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Behavior of the Solutions of a Partial Differential Equation with a Piecewise Constant Argument

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Abstract. In this paper, we consider a partial differential equation with a piecewise constant argument. We study existence and uniqueness of the solutions of this equation. We also investigate oscillation, instability and stability of the solutions.

1. Introduction

Since the early 1980's, differential equations with piecewise constant arguments have attracted great deal of attention of researchers in science. Differential equations with piecewise constant arguments appear in diverse areas such as engineering, physics and mathematics. The work [1] covers a systematical study on mathematical models with piecewise constant arguments. Differential equations with piecewise constant arguments are closely related to difference and differential equations. Therefore, they are stated as hybrid dynamical systems [2]. The qualitative works such as oscillation, periodicity and convergence of solutions of ordinary differential equations with piecewise constant arguments have been studied in ([3]-[11]) .

But, there are only a few papers ([12]-[24]) for partial differential equations with piecewise constant arguments. The first fundamental paper [12] in this direction appeared in 1991. It has been shown that partial differential equations (PDE) with piecewise constant time naturally arise in the process of approximating PDE by using piecewise constant arguments. Thus, if in the equation

$$u_t = a^2 u_{xx} - bu,$$

which describes heat flow in a rod with both diffusion $a^2 u_{xx}$ along the rod and heat loss (or gain) across the lateral sides of the rod, the lateral heat change is measured at discrete times, then we get an equation with piecewise constant argument (EPCA) for $t \in [nh, (n+1)h]$, $n = 0, 1, \ldots,$

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x, nh),$$

where h is a positive constant. This equation can be written in the form

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x, [t/h]\, h).$$

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In [13], it is considered the diffusion-convection equation

$$u_t(x,t) = a^2 u_{xx}(x,t) - r u_x(x, [t/h]\, h)$$

which describes, for instance, the concentration $u(x,t)$ of a pollutant carried along in a stream moving with velocity r. The term $a^2 u_{xx}$ is the diffusion contribution and $-ru_x$ is the convection component which is measured at discrete times nh.

In 1992, Wiener and Debnath [14] considered partial differential equations with piecewise constant argument of the form

$$u_t(x,t) = a^2 u_{xx}(x,t) + b u_{xx}(x,[t]) \tag{1}$$

and

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - b u_{xx}(x,[t]). \tag{2}$$

They investigated qualitative properties of the solution $u(x,t) = 0$ of Eq. (1) and Eq. (2).

In 1997, the same authors [15] studied the following equations

$$u_t(x,t) = a^2 u_{xx}(x,t) - b u(x,[t])$$

and

$$u_t(x,t) = a^2 u_{xx}(x,t) - b u(x,[t + \frac{1}{2}])$$

and compared the behavior of solutions to these equations. In addition they considered the equation of neutral type

$$u_t(x,t) = a^2 u_{xx}(x,t) + b u_t(x,[t])$$

and studied the behavior of the solutions.

In 1999, Wiener and Heller [16] studied the system of neutral type

$$U_t(x,t) = AU_{xx}(x,t) + BU_t(x,[t + \frac{1}{2}])$$

and investigated, oscillatory and periodic solutions in the scalar case of this system with respect to different values of A and B.

In 2014, Wang and Wen [17] investigated the asymptotic stability of the analytic solutions and the numerical solutions of

$$u_t(x,t) = a^2 u_{xx}(x,t) + b u_{xx}(x,[t]) + c u_{xx}(x,[t+1]).$$

In 2015, Veloz and Pinto [18] studied the following equation with piecewise constant argument of generalized type of the form

$$u_t(x,t) = a^2(t) u_{xx}(x,t) - b(t) u(x,\gamma(t)),$$

where $\gamma(t)$ is a step function.

Wiener's book [13] is a useful source with respect to both ordinary and partial differential equations with piecewise constant arguments.

It is well known that if in an insulated rod of length l the temperature flows from $x = 0$ to $x = l$ provided that the heat energy is neither created nor destroyed in the interior of the rod, then the temperature satisfies the heat equation. Such an equation becomes more meaningful but more complicated when the diffusion term depends on the present time and also time delays. Moreover, as we know, there is not much work in PDE with piecewise constant arguments. Due to these reasons, we have been motivated to consider the following initial boundary value problem (IBVP)

$$u_t(x,t) = a^2u_{xx}(x,t) + bu_{xx}(x,[t-1]),\ t \geq 1, \quad (3)$$
$$u(0,t) = u(1,t) = 0, \quad (4)$$
$$u(x,0) = f(x), \quad (5)$$

where $a, b \in \mathbb{R}$ and $a \neq 0$, $u : \Omega = [0,1] \times (\{0\} \cup [1,\infty)) \to \mathbb{R}$, $[.]$ denotes the greatest integer function and f is continuous function on $[0,1]$.

This paper is organised as follows. In Section 2, we obtain the existence of solutions. In Section 3, we get some results about asymptotic stability, instability and oscillation properties.

2. Existence of Solutions

Definition 2.1. *A function $u(x,t)$ is called a solution of IBVP* (3) − (5) *if it satisfies the following conditions:*

(i) *$u(x,t)$ and $\frac{\partial u}{\partial x}$ are continuous in $\Omega = [0,1] \times (\{0\} \cup [1,\infty))$,*

(ii) *$\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ are continuous in Ω, with the possible exception at the points $(x,[t]) \in E = [0,1] \times \mathbb{N}$, $\mathbb{N} = \{0,1,2,\ldots\}$, where one-sided derivatives exist with respect to second argument,*

(iii) *$u(x,t)$ satisfies Eq.* (3) *in Ω, with the possible exception at the points $(x,[t]) \in E$ and conditions* (4), (5)

Theorem 2.2. *If $f(x)$ is a continuously differentiable function on the interval $0 \leq x \leq 1$ and satisfies the conditions $f(0) = f(1) = 0$, then a formal solution of IBVP* (3) − (5) *has the form*

$$u(x,t) = \sum_{k=1}^{\infty} A_k X_k(x) T_k(t),$$

where

$$A_k = \frac{2}{T_0} \int_0^1 f(\xi) \sin \pi k \xi d\xi,$$

$X_k(x) = \sin \pi kx$, $0 \leq x \leq 1$, $k = 1, 2, \ldots$, and

$$T_k(t) = e^{-a^2\pi^2k^2(t-[t])}T_{[t]} - \frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-[t])}\right)T_{[t]-1},\ 1 \leq t < \infty, \quad (6)$$

here $T_{[t]}$, for $t \in [n, n+1)$, is the unique solution of the difference equation

$$T_{n+1} - e^{-a^2\pi^2k^2}T_n + \frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2}\right)T_{n-1} = 0,\ n \geq 1, \quad (7)$$

with the initial conditions $T(0) = T_0 \neq 0$, $T(1) = T_1$.

Proof. By using the method of separation of variables, we seek the nonzero solution of (3) − (5) in the form

$$u(x,t) = X(x)T(t). \quad (8)$$

Substituting (8) into Eq. (3), we have

$$X(x)T'(t) = X''(x)(a^2T(t) + bT([t-1]))$$

or

$$\frac{T'(t)}{a^2T(t)+bT([t-1])}=\frac{X''(x)}{X(x)}=-\lambda^2.$$

Therefore, we generate, respectively, the BVP

$$\begin{cases} X''(x)+\lambda^2X(x)=0, \\ X(0)=X(1)=0, \end{cases} \tag{9}$$

and the differential equation with piecewise constant argument

$$T'(t)+a^2\lambda^2T(t)=-b\lambda^2T([t-1]),\ t\geq 1. \tag{10}$$

The general solution of the equation in (9) is

$$X(x)=c_1\cos\lambda x+c_2\sin\lambda x,$$

where c_1 and c_2 are arbitrary constants. From $X(0)=X(1)=0$, we get

$$\lambda_k=\pi k, \tag{11}$$

and

$$X_k(x)=\sin\pi kx,\ k=1,2,\ldots, \tag{12}$$

that are eigenfunctions of (9) . On the interval $[n,n+1)$, $n\geq 1$, by (11) , the equation (10) takes the form

$$T'(t)+a^2\pi^2k^2T(t)=-b\pi^2k^2T(n-1). \tag{13}$$

The solution of (13) with the condition $T(n)=T_n$ is

$$T(t)=e^{-a^2\pi^2k^2(t-n)}T(n)-\frac{b}{a^2}\left(1-e^{-a^2\pi^2k^2(t-n)}\right)T(n-1),\ n\leq t<n+1, \tag{14}$$

which implies (6) for $t\in[1,\infty)$ when we replace n by $[t]$. Furthermore, on the interval $n+1\leq t<n+2$, we get from (10)

$$T'(t)+a^2\pi^2k^2T(t)=-b\pi^2k^2T(n)$$

which has the solution

$$T(t)=e^{-a^2\pi^2k^2(t-n-1)}T(n+1)-\frac{b}{a^2}\left(1-e^{-a^2\pi^2k^2(t-n-1)}\right)T(n), \tag{15}$$

where $T(n+1)=T_{n+1}$. Let us denote the solutions (14) and (15), respectively, by $T_n(t)$ and $T_{n+1}(t)$. Since $T(t)$ is continuous at $t=n+1$,

$$T_n(n+1)=T_{n+1}(n+1).$$

Hence, we obtain the second order difference equation

$$T_{n+1}-e^{-a^2\pi^2k^2}T_n+\frac{b}{a^2}\left(1-e^{-a^2\pi^2k^2}\right)T_{n-1}=0,\ n\geq 1, \tag{16}$$

which is Eq. (7). It is noted that for the unique solution of (16) , we need the initial conditions

$$T(0)=T_0,\ T(1)=T_1. \tag{17}$$

The characteristic equation of (16) is

$$\lambda^2 - e^{-a^2\pi^2k^2}\lambda + \frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2}\right) = 0. \tag{18}$$

Now, let us investigate the solutions of Eq. (16) with respect to the characteristic roots λ_1 and λ_2 of Eq. (18) .
(i) If

$$\frac{b}{a^2} < \frac{e^{-2a^2\pi^2k^2}}{4(1 - e^{-a^2\pi^2k^2})}, \tag{19}$$

then the discriminant of Eq. (18) Δ is positive and therefore the roots λ_1 and λ_2 of (18) are real and distinct. In this case, the general solution of (16) is

$$T_n = k_1\lambda_1^n + k_2\lambda_2^n, \tag{20}$$

where k_1 and k_2 arbitrary constants. Applying the initial conditions (17) to (20),

$$T_0 = k_1 + k_2, \qquad T_1 = k_1\lambda_1 + k_2\lambda_2$$

and so

$$k_1 = \frac{T_0\lambda_2 - T_1}{\lambda_2 - \lambda_1}, \qquad k_2 = \frac{T_1 - T_0\lambda_1}{\lambda_2 - \lambda_1}.$$

Substituting k_1 and k_2 into (20) , we get the unique solution of the difference equation (16) with subject to the initial conditions (17) as

$$T_n = \frac{T_0\lambda_2 - T_1}{\lambda_2 - \lambda_1}\lambda_1^n + \frac{T_1 - T_0\lambda_1}{\lambda_2 - \lambda_1}\lambda_2^n. \tag{21}$$

Putting (21) into (14) , we have on the interval $n \le t < n + 1$

$$\begin{aligned} T_k(t) &= e^{-a^2\pi^2k^2(t-n)}\left[\frac{T_0\lambda_2 - T_1}{\lambda_2 - \lambda_1}\lambda_1^n + \frac{T_1 - T_0\lambda_1}{\lambda_2 - \lambda_1}\lambda_2^n\right] \\ &\quad -\frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-n)}\right)\left[\frac{T_0\lambda_2 - T_1}{\lambda_2 - \lambda_1}\lambda_1^{n-1} + \frac{T_1 - T_0\lambda_1}{\lambda_2 - \lambda_1}\lambda_2^{n-1}\right], \ k = 1, 2, \dots. \end{aligned}$$

This $T_k(t)$ can be written on the interval $1 \le t < \infty$ as

$$\begin{aligned} T_k(t) &= e^{-a^2\pi^2k^2(t-[t])}\left[\frac{T_0\lambda_2 - T_1}{\lambda_2 - \lambda_1}\lambda_1^{[t]} + \frac{T_1 - T_0\lambda_1}{\lambda_2 - \lambda_1}\lambda_2^{[t]}\right] \\ &\quad -\frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-[t])}\right)\left[\frac{T_0\lambda_2 - T_1}{\lambda_2 - \lambda_1}\lambda_1^{[t]-1} + \frac{T_1 - T_0\lambda_1}{\lambda_2 - \lambda_1}\lambda_2^{[t]-1}\right], \ k = 1, 2, \dots. \end{aligned} \tag{22}$$

(ii) If

$$\frac{b}{a^2} = \frac{e^{-2a^2\pi^2k^2}}{4(1 - e^{-a^2\pi^2k^2})},$$

then $\Delta = 0$. Hence $\lambda_1 = \lambda_2 = \lambda$, the general solution of (16) is

$$T_n = (k_1 + k_2n)\lambda^n, \tag{23}$$

where k_1, k_2 arbitrary constants. Applying the initial conditions (17) to (23),

$$k_1 = T_0, \qquad k_2 = \frac{T_1 - T_0\lambda}{\lambda}.$$

Hence, we have the unique solution of (16) − (17) as

$$T_n = \left[T_0 + \left(\frac{T_1 - T_0\lambda}{\lambda}\right)n\right]\lambda^n, \tag{24}$$

which is also the limiting case of (21) as $\lambda_1 \to \lambda_2$. Combining (24) and (14), we get

$$\begin{aligned} T_k(t) &= e^{-a^2\pi^2k^2(t-n)}\left[T_0 + \left(\frac{T_1 - T_0\lambda}{\lambda}\right)n\right]\lambda^n \\ &-\frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-n)}\right)\left[T_0 + \left(\frac{T_1 - T_0\lambda}{\lambda}\right)(n-1)\right]\lambda^{n-1}, \quad n \le t < n+1. \end{aligned}$$

This $T_k(t)$ can be calculated on the interval $1 \le t < \infty$ similar to (22).

(*iii*) If

$$\frac{b}{a^2} > \frac{e^{-2a^2\pi^2}}{4(1 - e^{-a^2\pi^2})},$$

then $\Delta < 0$. So λ_1 and λ_2 are complex conjugate roots. Let us take $\lambda_{1,2} = \alpha \pm i\beta$. The general solution of (16) is

$$T_n = r^n(k_1 \cos n\theta + k_2 \sin n\theta), \tag{25}$$

where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$, $\alpha, \beta \in \mathbb{R}$, $\beta \ne 0$ and $\theta \ne m\pi$, $m = 0, 1, 2, \ldots$. Using initial conditions (17) in (25), the unique solution of IVP (16) − (17)

$$T_n = r^n\left[T_0 \cos n\theta + \left(\frac{T_1 - rT_0\cos\theta}{r\sin\theta}\right)\sin n\theta\right]. \tag{26}$$

Combining (26) and (14), we get, $n \le t < n + 1$,

$$\begin{aligned} T_k(t) &= e^{-a^2\pi^2k^2(t-n)}r^n\left[T_0 \cos n\theta + \left(\frac{T_1 - rT_0\cos\theta}{r\sin\theta}\right)\sin n\theta\right] \\ &-\frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-n)}\right)r^{n-1}\left[T_0 \cos(n-1)\theta + \left(\frac{T_1 - rT_0\cos\theta}{r\sin\theta}\right)\sin(n-1)\theta\right]. \end{aligned}$$

This $T_k(t)$ can be found on the interval $1 \le t < \infty$ similar to (22). So, for $k = 1, 2, \ldots,$

$$u_k(x,t) = X_k(x)T_k(t) \tag{27}$$

satisfy both Eq. (3) and the boundary conditions (4). Also, by virtue of the principle of superposition, the series

$$u(x,t) = \sum_{k=1}^{\infty} A_k \sin \pi kx \, T_k(t) \tag{28}$$

is a solution of BVP (3) − (4), where the coefficient A_k are arbitrary constants. Applying the initial condition (5) to (28), we obtain

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} A_k \sin \pi kx \, T_k(0), \quad 0 \le x \le 1,$$

where

$$A_kT_k(0) = 2\int_0^1 f(\xi)\sin \pi k\xi d\xi.$$

Since $T_k(0) = T_0 \neq 0$ in three cases (i), (ii) and (iii), A_k are calculated uniquely as

$$A_k = \frac{2}{T_0}\int_0^1 f(\xi)\sin \pi k\xi d\xi. \tag{29}$$

Therefore, the series (28) with (29) denotes the formal solution of IBVP (3) – (5) . □

3. Main Results

In this section, we study asymptotic stability, instability and oscillation of the solutions. We construct our results with respect to the characteristic roots λ_1 and λ_2 of (18) .

3.1. λ_1 and λ_2 are Real

We give the following theorems under the condition

$$\frac{b}{a^2} < \frac{e^{-2a^2\pi^2k^2}}{4(1-e^{-a^2\pi^2k^2})}. \tag{30}$$

Theorem 3.1. *If*

$$-1 < \frac{b}{a^2} < 0, \tag{31}$$

then the solution $T_n = 0$ of difference equation (16) *and the solution $T_k = 0$ of Eq.* (10) *are asymptotically stable.*

Proof. Since the characteristic roots of (18) are real, we have

$$\left|\lambda_{1,2}\right| = \left|\frac{e^{-a^2\pi^2k^2}}{2} \pm \frac{\sqrt{e^{-2a^2\pi^2k^2} - 4\frac{b}{a^2}(1-e^{-a^2\pi^2k^2})}}{2}\right|. \tag{32}$$

From (31) , we get

$$\begin{aligned}\left|\lambda_{1,2}\right| &< \frac{1}{2}\left|e^{-a^2\pi^2k^2} + \sqrt{\left(e^{-a^2\pi^2k^2}-2\right)^2}\right| \\ &= \frac{1}{2}\left|e^{-a^2\pi^2k^2} + (-e^{-a^2\pi^2k^2}+2)\right|\end{aligned}$$

and

$$\left|\lambda_{1,2}\right| < 1.$$

It is well-known that the solution $T_n = 0$ of difference equation (16) is asymptotically stable if and only if

$$|\lambda_1| < 1 \text{ and } |\lambda_2| < 1. \tag{33}$$

Therefore the solution $T_n = 0$ of Eq. (16) is asymptotically stable.

Next we prove that the solution $T_k = 0$ of Eq. (10) is also asymptotically stable. If the solution $T_k = 0$ of Eq. (10) is stable and $\lim_{t\to\infty} |T_k(t)| = 0$, then the solution $T_k = 0$ is asymptotically stable.

The solution $T_k = 0$ of Eq. (10) is stable if, for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $|T_0| < \delta$ and $|T_1| < \delta$, $|T_k(t)| < \epsilon$.

From (6) and (21),

$$\begin{aligned} |T_k(t)| &= \Bigg| e^{-a^2\pi^2k^2(t-[t])}\left[\frac{T_0\lambda_2 - T_1}{\lambda_2-\lambda_1}\lambda_1^{[t]} + \frac{T_1 - T_0\lambda_1}{\lambda_2-\lambda_1}\lambda_2^{[t]}\right] \\ &\quad -\frac{b}{a^2}\left(1-e^{-a^2\pi^2k^2(t-[t])}\right)\left[\frac{T_0\lambda_2 - T_1}{\lambda_2-\lambda_1}\lambda_1^{[t]-1} + \frac{T_1 - T_0\lambda_1}{\lambda_2-\lambda_1}\lambda_2^{[t]-1}\right]\Bigg|. \end{aligned} \tag{34}$$

Since $e^{-a^2\pi^2k^2(t-[t])} < 1$ and by (31) , (34) is written as

$$\begin{aligned} |T_k(t)| &< |T_0|\,|\lambda_1|\,|\lambda_2|\left|\frac{\lambda_1^{[t]-1} - \lambda_2^{[t]-1}}{\lambda_2-\lambda_1}\right| + |T_1|\left|\frac{\lambda_2^{[t]} - \lambda_1^{[t]}}{\lambda_2-\lambda_1}\right| \\ &\quad + |T_0|\,|\lambda_1|\,|\lambda_2|\left|\frac{\lambda_1^{[t]-2} - \lambda_2^{[t]-2}}{\lambda_2-\lambda_1}\right| + |T_1|\left|\frac{\lambda_2^{[t]-1} - \lambda_1^{[t]-1}}{\lambda_2-\lambda_1}\right|. \end{aligned} \tag{35}$$

We know $0 < \lambda_1 < 1$ and $-1 < \lambda_2 < 0$. So, from (35) , we get

$$|T_k(t)| < 2\,|T_0| + 2\,|T_1|$$

or

$$|T_k(t)| < 4\delta = \epsilon.$$

So, $T_k = 0$ is stable.

Now, we show that $\lim_{t\to\infty} |T_k(t)| = 0$. From (35) , we have

$$\begin{aligned} 0 &< \lim_{t\to\infty} |T_k(t)| \\ &< \lim_{t\to\infty}\frac{1}{|\lambda_2-\lambda_1|}\Big(\left|\lambda_1^{[t]-1}\right|(|T_0|\,|\lambda_1|\,|\lambda_2| + |T_1|) + \left|\lambda_2^{[t]-1}\right|(|T_0|\,|\lambda_1|\,|\lambda_2| + |T_1|) \\ &\quad + \left|\lambda_1^{[t]}\right||T_1| + \left|\lambda_2^{[t]}\right||T_1| + \left|\lambda_1^{[t]-2}\right||T_0|\,|\lambda_1|\,|\lambda_2| + \left|\lambda_2^{[t]-2}\right||T_0|\,|\lambda_1|\,|\lambda_2|\Big). \end{aligned} \tag{36}$$

Also,

$$\begin{aligned} |\lambda_1| &< 1 \Longrightarrow \lim_{[t]\to\infty}\lambda_1^{[t]} = 0, \\ |\lambda_2| &< 1 \Longrightarrow \lim_{[t]\to\infty}\lambda_2^{[t]} = 0 \end{aligned} \tag{37}$$

and so (36) implies

$$\lim_{t\to\infty}|T_k(t)| = 0.$$

Therefore, the solution $T_k = 0$ is asymptotically stable. □

Theorem 3.2. *The condition*

$$\frac{b}{a^2} < -1 \tag{38}$$

is necessary and sufficient so that the solution $T_n = 0$ *of difference equation* (16) *is unstable.*

Proof. First, let

$$\frac{b}{a^2} < -1.$$

Then the characteristic roots of (18) are real. So we have

$$\begin{aligned}\lambda_1 &= \frac{e^{-a^2\pi^2k^2}}{2} + \frac{\sqrt{e^{-2a^2\pi^2k^2} - 4\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})}}{2} \\ &> \frac{1}{2}\left(e^{-a^2\pi^2k^2} + \sqrt{e^{-2a^2\pi^2k^2} + 4(1 - e^{-a^2\pi^2k^2})}\right) \\ &> 1\end{aligned}$$

and

$$\begin{aligned}\lambda_2 &= \frac{e^{-a^2\pi^2k^2}}{2} - \frac{\sqrt{e^{-2a^2\pi^2k^2} - 4\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})}}{2} \\ &< \frac{1}{2}\left(e^{-a^2\pi^2k^2} - \sqrt{e^{-2a^2\pi^2k^2} + 4(1 - e^{-a^2\pi^2k^2})}\right) \\ &< 0.\end{aligned}$$

Hence we find that at least one of the characteristic roots (λ_1 and λ_2) is bigger than 1. It is known that the solution $T_n = 0$ of difference equation (16) is unstable if and only if

$$|\lambda_1| > 1 \text{ or } |\lambda_2| > 1.$$

So the solution $T_n = 0$ of difference equation (16) is unstable.

Now, assume that the solution $T_n = 0$ of difference equation (16) is unstable, i.e, $\lambda_1 > 1$. So

$$\frac{e^{-a^2\pi^2k^2}}{2} + \frac{\sqrt{e^{-2a^2\pi^2k^2} - 4\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})}}{2} > 1. \tag{39}$$

From (39) , we get (38) . □

Theorem 3.3. *If* (38) *holds, then the solution* $T_k = 0$ *of Eq.* (10) *is unstable.*

Proof. Under this condition we know, from Theorem 3.2, that the solution $T_n = 0$ of the difference equation (16) is unstable, i.e, $\left|T_{[t]}\right| \to \infty$ and $\left|T_{[t]-1}\right| \to \infty$ as $t \to \infty$ with $T_0 \neq 0$ or $T_1 \neq 0$, considering (6) gives us

$$T_k(t) = e^{-a^2\pi^2k^2(t-[t])}T_{[t]} - \frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-[t])}\right)T_{[t]-1} \tag{40}$$

and so, $T_k(t) \to \infty$ as $t \to \infty$. Hence the solution $T_k = 0$ of Eq. (10) is unstable. □

3.2. λ_1 and λ_2 are Equal

Theorem 3.4. *If*

$$\frac{b}{a^2} = \frac{e^{-2a^2\pi^2k^2}}{4(1 - e^{-a^2\pi^2k^2})},\ k = 1, 2, \ldots, \tag{41}$$

then the solution $T_n = 0$ *of difference equation* (16) *is asymptotically stable, while the solution* $T_k = 0$ *of Eq.* (10) *is unstable.*

Proof. From (41), we get $\Delta = 0$, where Δ is the discriminant of Eq. (18). So we get

$$\lambda_{1,2} = \lambda = \frac{e^{-a^2\pi^2k^2}}{2}.$$

Since $0 < e^{-a^2\pi^2k^2} < 1$, we have

$$0 < \lambda < \frac{1}{2}.$$

Hence, the solution $T_n = 0$ is asymptotically stable.

Next we prove that the solution $T_k = 0$ of Eq. (10) is unstable. In this case, from (6) and (24), we get

$$\begin{aligned} T_k(t) &= (T_0\lambda + (T_1 - T_0\lambda)[t])\,\lambda^{[t]-2}\left(\lambda e^{-a^2\pi^2k^2(t-[t])} - \frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-[t])}\right)\right) \\ &\quad + \frac{b}{a^2}\left(1 - e^{-a^2\pi^2k^2(t-[t])}\right)(T_1 - T_0\lambda)\,\lambda^{[t]-2}. \end{aligned} \tag{42}$$

The term $[t]$ in (42) makes $T_k(t)$ unbounded, since $t \in [1,\infty)$. Hence $T_k = 0$ is unstable. □

3.3. λ_1 and λ_2 are Complex

We give the following theorems under the condition

$$\frac{b}{a^2} > \frac{e^{-2a^2\pi^2}}{4(1 - e^{-a^2\pi^2})}. \tag{43}$$

Theorem 3.5. *The condition*

$$\frac{b}{a^2} < \frac{1}{1 - e^{-a^2\pi^2k^2}} \tag{44}$$

is necessary and sufficient so that the solution $T_n = 0$ of difference equation (16) *and also the solution $T_k = 0$ of differential equation* (10) *are asymptotically stable.*

Proof. From (43), we know that $\Delta < 0$, so we have

$$\lambda_{1,2} = \frac{e^{-a^2\pi^2k^2}}{2} \pm \frac{i\sqrt{-e^{-2a^2\pi^2k^2} + 4\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})}}{2} \tag{45}$$

and

$$\begin{aligned} r &= \sqrt{\frac{1}{4}\left[e^{-2a^2\pi^2k^2} + \left(-e^{-2a^2\pi^2k^2} + 4\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})\right)\right]} \\ &= \sqrt{\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})}. \end{aligned} \tag{46}$$

From (46) and (44), we have

$$0 < r < 1$$

and the solution $T_n = 0$ of difference equation (16) is asymptotically stable.

Now, let the solution $T_n = 0$ of difference equation (16) is asymptotically stable. So we have

$$\left|\sqrt{\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})}\right| < 1.$$

By using the condition (43) , we get

$$\sqrt{\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})} < 1.$$

Taking square both side, we obtain (44) .

Next we prove that the solution $T_k = 0$ of Eq. (10) is asymptotically stable. From (6) and (26) , we have

$$\begin{aligned} |T_k(t)| &= \left| e^{-a^2\pi^2k^2(t-[t])} r^{[t]-1} \left(rT_0 \cos([t]\,\theta) + \left(\frac{T_1 - rT_0\cos\theta}{\sin\theta} \right) \sin([t]\,\theta) \right) \right. \\ &\quad \left. - \frac{b}{a^2} \left(1 - e^{-a^2\pi^2k^2(t-[t])} \right) r^{[t]-2} \left(rT_0 \cos(([t]-1)\theta) + \left(\frac{T_1 - rT_0\cos\theta}{\sin\theta} \right) \sin(([t]-1)\theta) \right) \right|. \end{aligned} \tag{47}$$

Since $e^{-a^2\pi^2k^2(t-[t])} < 1$, $\sin([t]\,\theta) < 1$, $\cos([t]\,\theta) < 1$ and (44) , we get

$$\begin{aligned} |T_k(t)| &< \left| r^{[t]-1} \right| \left(|r|\,|T_0| + |T_1| \left| \frac{1}{\sin\theta} \right| + |r|\,|T_0|\,|\cot\theta| \right) \\ &\quad + \frac{1}{1 - e^{-a^2\pi^2k^2}} \left| r^{[t]-2} \right| \left(|r|\,|T_0| + |T_1| \left| \frac{1}{\sin\theta} \right| + |r|\,|T_0|\,|\cot\theta| \right). \end{aligned} \tag{48}$$

Since $|r| < 1$, $|T_0| < \delta$ and $|T_1| < \delta$,

$$|T(t)| < \left(1 + \frac{1}{1 - e^{-a^2\pi^2k^2}} \right) \left(1 + |\cot\theta| + \left| \frac{1}{\sin\theta} \right| \right) \delta = \epsilon.$$

Hence $T_k = 0$ is stable.

We show $\lim_{t\to\infty} |T_k(t)| = 0$. If we take limit of (48) , then we get

$$\begin{aligned} 0 &< \lim_{t\to\infty} |T_k(t)| \\ &< \lim_{t\to\infty} \left[\left| r^{[t]-1} \right| \left(|r|\,|T_0| + |r|\,|T_0|\,|\cot\theta| + |T_1| \left| \frac{1}{\sin\theta} \right| \right) \right. \\ &\quad \left. + \frac{1}{1 - e^{-a^2\pi^2k^2}} \left| r^{[t]-2} \right| \left(|r|\,|T_0| + |r|\,|T_0|\,|\cot\theta| + |T_1| \left| \frac{1}{\sin\theta} \right| \right) \right] = 0, \end{aligned} \tag{49}$$

since

$$|r| < 1 \Leftrightarrow \lim_{[t]\to\infty} r^{[t]} = 0. \tag{50}$$

So, the solution $T_k = 0$ of Eq. (10) is asymptotically stable.

Conversely, if $T_k = 0$ is asymptotically stable, then, from (49) and (50) , $|r| < 1$. Hence we get (44) . □

Theorem 3.6. *The condition*

$$\frac{b}{a^2} > \frac{1}{1 - e^{-a^2\pi^2}} \tag{51}$$

is necessary and sufficient so that the solution $T_n = 0$ of difference equation (16) is unstable.

Proof. From (43) , we know that $\Delta < 0$, so we have (45) and (46). Since (51) is hold, we get

$$\frac{b}{a^2} > \frac{1}{1 - e^{-a^2\pi^2k^2}}, \; k = 1, 2 \ldots. \tag{52}$$

In view of (52) , (46) yields

$$r > 1. \tag{53}$$

We know that the solution $T_n = 0$ of difference equation (16) is unstable if and only if

$$|r| > 1.$$

Hence the solution $T_n = 0$ of Eq. (16) is unstable.

Now, let the solution $T_n = 0$ of difference equation (16) is unstable. So we have (53) and

$$\sqrt{\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})} > 1.$$

Hence we get

$$\frac{b}{a^2} > \frac{1}{1 - e^{-a^2\pi^2k^2}}, \; k = 1, 2 \ldots. \tag{54}$$

Since the function $\frac{1}{1 - e^{-a^2\pi^2k^2}}$ is decreasing, we get (51). □

Theorem 3.7. *If the inequality* (51) *is satisfied, then the solution* $T_k = 0$ *of Eq.* (10) *is unstable.*

Proof. Let (51) is satisfied. From Theorem (3.6) , we get $\left|T_{[t]}\right| \to \infty$ and $\left|T_{[t]-1}\right| \to \infty$ as $t \to \infty$ with $T_0 \neq 0$ or $T_1 \neq 0$. Hence, from (40) , $|T_k(t)| \to \infty$ as $t \to \infty$. So the solution $T_k = 0$ of Eq. (10) is unstable. □

Theorem 3.8. *If* (43) *is satisfied, then the solutions* T_n *of difference equation* (16) *are oscillatory.*

Proof. By (43), we get

$$\frac{4b}{a^2} > \frac{e^{-2a^2\pi^2}}{1 - e^{-a^2\pi^2}} \geq \frac{e^{-2a^2\pi^2k^2}}{1 - e^{-a^2\pi^2k^2}}, \; k = 1, 2, \ldots, \tag{55}$$

From (55),

$$\Delta = \sqrt{e^{-2a^2\pi^2k^2} - 4\frac{b}{a^2}(1 - e^{-a^2\pi^2k^2})} < 0$$

and the characteristic roots λ_1, λ_2 are conjugate complex. So the solutions are oscillatory.

Hence the solutions $T_k(t)$ of Eq. (10) are oscillatory. □

Theorem 3.9. *If* $\frac{b}{a^2} > 0$, *then the solutions* T_n *of difference equation* (16) *are oscillatory, for sufficiently large k.*

Proof. For sufficiently large k, we have

$$\lambda_{1,2} = \pm\sqrt{-\frac{b}{a^2}}.$$

Since $-\frac{b}{a^2} < 0$, λ_1 and λ_2 are conjugate complex. So the solutions T_n of Eq. (16) and the solutions $T(t)$ of Eq. (10) are oscillatory. □

Remark 3.10. *Since* $|\sin \pi kx| < 1$, *asymptotic stability or instability of solution* $u(x, t) = 0$ *of BVP* (3) – (5) *coincide with asymptotic stability or instability of solution* $T_k(t) = 0$ *of differential equations* (10) .

Remark 3.11. *Since* $|\sin \pi kx| < 1$, *oscillation of the solutions* $u(x, t)$ *of BVP* (3) – (5) *coincide with oscillation of the solutions* $T_k(t)$ *of differential equations* (10) .

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