0;-1;u,\vartheta)=\frac{1}{\left(\vartheta^{-1}-u^{-1}\right)^{2}}\left[-2\ln\left(\frac{M_{-1}(u,\vartheta)}{\vartheta}\right)+2\vartheta^{-1}\left(M_{-1}(u,\vartheta)-\vartheta\right)\right],$$

and for $p\in\mathbb{R}\backslash\{-1,0\}$

$$C(0;p;u,\vartheta)=\frac{p}{(p+1)\left(\vartheta^{p}-u^{p}\right)^{2}}\left[2\left(M_{p}^{p+1}(u,\vartheta)-\vartheta^{p+1}\right)-2(p+1)\vartheta^{p}\left(M_{p}(u,\vartheta)-\vartheta\right)\right].$$

Corollary 2.11. *Under the assumptions Theorem 2.6 with* $\lambda=1$*, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1-p}}dx\right|\leq\frac{b^{p}-a^{p}}{2p}\left(C(1;p;a,b)+C(1;p;b,a)\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{1/q}$$

where for $p=-1$

$$C(1;-1;u,\vartheta)=\frac{1}{\left(\vartheta^{p}-u^{p}\right)^{2}}\left[-2\ln\left(\frac{\vartheta}{M_{-1}(u,\vartheta)}\right)+\left(\vartheta^{-1}+M_{\lambda,-1}^{-1}(u,\vartheta)\right)\left(\vartheta-M_{-1}(u,\vartheta)\right)\right],$$

and for $p\in\mathbb{R}\backslash\{-1,0\}$

$$C(1;p;u,\vartheta)=\frac{p}{(p+1)\left(\vartheta^{p}-u^{p}\right)^{2}}\left[2\left(\vartheta^{p+1}-M_{p}^{p+1}(u,\vartheta)\right)-(p+1)\left(\vartheta^{p}+M_{\lambda,p}^{p}(u,\vartheta)\right)\left(\vartheta-M_{p}(u,\vartheta)\right)\right].$$

Corollary 2.12. *Under the assumptions Theorem 2.6 with* $\lambda=1/3$*, we have*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(M_{p}(a,b)\right)\right]-\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1-p}}dx\right|$$
$$\leq\frac{b^{p}-a^{p}}{2p}\left(C(1/3;p;a,b)+C(1/3;p;b,a)\right)\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{1/q}$$

wherewhere for $p=-1$

$$C(1/3;-1;u,\vartheta)=\frac{1}{\left(\vartheta^{-1}-u^{-1}\right)^{2}}\left[-2\ln\left(\frac{\vartheta M_{-1}(u,\vartheta)}{M_{\frac{1}{6},-1}^{2}(u,\vartheta)}\right)+\left(\vartheta^{-1}+M_{\frac{1}{3},-1}^{-1}(u,\vartheta)\right)\left(\vartheta+M_{-1}(u,\vartheta)-2M_{\frac{1}{6},-1}(u,\vartheta)\right)\right],$$

and for $p\in\mathbb{R}\backslash\{-1,0\}$

$$\begin{aligned}C(1/3;p;u,\vartheta)=&\frac{p}{(p+1)\left(\vartheta^{p}-u^{p}\right)^{2}}\left[2\left(\vartheta^{p+1}+M_{p}^{p+1}(u,\vartheta)-2M_{\frac{1}{6},p}^{p+1}(u,\vartheta)\right)\right.\\&\left.-(p+1)\left(\vartheta^{p}+M_{\frac{1}{3},p}^{p}(u,\vartheta)\right)\left(\vartheta+M_{p}(u,\vartheta)-2M_{\frac{1}{6},p}(u,\vartheta)\right)\right].\end{aligned}$$

3. Some Applications for Special Means

Let us recall the following special means of two nonnegative number a,b with $b>a$:

1. The arithmetic mean

$$A=A(a,b):=\frac{a+b}{2}.$$

2. The geometric mean

$$G = G(a,b) := \sqrt{ab}.$$

3. The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}.$$

4. The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

5. The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \ \ p \in \mathbb{R}\setminus\{-1,0\}.$$

6. The Identric mean

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

7. The power mean

$$M_p = M_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}, \ p \in \mathbb{R}\setminus\{0\}.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \le G \le L \le I \le A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 3.1. *Let $0 < a < b$, $p \in \mathbb{R}\setminus\{-1,-1/2,0\}$ and $\lambda \in [0,1]$. Then we have the following inequality*

$$\left|(1-\lambda)M_p^{p+1} + \lambda M_{p+1}^{p+1} - L_{2p}^{2p}L_{p-1}^{1-p}\right| \le \frac{(p+1)(b^p-a^p)}{2p}\left(C(\lambda;p;a,b)+C(\lambda;p;b,a)\right)\left(\max\{a^p,b^p\}\right),$$

where $C(\lambda;p;a,b)$ is defined as in Theorem 2.6.

Proof. The assertion follows from the inequality (6) in Corollary 2.9, for $f:(0,\infty)\to\mathbb{R}$, $f(x) = x^{p+1}/p+1$. □

Proposition 3.2. *Let $0 < a < b$ and $\lambda \in [0,1]$. Then we have the following inequality*

$$|(1-\lambda)\ln A + \lambda \ln G - \ln I| \le \frac{b-a}{2a}\left(C(\lambda;1;a,b)+C(\lambda;1;b,a)\right),$$

where $C(\lambda;p;a,b)$ is defined as in Theorem 2.6.

Proof. The assertion follows from the inequality (6) in Corollary 2.9, for $p = 1$ and $f:(0,\infty)\to\mathbb{R}$, $f(x) = \ln x$. □

4. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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