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Pointwise Pseudo-Slant Submanifolds of a Kenmotsu Manifold

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Abstract. In the present article, we have investigated pointwise pseudo-slant submanifolds of Kenmotsu manifolds and have sought conditions under which these submanifolds are warped products. To this end first, it is shown that these submanifolds can not be expressed as non-trivial doubly warped product submanifolds. However, as there exist non-trivial (single) warped product submanifolds of a Kenmotsu manifold, we have worked out characterizations in terms of a canonical structure T and the shape operator under which a pointwise pseudo slant submanifold of a Kenmotsu manifold reduces to a warped product submanifold.

1. Introducion

Slant immersions in complex geometry were defined by B.Y.Chen [6] as a natural generalization to both holomorphic and totally real immersions . The notion was further generalized by him in [9] when he considered pointwise slant submanifolds in almost Hermitian manifolds. In [16], A.Lotta extended the notion to the settings of almost contact metric manifold and obtained some important properties of such immersions.

There are two important classes of submanifolds in Kaehlerian as well as in contact settings, the first one is the class of submanifolds which admit an invariant distribution and the other one consists of submanifolds which admit an anti-invariant distribution. If the complementary distribution is slant, the submanifolds of class one are semi-slant submanifolds whereas submanifolds of the other class are pseudo-slant submanifolds. J.L. Cabrerizo et al [3] defined and studied semi-slant submanifolds in the setting of almost contact metric manifolds whose study in almost Hermitian manifolds was initiated by N.Papaghiuc [18]. Recently, K.S.Park [19] studied pointwise slant and pointwise semi-slant submanifolds of almost contact metric manifolds. He obtained some geometrically important properties of these manifolds. Now, it is natural seek differential geometric properties of pseudo-slant or more generally pointwise pseudo-slant submanifolds of almost contact metric manifolds. Since, proper slant distribution is not integrable on a submanifold of a Sasakian manifold, we aim to investigate pseudo-slant submanifolds of a Kenmotsu manifold.The paper is organised as follows:

Section 2 deals with basic concepts, formulas and some known result that are relevant for the subsequent sections. In section 3, point wise pseudo-slant submanifolds of a Kenmotsu manifold are studied. Some formulas are derived that helped in obtaining integrability conditions for the distributions on a pseudo-slant

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submanifold of Kenmotsu manifold and revealing the geometry of the leaves of the distributions. Section 4 is devoted to study doubly warped product submanifolds of a Kenmotsu manifold. After going through various properties, we established that non-trivial doubly warped products are non-existent in a Kenmotsu manifold. This leads us to take up single warped product submanifolds of Kenmotsu manifolds. Along the years, there has been interests to find analogous of classical deRahm's Theorem to warped products, we have considered in section 5 warped product submanifolds of a Kenmotsu manifold $\bar{M}$ whose one of the factors is a $\phi-$ anti-invariant submanifold of $\bar{M}$. We worked out formulas for ∇T. Since point wise pseudo slant submanifolds are a special case of these submanifolds, these formulas are used to obtain characterizations under which a pointwise pseudo slant submanifold is a warped product submanifold.

2. Preliminaries

All manifolds, vector bundles, functions etc. are assumed to be of class C^∞. The set of locally defined sections of a vector bundle E is denoted by $\Gamma(E)$.

An almost contact structure on a $(2n+1)$-dimensional manifold $\bar{M}$ is defined by a $(1,1)$ tensor field ϕ, a vector field ξ and the dual 1-form η of ξ satisfying the following properties

$$\phi^2 = -I + \eta \otimes \xi, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0, \;\; \eta(\xi) = 1.$$

There always exist a Riemannian metric g on $\bar{M}$ satisfying the following compatibility condition

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V) \tag{1}$$

for any vector fields U, V on $\bar{M}$. An almost contact manifold endowed with a compatible Riemannian metric is called *an almost contact metric manifold*. It is easy to observe that the Riemannian metric defined in (1) satisfies

$$g(\phi U, V) + g(U, \phi V) = 0, \;\; g(U, \xi) = \eta(U). \tag{2}$$

If $\bar{\nabla}$ is the Levi-Civita connection on $(\bar{M}, g)$, then the covariant derivative of ϕ is defined as

$$(\bar{\nabla}_U \phi)V = \bar{\nabla}_U \phi V - \phi \bar{\nabla}_U V. \tag{3}$$

Let Ω be the fundamental 2-form on $\bar{M}$, i.e, $\Omega(U, V) = g(U, \phi V)$. If $\Omega = d\eta$, $\bar{M}$ is said to be a *contact manifold*. If ξ is a Killing vector field with respect to g, the contact metric structure is called a *K-contact structure*. It is easy to show that a contact metric manifold is K-contact if $\bar{\nabla}_U \xi = -\phi U$ for each vector field U on $\bar{M}$. The almost contact structure on $\bar{M}$ is said to be *normal* if $[\phi, \phi] + 2d\eta \otimes \xi = 0$ where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. A *Sasakian manifold* is a normal contact metric manifold. It is known that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\bar{\nabla}_U \phi)V = g(U, V)\xi - \eta(V)U.$$

S.Tanno [20] classified connected almost contact metric manifolds whose automorphism groups posses the maximum dimension. For such a manifold, the sectional curvature of a plane section containing ξ is a constant c. One of the classes of this classification consists of warped product $R \times_f \mathbb{C}^n$ with $c < 0$. These manifolds are not Sasakian and are characterized by a tensorial equation :

$$(\bar{\nabla}_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U. \tag{4}$$

Kenmotsu [15] explored some fundamental differential geometric properties of these spaces and therefore they are named as *Kenmotsu manifolds*.

It can also be seen that on a Kenmotsu manifold $\bar{M}$,

$$\bar{\nabla}_U \xi = -\phi^2 U = U - \eta(U)\xi \tag{5}$$

for all vector fields U, V on $\bar{M}$.

Throughout, we denote by M a submanifold of an almost contact metric manifold $\bar{M}$ with TM and $T^{\perp}M$ as the tangent and normal bundles on M respectively. If ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on TM and $T^{\perp}M$ then Gauss and Weingarton formulae are

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{6}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \tag{7}$$

for any $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$. A_N and h respectively denote the shape operator (corresponding to the normal vector field N) and the second fundamental form of the immersion of M into $\bar{M}$. The two are related as

$$g(A_N U, V) = g(h(U, V), N), \tag{8}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the induced Riemannian metric on M.

If the structure vector field ξ is tangential to the submanifold M of a Kenmotsu manifold $\bar{M}$, then for any $U \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$ by formula (5), (6) and (8)

$$h(U, \xi) = 0, \quad A_N\xi = 0 \tag{9}$$

If ξ is normal to the submanifold, then by (5) and (7), it follows that

$$A_\xi = -Id \quad \text{and} \quad \nabla^{\perp}_U \xi = 0. \tag{10}$$

For any $U \in \Gamma(TM)$, we write

$$TU = tan(\phi U) \quad \text{and} \quad FU = nor(\phi U).$$

where $'tan'$ and $'nor'$ are the natural projections associated with the direct decomposition:

$$T_p\bar{M} = T_pM \oplus T^{\perp}_p M, \quad p \in M.$$

Similarly, for $N \in \Gamma(T^{\perp}M)$, we write

$$tN = tan(\phi N) \quad \text{and} \quad fN = nor(\phi N)$$

The tensor fields on M determined by the endomorphism T and the normal valued 1-form F are denoted by the same letters T and F respectively. Similarly, t and f are tangential and normal valued (1,1)-tensor fields on the normal bundle of M. The covariant differentiations of the tensor fields T and F are defined respectively as:

$$(\bar{\nabla}_U T)V = \nabla_U TV - T\nabla_U V \tag{11}$$

$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V. \tag{12}$$

If ξ is tangential to a submanifold M of a Kenmotsu manifold, then by virtue of Gauss-Weingarten formulae and (4), we obtain

$$(\bar{\nabla}_U T)V = A_{FV}U + th(U, V) - \eta(V)TU + g(TU, V)\xi \tag{13}$$

for any $U, V \in \Gamma(TM)$. Taking account of (9) in the above formula, we deduce that

$$(\bar{\nabla}_\xi T)U = 0, \quad and \tag{14}$$

$$(\bar{\nabla}_U T)\xi = -TU \tag{15}$$

Let D be a differentiable distribution on M. For any $p \in M$ and $U \in D_p$, if the vectors U and ξ_p are linearly independent then the angle $\theta(U) \in [0, \frac{\pi}{2}]$ between ϕU and D_p is known as slant angle of U. If $\theta(U)$ does not depend on the choice of $p \in M$ and $U \in D_p$, then D is said to be a *slant distribution* on M (with the slant angle θ). Usually, a slant distribution with slant angle θ is denoted by D^θ. Invariant and anti-invariant distributions are slant distributions with slant angle $\theta = 0$ and $\frac{\pi}{2}$ respectively [16]. A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant submanifold if the tangent bundle TM is slant. A slant submanifold which is neither invariant nor anti-invariant is called a *proper slant submanifold*. It is easy to observe that :

Theorem 2.1. [24] *Let $\bar{M}$ be an almost contact metric manifold with $dim(\bar{M}) = 2n+1$ and M, an $(n+1)$-dimensional anti-invariant submanifold of $\bar{M}$ then ξ is tangent to M.*

Conversely if the structure vector field ξ is tangential to a submanifold M of an almost contact metric manifold $\bar{M}$, then TM admits the orthogonal direct decomposition

$$TM = D \oplus < \xi >$$

where D denotes the distribution orthogonal complementary to the one dimensional distribution generated by ξ. A submanifold M of an almost contact metric manifold (tangent to ξ) is slant if and only if D is a slant distribution on M. In particular, D may be $\phi-$invariant or $\phi-$anti invariant distribution accordingly M is invariant or ant-invariant submanifold of $\bar{M}$ tangent to ξ. More generally semi-invariant, semi-slant and pseudo slant submanifolds of almost contact metric manifolds tangential to ξ are studied (cf. [3], [11], [13],[14], etc.). In the setting of contact metric manifolds, we have

Proposition 2.2. [4] *Let M be a submanifold of a contact metric manifold $\bar{M}$ tangential to the structure vector field ξ. Then M is anti-invariant if and only if D is involutive.*

Proposition 2.3. *A proper slant distribution on a submanifold of contact metric manifold tangent to the structure vector field ξ is not involutive.*

With regards to the case when $\xi \in T^\perp M$, A.Lotta proved the following which generalise a well known result of Yano and Kon [24].

Theorem 2.4. [16] *Let M be a submanifold of a contact metric manifold $\bar{M}$. If ξ is orthogonal to M, then M is anti-invariant.*

Corollary 2.5. *Proper slant, semi-slant and pseudo-slant submanifolds orthogonal to ξ are non existent in Sasakian manifolds.*

For the existence of slant submanifolds of almost contact metric manifold, we have the following characterization :

Theorem 2.6. [4] *Let M be a submanifold of an almost contact metric manifold tangent to the structure vector field ξ. Then M is slant if and only if there exits a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi).$$

Furthermore, in such case if θ is the slant angle of M, then $\lambda = cos^2\theta$.

If M is a slant submanifold of an almost contact metric manifold $\bar{M}$ with slant angle θ, then it follows from the above Theorem that

$$g(TX, TY) = cos^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{16}$$

$$g(FX, FY) = sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{17}$$

B.Y. Chen [9] considered a generalised version of slant submanifolds by defining pointwise slant submanifolds of almost Hermitian manifolds. K.S. Park [19] extended the notion to the setting of almost contact metric manifold as follows:

Let M be a submanifold of an almost contact metric manifold and let $M_p = \{U \in T_pM / g(U, \xi_p) = 0\}$. Then M is called a *pointwise slant submanifold* if at each given point $p \in M$, the angle $\theta = \theta(U)$ between ϕU and the space M_p is independent of the choice of $U \in M_p$. In this case the angle θ is called a *slant function* on M. If the slant function θ on a pointwise slant submanifold M is non-constant, then M is called a *proper pointwise slant submanifold*.

The advantage of defining the subspace M_p is that many of the results on slant distribution have a simpler version e.g, if we denote by D, the space $\bigcup_{p\in M} M_p$ i.e. $D = \bigcup_{p\in M} \{U \in T_p(M)/g(U, \xi_p) = 0\}$, then Theorem 2.6 is extended as :

Theorem 2.7. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$. Then M is a pointwise slant submanifold of $\bar{M}$ if and only if $T^2 = -cos^2\theta I$ on D for some function $\theta : M \to R$*

Park obtained some important properties of pointwise slant submanifold of almost contact metric manifold e.g.,

Proposition 2.8. [19] *Any two dimensional submanifold of an almost contact metric manifold is a pointwise slant submanifold.*

Proposition 2.9. [19] *A submanifold M of an almost contact metric manifold is a pointwise slant submanifold if and only if*

$$g(TX, TY) = 0$$

whenever $g(X, Y) = 0$ for $X, Y \in D$.

3. Pointwise Pseudo-slant submanifolds of a Kenmotsu manifold

Let $(\bar{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold. A submanifold M of $\bar{M}$ is called a *pointwise pseudo-slant submanifold* if there is a pointwise slant distribution $D_1 \subseteq TM$ such that its orthogonal complement D_2 is ϕ-anti invariant. That is

$$TM = D_1 \oplus D_2,$$

where $\phi D_2 \subset \Gamma(T^{\perp}M)$, and at each given point $p \in M$, the angle $\theta = \theta(X)$ between ϕX and the space $(D_1)_p$ is constant for non zero $X \in (D_1)_p$.

If the structure vector field ξ is tangential to the submanifold M, then for any $U \in \Gamma(TM)$, we may write

$$U = BU + CU + \eta(U)\xi,$$

where $BU \in D_1$ and $CU \in D_2$.

Note 3.1. *If the structure vector field ξ is tangential to the submanifold M, then it can not lie in D_1 as $\theta(\xi_p)$ is not defined for any $p \in M$. In fact, no part of ξ can be tangential to D_1. In this case, ϕ-anti invariant distribution orthogonal to D_1 is infact $D_2 \oplus < \xi >$.*

If we denote the dimensions of D_1 and D_2 by d_1 and d_2 respectively, then we have the following cases :

(i) If $d_1 = 0$, then M is a ϕ-anti invariant submanifold of $\bar{M}$.
(ii) If $d_2 = 0$ and $\theta = 0$, then M is a ϕ-invariant submanifold of $\bar{M}$.

(iii) If $d_2 = 0$ and $\theta \neq 0$, then M is a proper pointwise slant submanifold with slant angel θ (In this case $TM = D_1 \oplus <\xi>$).

(iv) If $d_1, d_2 \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a *proper pointwise pseudo-slant submanifold* of $\bar{M}$.

On a pointwise pseudo-slant submanifold, the following relations can be checked easily

$$T(D_1) \subseteq D_1, \ \ T(D_2) = 0, \ \ tFY = -sin^2\theta \, Y \tag{18}$$

for each $Y \in D_1$.

For any $U \in \Gamma(T\bar{M})$ we write

$$U = \mathcal{H}U + \mathcal{V}U \tag{19}$$

where $\mathcal{H}U \in \Gamma(TM)$ and $\mathcal{V}U \in \Gamma(T^{\perp}M)$. Then

$$T^2 + tf = -I + \eta \otimes \mathcal{H}\xi, \ \ FT + fF = \eta \otimes \mathcal{V}\xi \tag{20}$$

$$Tt + tf = \eta \otimes \mathcal{H}\xi, \ \ FT + f^2 = -I + \eta \otimes \mathcal{V}\xi. \tag{21}$$

The normal bundle of a pointwise pseudo-slant submanifold of an almost contact metric manifold admits the following orthogonal direct decomposition

$$T^{\perp}M = FD_1 \oplus FD_2 \oplus \mu \tag{22}$$

where μ is the orthogonal complement of $\phi(TM)$ in $T^{\perp}M$.

Now, For any $U_1 \in D_1$ and $U_2 \in D_2$,

$$g(FU_1, FU_2) = g(\phi U_1, \phi U_2) = g(U_1, U_2) = 0.$$

That means FD_1 and FD_2 are orthogonal to each other.

Proposition 3.2. *Let M be a pointwise pseudo-slant submanifold of an almost contact metric manifold. Then μ is a ϕ-invariant normal sub bundle if and only if either $\mu \subseteq ker(\eta)$ or $D_2 \subseteq ker(\eta)$.*

Proof. For any $N \in \Gamma(\mu)$ and $U \in \Gamma(TM)$,

$$g(\phi N, U) = 0 \ \text{ as } \ g(N, FBU) = 0 = g(N, \phi CU).$$

That shows that

$$\phi\mu \subseteq T^{\perp}M. \tag{23}$$

Further, for $X \in \Gamma(D_1)$,

$$g(\phi N, FX) = g(\phi N, \phi X) = g(N, X) - \eta(N)\eta(X) = 0, \tag{24}$$

That is, $\phi\mu$ is orthogonal to FD_1. Now, for $Z \in \Gamma(D_2)$,

$$\begin{aligned} g(\phi N, \phi Z) &= g(N, Z) - \eta(N)\eta(Z) \\ &= -\eta(N)\eta(Z). \end{aligned} \tag{25}$$

That is, $g(\phi N, \phi Z) = 0$ if and only if $\eta(N)\eta(Z) = 0$ The assertion follows from (23), (24) and (25). □

Proposition 3.3. *Let M be a pointwise pseudo-slant submanifold of a Kenmotsu manifold, then*

(i) $A_{\phi Z}W = A_{\phi W}Z$

(ii) $g(\nabla^{\perp}_X FY, \phi Z) = sin^2\theta \, \{g(\nabla_X Y, Z) + \eta(Z)g(X, Y)\} - g(h(X, Z), FTY)$

(iii) $g(\nabla^{\perp}_Z \phi W, FX) = sin^2\theta g(\nabla_Z W, X) + g(h(Z, W), FTX)$

for each $X, Y \in \Gamma(D_1)$ *and* $Z, W \in \Gamma(D_2)$ *.*

Proof. Consider $g(A_{\phi Z} W, U)$ for any $U \in \Gamma(TM)$,

$$\begin{aligned} g(A_{\phi Z} W, U) =& g(h(U, W), \phi Z) \\ =& g(\bar{\nabla}_U W, \phi Z) \\ =& - g(\phi \bar{\nabla}_U W, Z). \end{aligned}$$

Now, as $g((\bar{\nabla}_U \phi)W, Z) = 0$ by virtue of formula (4) and the fact that $g(\phi U, Z) = 0$, the right hand side of the above equation on using (3), reduces to -$g(\bar{\nabla}_U \phi W, Z)$, which oin view of Weingarten formula is same as: $g(A_{\phi W} Z, U)$. This prves that $A_{\phi Z} W = A_{\phi W} Z$.

By using (2) and (3), we have

$$\begin{aligned} g(\nabla^{\perp}_X FY, \phi Z) =& g(\bar{\nabla}_X FY, \phi Z) \\ =& g((\bar{\nabla}_X \phi)FY, Z) - g(\bar{\nabla}_X \phi FY, Z) \end{aligned}$$

Writing $\phi FY = tFY + fFY$ and making use of (4) and (17), the right hand side of the above equation reduces to

$$\eta(Z) sin^2\theta g(X, Y) - g(\nabla_X tFY, Z) - g(\bar{\nabla}_X fFY, Z)$$

which on making use of (18),(20),(7) and (8) takes the form

$$\eta(Z) sin^2\theta g(X, Y) + sin2\theta\ X(\theta) g(Y, Z) + sin^2\theta g(\nabla_X Y, Z) - g(h(X, Z), FTY).$$

As $g(Y, Z) = 0$, we obtain that

$$g(\nabla^{\perp}_X FY, \phi Z) = sin^2\ \theta(g(\nabla_X Y, Z) + \eta(Z) g(X, Y)) - g(h(X, Z), FTY).$$

This proves the second part. For the third part, as FD_1 and FD_2 are orthogonal, we have

$$\begin{aligned} g(\bar{\nabla}_Z \phi W, FX) = -g(\phi W, \bar{\nabla}_Z FX) =& g(W, \phi \bar{\nabla}_Z FX) \\ =& g(W, \bar{\nabla}_Z \phi FX) - g(W, (\bar{\nabla}_Z \phi) FX) \end{aligned}$$

The second term in the right hand side of the above equation is zero by virtue of (4), whereas the first term is written as $g(W, \bar{\nabla}_Z tFX) + g(W, \bar{\nabla}_Z fFX)$. Therefore, the equation takes the from :

$$\begin{aligned} g(\bar{\nabla}_Z \phi W, FX) =& g(W, \bar{\nabla}_Z - sin^2\theta X) - g(W, \bar{\nabla}_Z FTX) \\ =& -sin2\theta\ Z(\theta) g(W, X) - sin^2\theta g(W, \nabla_Z X) + g(A_{FTX} Z, W) \\ =& sin^2\theta\ g(\nabla_Z W, X) + g(h(Z, W), FTX). \end{aligned}$$

This proves the third part and the proposition completely. □

Theorem 3.4. *Let M be a proper pointwise pseudo-slant submanifold of a Kenmotsu manifold* $\bar{M}$*. Then the distribution* D_1 *is involutive on M if and only if*

$$g(A_{FZ} X, TY) - g(A_{FZ} TX, Y) = g(A_{FTY} X, Z) - g(A_{FTX} Y, Z)$$

for $X, Y \in \Gamma(D_1)$ *and* $Z \in \Gamma(D_2)$*.*

Proof. By virtue of (5) and the fact that $\eta(X) = 0$, we have

$$g(\nabla_X Y, \xi) = \eta(\nabla_X Y) = -\, g(Y, \nabla_X \xi) = -\, g(X, Y).$$

Therefore,

$$g([X, Y], \xi) = 0.$$

Further for any $Z \in \Gamma(D_2)$,

$$g(\nabla_X Y, Z) = g(\phi\bar{\nabla}_X Y, \phi Z) + \eta(\bar{\nabla}_X Y)\eta(Z).$$

By formula (4), $g((\bar{\nabla}_X \phi)Y, \phi Z) = 0$ and $\eta(\nabla_X Y) = -g(X, Y)$. Taking account of these observations while using (3), the above equation takes the form :

$$g(\nabla_X Y, Z) = g(\bar{\nabla}_X \phi Y, \phi Z) - \eta(Z) g(X, Y) = g(\bar{\nabla}_X TY, \phi Z) + g(\bar{\nabla}_X FY, \phi Z) - \eta(Z) g(X, Y).$$

Making use of Gauss formula and Proposition3.3 in the above equation, we obtain

$$cos^2\theta \; g(\nabla_X Y, Z) = g(A_{\phi Z} X, TY) - g(A_{FTY} X, Z) - cos^2\theta \; \eta(Z) g(X, Y).$$

Interchanging X and Y and subtracting the obtained equation from the above, we get

$$cos^2\theta \; g([X, Y], Z) = g(A_{\phi Z} X, TY) - g(A_{\phi Z} TX, Y) + g(A_{FTX} Y, Z) - g(A_{FTY} X, Z).$$

Since M is proper pointwise pseudo-slant, D_1 is involutive if and only if

$$g(A_{\phi Z} X, TY) - g(A_{\phi Z} TX, Y) = g(A_{FTY} X, Z) - g(A_{FTX} Y, Z).$$

This proves the Theorem. □

Proposition 3.5. *Let M be a proper pointwise pseudo-slant submanifold of a Kenmotsu $\bar{M}$ such that ξ is tangential to the submanifold M. Then*

$$g(\nabla_\xi TX, Z) = 0$$

for any $X \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof.

$$g(\nabla_\xi TX, Z) = g(\bar{\nabla}_\xi TX, Z) = g(\bar{\nabla}_\xi \phi X, Z) - g(\bar{\nabla}_\xi FX, Z) = g((\bar{\nabla}_\xi \phi)\, X, Z) - g(\bar{\nabla}_\xi X, \phi Z) + g(A_{FX}\xi, Z).$$

The right hand side is identically zero by virtue of (4),(6),(8) and (9). This proves the proposition. □

Lemma 3.6. *Let M be a proper pointwise pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ tangent to the structure vector field ξ. Then $[Z, \xi] \in \Gamma(D_2)$ for any $Z \in \Gamma(D_2)$.*

Proof. For $X \in \Gamma(D_1)$, we have

$$g([Z,\xi],TX) = g(\bar{\nabla}_Z\xi,TX) - g(\bar{\nabla}_\xi Z,TX).$$

The first term in the right hand side of the above equation is zero by virtue of (5) whereas the second term on using the fact that D_1 and D_2 are orthogonal complementary, reduces to $g(\bar{\nabla}_\xi TX, Z)$ which is zero in view of proposition 3.5. Hence

$$g([Z,\xi],TX) = 0$$

This proves the Lemma. □

Lemma 3.7. *Let M be a pointwise pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then*

$$g(\nabla_Z W, X) = sec^2\theta\{g(h(Z,W),FTX) - g(h(TX,Z),FW) \tag{26}$$

for each $X \in D_1$ and $Z, W \in D_2$.

Proof. .

$$\begin{aligned} g(\nabla_Z W,X) = g(\bar{\nabla}_Z W,X) &= g(\phi\bar{\nabla}_Z W,\phi X) + \eta(\bar{\nabla}_Z W)\eta(X) \\ &= g(\phi\bar{\nabla}_Z W,\phi X) \\ &= g(\bar{\nabla}_Z\phi W,\phi X) - g((\bar{\nabla}_Z\phi)W,\phi X) \\ &= g(\bar{\nabla}_Z\phi W,TX) + g(\bar{\nabla}_Z\phi W,FX) \\ &\quad - g(\phi Z,W)\eta(\phi X) + \eta(W)g(\phi Z,\phi X) \\ &= -g(A_{\phi W}Z,TX) + g(\bar{\nabla}_Z\phi W,FX) \end{aligned}$$

substituting from part (*iii*) of proposition 3.3, the above equation reduces to

$$cos^2\theta\ g(\nabla_Z W,X) = g(h(Z,W),FTX) - g(A_{\phi W}Z,TX)$$

That proves the lemma. □

As an immediate consequence of the above lemma and proposition 3.3 part (*i*), we have

$$g([Z,W],X) = 0 \tag{27}$$

Further it is easy to show that

$$g(\nabla_Z W,\xi) = Z\eta(W) + \eta(Z)\eta(W) - g(Z,W).$$

Hence,

$$g([Z,W],\xi) = Z\eta(W) - W\eta(Z). \tag{28}$$

If ξ is normal to the submanifold M, then from (27) and (28), we deduce that D_2 is involutive on M. However, if $\xi \in TM$, then the integrability of the distribution $D_2\oplus < \xi >$ follows from (27) and (28). Thus, we have

Theorem 3.8. *Let $\bar{M}$ be a Kenmotsu manifold and M a proper pointwise pseudo-slant submanifold of $\bar{M}$. Then D_2 as well as $D_2\oplus < \xi >$ are involutive on M.*

As a consequence of formula (26),we have

Corollary 3.9. *Let M be a pointwise pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$. Then the anti-invariant distribution defines a totally geodesic foliation on M if and only if*

$$g(h(Z,X),\phi W) = g(h(Z,W),FX)$$

for any $X \in \Gamma(D_1)$ and $Z, W \in \Gamma(D_2\oplus < \xi >)$.

4. Doubly warped product submanifolds of a Kenmotsu manifold

Our aim in this section is to study submanifolds of a Kenmotsu manifold which are doubly warped products. That is, doubly warped product manifolds isometrically immersed in a Kenmotsu manifold. On analysing these submanifolds, we deduce that non-trivial doubly warped products are non-existent in a Kenmotsu manifold. In fact, it is shown that pseudo-slant submanifolds of Kenmotsu manifolds can be realized only as single warped product submanifolds.

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and let $f_1 : N_1 \to (0, \infty)$ and $f_2 : N_2 \to (0, \infty)$ be smooth functions. The *doubly warped product* $M =_{f_2} N_1 \times_{f_1} N_2$ is the product manifold $N_1 \times N_2$ endowed with the metric $g = f_2^2 g_1 + f_1^2 g_2$. More precisely, if $\pi : N_1 \times N_2 \to N_1$ and $\tau : N_1 \times N_2 \to N_2$ are natural projections, the metric g is defined by

$$g = (f_2 \circ \tau)^2 \pi^* g_1 + (f_1 \circ \pi)^2 \tau^* g_2. \tag{29}$$

The functions f_1 and f_2 are called *warping functions*.

If either $f_1 = 1$ or $f_2 = 1$ but not both, then we obtain a (single) warped product. If both $f_1 = 1 = f_2$, we have a Riemannian product $N_1 \times N_2$, usually we call it a *trivial warped product*. If neither f_1 nor f_2 is constant, we have a *non-trivial doubly warped product*.

If ∇' and ∇'' are the Levi-Civita connections of the Riemannian metric g_1 and g_2 respectively, then the Levi-Civita connection ∇ of the doubly warped product metric g on M is expressed as :

$$\nabla_{U_1} V_1 = \nabla'_{U_1} V_1 - \frac{f_2^2}{f_1^2} g_1(U_1, V_1) \nabla''(ln f_2), \tag{30}$$

$$\nabla_{U_2} V_2 = \nabla''_{U_2} V_2 - \frac{{f_1}^2}{{f_2}^2} g(U_2, V_2) \nabla'(ln f_1) \tag{31}$$

$$\nabla_{U_1} U_2 = (U_2 ln f_2) U_1 + (U_1 ln f_1) U_2. \tag{32}$$

for all $U_1, V_1 \in \Gamma(TN_1)$ and $U_2, V_2 \in \Gamma(TN_2)$. Here $\nabla'(ln f_1)$ and $\nabla''(ln f_2)$ denote the gradient of $ln f_1$ and $ln f_2$ with respect to the metrics g_1 and g_2 respectively [22]. In terms of the Riemannian metric g on M and the Levi-Civita connection ∇ on M, formulae (30) and (31) are respectively written as :

$$\nabla_{U_1} V_1 = \nabla'_{U_1} V_1 - g(U_1, V_1) \nabla ln f_2, \tag{33}$$

$$\nabla_{U_2} V_2 = \nabla''_{U_2} V_2 - g(U_2, V_2) \nabla ln f_1. \tag{34}$$

where $\nabla ln f$ is defined as $g(\nabla ln f, U) = U ln f$.

By using the covariant derivative formula for the doubly warped products (c.f. [1]), the following result is obtained in [23] .

Proposition 4.1. *Let $M =_{f_2} N_1 \times_{f_1} N_2$ be a doubly warped product manifold with metric $g = {f_2}^2 g_1 + {f_1}^2 g_2$. Then*

(i) The leaves $N_1 \times \{q\}$ and the fibers $\{p\} \times N_2$ of the doubly warped products are totally umbilic.

(ii) The leaf $N_1 \times \{q\}$ is totaly geodesic if $grad_{N_2}(f_2)|_q = 0$. Similarly, the fibers $\{p\} \times N_2$ is totally geodesic if $grad_{N_1}(f_1)|_p = 0$.

Let $M =_{f_2} N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\bar{M}$. If we denote by σ_1 and σ_2 the second fundamental form of N_1 and N_2 respectively in M, then

$$\sigma_1(U_1, V_1) = -\frac{f_2^2}{f_1^2} g_1(U_1, V_1) \nabla''(ln f_2) = -g(U_1, V_1) \nabla ln f_2 \tag{35}$$

and

$$\sigma_2(U_2, V_2) = -\frac{f_1^2}{f_2^2} g_2(U_2, V_2) \nabla'(ln f_1) = -g(U_2, V_2) \nabla ln f_1, \tag{36}$$

for all $U_1, V_1 \in \Gamma(TN_1)$ and $U_2, V_2 \in \Gamma(TN_2)$. That is, both the factors namely N_1 and N_2 of M are totally umbilical in M with mean curvature vectors $\nabla ln f_2$ and $\nabla ln f_1$ respectively.

Proposition 4.2. *Let $M =_{f_2} N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\bar{M}$. Then*

$$g(h(U_1, U_2), FV_1) = g(h(V_1, U_2), FU_1), \tag{37}$$

and

$$g(h(U_1, U_2), FV_2) = g(h(U_1, V_2), FU_2) \tag{38}$$

for $U_1, V_1 \in \Gamma(TN_1)$ and $U_2, V_2 \in \Gamma(TN_2)$.

Proof. By virtue of Gauss formula, we may write

$$g(h(U_1, U_2), FV_1) = g(\bar{\nabla}_{U_2} U_1, \phi V_1) - g(\nabla_{U_2} U_1, TV_1)$$

Making use of (2),(3),(4),(32) and the fact that $TU_i \in \Gamma(TN_i)$ $(1 \leq i \leq 2)$ the above equation takes the form

$$\begin{aligned} g(h(U_1, U_2), FV_1) =&(U_2 lnf_2) g(TU_1, V_1) - g(\bar{\nabla}_{U_2} \phi U_1, V_1) \\ =&(U_2 lnf_2) g(TU_1, V_1) - g(\nabla_{U_2} TU_1, V_1) \\ &- g(\bar{\nabla}_{U_2} FU_1, V_1). \end{aligned}$$

Further applying (7),(8),(32) and the fact that $TU_i \in \Gamma(TN_i)$, the right hand side reduces to $g(h(V_1, U_2), FU_1)$. This proves the first part of the Proposition.

For the second part, on writing

$$g(h(U_1, U_2), FV_2) = g(\bar{\nabla}_{U_1} U_2, \phi V_2) - g(\nabla_{U_1} U_2, TV_2)$$

and working along the same lines as in the proof of (37), we obtain that

$$g(h(U_1, U_2), FV_2) = g(h(U_1, V_2), FU_2).$$

□

Lemma 4.3. *Let $M =_{f_2} N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\bar{M}$ with only one of the factors a ϕ-anti invariant submanifold of $\bar{M}$. Then $U_1 lnf_1 = \eta(U_1)$ (resp. $U_2 lnf_2 = \eta(U_2)$) if N_1 (resp. N_2) is ϕ-anti invariant.*

Proof. For $U_1, V_1 \in \Gamma(TN_1)$ and $U_2, V_2 \in \Gamma(TN_2)$,

$$\begin{aligned} g(h(U_1, U_2), FV_2) =&g(\bar{\nabla}_{U_2} U_1, \phi V_2) - g(\nabla_{U_2} U_1, TV_2) \\ =& - g(\phi \bar{\nabla}_{U_2} U_1, V_2) + (U_1 lnf_1) g(TU_2, V_2) \end{aligned}$$

The first term in the right hand side of the above equation on making use of (4),(6), (7),(8) and (32) is written as

$$-\eta(U_1) g(TU_2, V_2) - (TU_1 lnf_1) g(U_2, V_2) + g(h(U_2, V_2), FU_1).$$

That makes the equation to take the form:

$$\begin{aligned} g(h(U_1, U_2), FV_2) =&(U_1 lnf_1 - \eta(U_1)) g(TU_2, V_2) \\ &- (TU_1 lnf_1) g(U_2, V_2) + g(h(U_2, V_2), FU_1). \end{aligned} \tag{39}$$

Similarly, on writing

$$g(h(U_1, U_2), FV_1) = g(\bar{\nabla}_{U_1} U_2, \phi V_1) - g(\nabla_{U_1} U_2, TV_1),$$

and making use of (2),(3),(4) and (32), the above equation takes the form

$$g(h(U_1, U_2), FV_1) = (U_2 lnf_2)g(TU_1, V_1) - \eta(U_2)g(TU_1, V_1) \\ - g(\bar{\nabla}_{U_1}\phi U_2, V_1).$$

On applying (6),(7),(8) and (32), the last term in the right hand side of the above equation reduces to

$$-(TU_2 lnf_2)g(U_1, V_1) + g(h(U_1, V_1), FU_2),$$

and thus we obtain

$$g(h(U_1, U_2), FV_1) = g(h(U_1, V_1), FU_2) - (TU_2 lnf_2)g(U_1, V_1) \\ + (U_2 lnf_2 - \eta(U_2))g(TU_1, V_1). \tag{40}$$

If N_1 is ϕ-anti invariant, then (39) can be written as

$$g(h(U_1, U_2), FV_2) = (U_1 lnf_1 - \eta(U_1))g(TU_2, V_2) + g(h(U_2, V_2), FU_1),$$

or

$$g(h(U_1, U_2), FV_2) - g(h(U_2, V_2), FU_1) = (U_1 lnf_1 - \eta(U_1))g(TU_2, V_2). \tag{41}$$

The left hand side of (41) is symmetric in U_2, V_2 whereas the right hand side is skew-symmetric in U_2, V_2. This fact together with the assumption that N_2 is not anti-invariant, gives

$$U_1 lnf_1 = \eta(U_1). \tag{42}$$

If N_2 is ϕ-anti-invariant, then equation (40) takes the form :

$$g(h(U_1, U_2), FV_1) - g(h(U_1, V_1), FU_2) = (U_2 lnf_2 - \eta(U_2))g(TU_1, V_1). \tag{43}$$

Since, the left hand side of (43) is symmetric in U_1, V_1, the right hand side is skew-symmetric in U_1, V_1 and N_1 is not anti-invariant, we deduce that

$$U_2 lnf_2 = \eta(U_2). \tag{44}$$

This proves the Lemma. □

Theorem 4.4. *There does not exist a non-trivial doubly warped product submanifold of a Kenmotsu manifold normal to the structure vector field ξ.*

Proof. Let $M = {}_{f_2}N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\bar{M}$ with $\xi \in \Gamma(T^\perp M)$. Then by (40), we have

$$g(h(U_1, U_2), FV_1) = g(h(U_1, V_1), FU_2) - (TU_2 lnf_2)g(U_1, V_1) \\ + (U_2 lnf_2)g(TU_1, V_1)$$

for any $U_1, V_1 \in \Gamma(TN_1)$ and $U_2 \in \Gamma(TN_2)$. Interchanging U_1, V_1 and subtracting the obtained equation from the above while using (37) and the symmetry of g and h we obtain

$$(U_2 lnf_2)g(TU_1, V_1) = 0$$

It follows from the above that either N_1 is ϕ-anti invariant or f_2 is constant on N_2. If N_1 is ϕ-anti invariant then by Lemma 4.3 and the assumption that $\xi \in T^\perp M$, $U_1 lnf_1 = \eta(U_1) = 0$. That is f_1 is constant on N_1, which means either f_1 is constant or f_2 is constant.This proves that M is infact a single warped product. Similarly, by considering equation (39), one can argue on the same lines that either f_1 is constant on N_1 or f_2 is constant on N_2, proving that there does not exist a non trivial doubly warped product submanifold M of a Kenmotsu manifold such that ξ is normal to M. □

This leads us to consider doubly warped product submanifolds of Kenmotsu manifolds with structure vector field tangential to the submanifold.

Let $M =_{f_2} N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\bar{M}$ such that $\xi \in \Gamma TM$.

If ξ_1 and ξ_2 are components of ξ along N_1 and N_2 respectively then by virtue of formulae (5),(6) and (9),

$$\nabla_{U_1}\xi = U_1 - \eta(U_1)\xi. \tag{45}$$

On writing $\xi = \xi_1 + \xi_2$ and making use of (32) and (33), the above equation reduces to

$$\nabla'_{U_1}\xi_1 - \eta(U_1)\nabla lnf_2 + (U_1 lnf_1)\xi_2 + (\xi_2 lnf_2) = U_1 - \eta(U_1)\xi_1 - \eta(U_1)\xi_2$$

On comparing components along N_1 and N_2 in the above equation, we obtain

$$\nabla'_{U_1}\xi_1 + \eta(U_1)\xi_1 = (1 - \xi_2 lnf_2)U_1 \tag{46}$$

and

$$\eta(U_1)\nabla lnf_2 = (\eta(U_1) + U_1 lnf_1)\xi_2. \tag{47}$$

Similarly writing $\nabla_{U_2}\xi = U_2 - \eta(U_2)\xi$ and proceeding along the same lines, we obtain

$$\nabla''_{U_2}\xi_2 + \eta(U_2)\xi_2 = (1 - \xi_1 lnf_1)U_2, \tag{48}$$

and

$$\eta(U_2)\nabla lnf_1 = (\eta(U_2) + U_2 lnf_2)\xi_1. \tag{49}$$

From (47) and (49), we observe that

$$\begin{cases} \text{If } \xi_1 \text{ and } \xi_2 \text{ are both non zero, then } \nabla(\mathrm{lnf_1}) \text{ is along} \\ \xi_1 \text{ and } \nabla(\mathrm{lnf_2}) \text{ is along } \xi_2. \end{cases} \tag{50}$$

Now, taking $U_1 = \xi_1$ in (46), we get

$$\xi_1 = (1 - \xi_2 lnf_2)\xi_1$$

which implies that

$$\xi_2 lnf_2 = 0$$

That is,

$$g(\nabla lnf_2, \xi_2) = 0. \tag{51}$$

From observation (50) and equation (51), we find that $\nabla lnf_2 = 0$. That is f_2 is constant.

Similarly, taking $U_2 = \xi_2$ in equation (48), we find that

$$\xi_1 lnf_1 = 0$$

i.e.,

$$g(\nabla lnf_1, \xi_1) = 0.$$

Again, the above equation together with observation (50) yields that f_1 is constant on N_1. Hence, in this case M is simply a Riemannian product of N_1 and N_2.

In particular, if ξ lies completely along one of the factors of M, then we have

Case (*i*) when $\xi_2 = 0$, then by equation (47), $\nabla lnf_2 = 0$. That is, f_2 is constant along N_2.

Case (*ii*) when $\xi_1 = 0$, then by equation (49), $\nabla lnf_1 = 0$, which means f_1 is constant along N_1.

The above findings can be summarized as :

Theorem 4.5. *Let $M = _{f_2} N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\bar{M}$ such that the structure vector field ξ is tangent to M. If ξ has a non-trivial components along N_1 and N_2, then M is a Riemannian product of N_1 and N_2 (i.e. a trivial warped product). However, if ξ is tangent to the first factor of M, then f_2 is constant whereas if ξ is tangent to N_2, then f_1 is constant.*

Corollary 4.6. *There does not exist a non-trivial doubly warped product submanifold of a Kenmotsu manifold tangent to the structure vector field ξ.*

5. Warped product submanifolds of a Kenmotsu manifold with one of the factors a ϕ-anti invariant submanifold

In view of Theorem 4.5 doubly warped product submanifolds $_{f_2}N_1 \times_{f_1} N_2$ of Kenmotsu manifolds are trivial i.e. in this case, either f_1 or f_2 is constant. Therefore, the only warped products in a Kenmotsu manifold are single warped products with structure vector field ξ tangent to the submanifold (c.f. Theorem4.4).

Warped product manifolds were introduced by R.L.Bishop and B.O'Neill [2] as a generalized version of product manifolds by homothetically warping the product metric on to the fibers. The study of warped products with extrinsic geometric point of view was initiated by B.Y.Chen [7, 8] when he considered *CR*-submanifolds of a Kaehler manifold as warped products.Our aim in this section is consider pseudo-slant submanifolds of a Kenmotsu manifold as warped products.

We begin the proceedings by stating an immediate consequence of Theorem 4.5

Proposition 5.1. *There does not exist a (single) non-trivial warped product submanifold of a Kenmotsu manifold such that the structure vector field ξ is tangential to the second factor of the submanifold.*

Hence, the possible non-trivial warped product submanifold M of a Kenmotsu manifold has the form $N_1 \times_f N_2$ with structure vector field ξ tangent to the first factor N_1 of the warped product. In this case, as an immediate consequence of formula (32), we have

$$\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = (U_1 lnf) U_2 \tag{52}$$

for each $U_1 \in \Gamma(TN_1)$ and $U_2 \in \Gamma(TN_2)$. Further, it follows from formula (35) that $\sigma_1(U_1, V_1) = 0$ for all $U_1, V_1 \in \Gamma(TN_1)$ i.e., N_1 is totally geodesic in M and by formula (36), we have

$$\sigma_2(U_2, V_2) = -g(U_2, V_2)\nabla lnf, \tag{53}$$

which shows that N_2 is totaly umbilical in M with mean curvature vector ∇lnf.

Proposition 5.2. *Let $N_1 \times_f N_2$ be a warped product submanifold of a Kenmotsu manifold $\bar{M}$. If the structure vector field $\xi = \frac{\partial}{\partial t}$, then the warping function f satisfies: $f(t) = e^t$.*

Proof. From formula (5), (52) and the fact that ξ is tangential to N_1, we have

$$(\xi lnf) U_2 = U_2,$$

which implies that $f(t) = e^t$.
□

From now on, we assume that M is a warped product submanifold of a Kenmotsu manifold $\bar{M}$ with only one of the factors a ϕ-anti invariant submanifold of $\bar{M}$.

First, we prove few preparatory results.

Theorem 5.3. *If M is a warped product submanifold of a Kenmotsu manifold $\bar{M}$, with only one of the factors a ϕ-anti invariant submanifold of $\bar{M}$, then for each vector field Z tangential to the anti-invariant factor of M and $U, V \in \Gamma(TM)$,*

$$g(h(U,V),\phi Z) = g(h(U,Z),FV) = g(h(V,Z),FU), \tag{54}$$

when N_1 is ϕ-anti invariant and

$$g(h(U,V),\phi Z) = g(h(U,Z),FV) - (TVlnf)g(U,Z), \tag{55}$$

when N_2 is ϕ-anti invariant.

Proof. Writing $U = U_1 + U_2$ with $U_1 \in \Gamma(TN_1)$ and $U_2 \in \Gamma(TN_2)$, we have

$$g(\nabla_U TV, Z) = g(\nabla_{U_1} TV, Z) + g(\nabla_{U_2} TV, Z).$$

If N_1 is ϕ-anti invariant, then the first term in the right hand side of the above equation vanishes by virtue of formula (52) and the fact that in this case either $TV = 0$ or it lies in $\Gamma(TN_2)$, whereas the second term on using (52) takes the form: $(Zlnf)g(TU,V)$. Thus, we have

$$g(\nabla_U TV, Z) = (Zlnf)g(TU,V). \tag{56}$$

Similarly, when N_2 is ϕ-anti invariant, then making use of formula (52) and the fact that N_1 is totally geodesic in M, we obtain

$$g(\nabla_U TV, Z) = (TVlnf)g(U,Z). \tag{57}$$

Now, by formula (2) and (6), we may write

$$\begin{aligned} g(h(U,V),\phi Z) &= g(\bar\nabla_U V,\phi Z) = -g(\phi\bar\nabla_U V, Z) \\ &= g((\bar\nabla_U\phi)V, Z) - g(\bar\nabla_U\phi V, Z). \end{aligned}$$

Making use of (4),(6),(7) and (8) on the right hand side of the above equation gives

$$g(h(U,V),\phi Z) = g(TU,V)\eta(Z) - g(\nabla_U TV, Z) + g(h(U,Z),FV) \tag{58}$$

If N_1 is ϕ-anti invariant, then on substituting from (56), the above equation yields

$$g(h(U,V),\phi Z) = (Zlnf - \eta(Z))g(U,TV) + g(h(U,Z),FV).$$

The first term in the right hand side of the above equation vanishes by virtue of Lemma 4.3. That is, in this case, we have

$$g(h(U,V),\phi Z) = g(h(U,Z),FV)$$

Similarly, when N_2 is ϕ-anti invariant, then $\eta(Z) = 0$ as ξ is tangential to N_1. Taking account of this fact and substituting from (57) into (58), we obtain that in this case,

$$g(h(U,V),\phi Z) = g(h(U,Z),FV) - TVlnfg(U,Z)$$

□

Corollary 5.4. *Let $M = N_1 \times_f N_2$ be a warped product submanifold of a Kenmotsu manifold $\bar M$ with only one of the factors a ϕ-anti invariant submanifold of $\bar M$. Then for $U_1 \in \Gamma(TN_1)$ and $U_2 \in \Gamma(TN_2)$,*

$$A_{FU_1}U_2 = A_{FU_2}U_1$$

when N_1 is ϕ-anti invariant and

$$A_{FU_1}U_2 - A_{FU_2}U_1 = (TU_1lnf)U_2$$

when N_2 is ϕ-anti invariant.

Proof. If N_1 is ϕ-anti invariant, then on taking $V = U_2$ and $Z = U_1$ in (54) gives

$$g(h(U, U_2), FU_1) = g(h(U, U_1), FU_2)$$

for each $U \in \Gamma(TM)$, which in view of (8) yields

$$A_{FU_1}U_2 = A_{FU_2}U_1$$

Similarly, if N_2 is ϕ-anti invariant, then on taking $V = U_1$ and $Z = U_2$ in (55) and using (8), we obtain

$$A_{FU_1}U_2 - A_{FU_2}U_1 = (TU_1 lnf)U_2$$

This proves the Corollary. □

Now, we prove some formulas for later use.

Lemma 5.5. *Let M be a warped product submanifold of a Kenmotsu manifold $\bar{M}$ with first factor a ϕ-anti invariant submanifold of $\bar{M}$. Then*

(i) $(\bar{\nabla}_U T)Z = -(Zlnf)TU$
(ii) $(\bar{\nabla}_Z T)U = 0$
(iii) $C(\bar{\nabla}_X T)Y = g(TX, Y)\nabla lnf$

for any $U \in \Gamma(TM)$, $Z \in \Gamma(TN_1)$ and $X, Y \in \Gamma(TN_2)$.

Proof. By (5.3), we have

$$g(A_{FZ}U, V) = -g(th(U, Z), V)$$

i.e.,

$$g(A_{FZ}U + th(U, Z), V) = 0$$

for any $U, V \in \Gamma(TM)$. That means,

$$A_{FZ}U + th(U, Z) = 0. \tag{59}$$

Making use of this identity and the fact that $g(TU, Z) = 0$ in formula (13), we get

$$(\bar{\nabla}_U T)Z = -\eta(Z)TU.$$

Using now that $\eta(Z) = Zlnf$, we obtain the statement (*i*).
Now,by formula(2.11), we have

$$(\bar{\nabla}_Z T)U = \nabla_Z TU - T\nabla_Z U \tag{60}$$

Writing $U = U_1 + U_2$ with $U_1 \in \Gamma(TN_1)$ and $U_2 \in \Gamma(TN_2)$ and using (52),we get

$$\nabla_Z TU = \nabla_Z TU_1 + \nabla_Z TU_2$$

The first term in the right hand side of the above equation is zero as $TU_1 = 0$, whereas by formula (52), $\nabla_Z TU_2 = (Zlnf)TU_2$. It then follows that

$$\nabla_Z TU = (Zlnf)TU_2$$

On the other hand, since $\nabla_Z U_1 \in \Gamma(TN_1)$ and N_1 is $\phi-$ anti invariant, $T\nabla_Z U_1 = 0$. Further, by (52) $T\nabla_Z U_2 = (Zlnf)TU_2$. That gives

$$T\nabla_Z U = (Zlnf)TU_2$$

Taking accounts of these observations in equation (60), we get $(\bar{\nabla}_Z T)U = 0$, proving the statement (*ii*) of the Lemma.

Now, for any $X, Y \in \Gamma(TN_2)$, by formula (11), we may write

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y.$$

If ∇'' denotes the Levi-Civita connection on TN_2 and σ_2 the second fundamental form of the immersion of N_2 into M, then the above equation , on using the Gauss formula, is expressed as

$$(\bar{\nabla}_X T)Y = \nabla''_X TY + \sigma_2(X, TY) - T\nabla''_X Y - T\sigma_2(X, Y).$$

As $\sigma_2 \in \Gamma(TN_1)$, $T\sigma_2(X, Y) = 0$ and by formula (53),$\sigma_2(X, TY) = -g(X, TY)\nabla lnf$, the above equation yields,

$$(\bar{\nabla}_X T)Y = (\nabla_X T)Y - g(X, TY)\nabla lnf, \tag{61}$$

where $\nabla''_X TY - T\nabla''_X Y$ is denoted by $(\nabla_X T)Y$. It follows from (61) that

$$C(\bar{\nabla}_X T)Y = g(TX, Y)\nabla lnf.$$

This proves part (*iii*) and the Lemma completely. □

Pseudo-slant submanifolds of Kenmotsu manifolds are a special case of submanifolds admitting a ϕ-anti invariant distribution. Our aim in the remainder of this section, is to study these submanifolds in a Kenmotsu manifold.

If $N_\perp$ and N_θ denote respectively ϕ-anti invariant and pointwise slant submanifold (with slant function θ) of a Kenmotsu manifold $\bar{M}$ such that $M_1 = N_\perp \times_f N_\theta$ admits an isometric immersion into $\bar{M}$, then M_1 is a point wise pseudo-slant warped product submanifold of $\bar{M}$. By virtue of Proposition (5.1), the structure vector field in this case, is tangential to the submanifold $N_\perp$. More precisely, if $D^\perp$ and D^θ denote ϕ-anti invariant and point wise slant distributions on M_1 such that both are involutive, then

$$TN_\perp = D^\perp \oplus <\xi> \quad \text{and} \quad TN_\theta = D^\theta$$

In this section, we investigate characterizations under which a pseudo-slant submanifold of a Kenmotsu manifold is a warped product submanifold. The following Theorem was proved by S.Hiepko [10] that we will be using to obtain the characterizations.

Theorem 5.6. *Let F be a vector sub bundle in the tangent bundle of a Riemannian manifold M and let $F^\perp$ be its normal bundle. Assume that the two distributions are both involutive and the integral manifold of F (resp.$F^\perp$) are extrinsic spheres (resp. totally geodesic). Then M is locally isometric to a warped product $M_1 \times_f M_2$. Moreover, if M is simply connected and complete there exists a global isometry of M with a warped product.*

Now, we prove

Theorem 5.7. *A proper pointwise pseudo-slant submanifold M_1 of a Kenmotsu manifold is a warped product submanifold of the type $N_\perp \times_f N_\theta$ if and only if there is a function μ on M_1 with $X\mu = 0$ for all $X \in \Gamma(D^\theta)$ such that*

$$g((\bar{\nabla}_U T)TV, Z) = cos^2(\theta)(Z\mu)\, g(BU, BV) \tag{62}$$

for all $U, V \in \Gamma(TM_1)$ and $Z \in \Gamma(D^\perp \oplus <\xi>)$.

Proof. If $M_1 = N_\perp \times_f N_\theta$ be a warped product submanifold of a Kenmotsu manifold $\bar{M}$, then we write

$$(\bar{\nabla}_U T)V = (\bar{\nabla}_U T)BV + (\bar{\nabla}_U T)CV + \eta(V)(\bar{\nabla}_U T)\xi \tag{63}$$

By Lemma 5.5 and formula (15), the last two terms in the right hand side of the above equation are given by

$$(\bar{\nabla}_U T)CV = -(CVlnf)TU, \text{ and } (\bar{\nabla}_U T)\xi = -TU. \tag{64}$$

Therefore, (63)can be re-written as:

$$(\bar{\nabla}_U T)V = (\bar{\nabla}_U T)BV - ((CVlnf) + \eta(V))TU \tag{65}$$

Now,

$$(\bar{\nabla}_U T)BV = (\bar{\nabla}_{BU} T)BV + (\bar{\nabla}_{CU} T)BV + \eta(U)(\bar{\nabla}_\xi T)BV$$

By virtue of Lemma 5.5 and formula (14) $(\bar{\nabla}_{CU} T)BV = 0 = (\bar{\nabla}_\xi T)BV$. Thus, the above equation reduces to

$$(\bar{\nabla}_U T)BV = (\bar{\nabla}_{BU} T)BV = B[(\bar{\nabla}_{BU} T)BV] + C[(\bar{\nabla}_{BU} T)BV]$$

Also,as $C[(\bar{\nabla}_{BU} T)BV] = g(TBU, BV)\nabla lnf$, we may express $(\bar{\nabla}_U T)BV$ as:

$$(\bar{\nabla}_U T)BV = B[(\bar{\nabla}_{BU} T)BV] + g(TU, V)\nabla lnf \tag{66}$$

Substituting from (64)and(66) into (65) and taking product with $Z \in \Gamma(TN_1)$, we obtain that

$$g((\bar{\nabla}_U T)V, Z) = Zlnf\ g(TBU, BV). \tag{67}$$

Replacing V by TV in the above equation and using (1) and (16), we obtain (62)

Conversely, suppose that M_1 is a point wise pseudo-slant submanifold of a Kenmotsu manifold with ϕ-anti invariant distribution $D^\perp$ and pointwise slant distribution D^θ such that for a smooth function μ on M_1 (with $X\mu = 0$), the condition (62) holds. Then for any $Z, W \in D^\perp \oplus < \xi >$, we have

$$g((\bar{\nabla}_Z T)W, Z') = 0,$$

as well as $g((\bar{\nabla}_Z T)W, X) = -g(W, (\bar{\nabla}_Z T)X) = 0$, for any $X \in D^\theta$ and $Z' \in D^\perp \oplus < \xi >$. This means, $(\bar{\nabla}_Z T)W = 0$ i.e.,

$$T\nabla_Z W = \nabla_Z TW = 0$$

which means $\nabla_Z W \in D^\perp \oplus < \xi >$. That is, the distribution $D^\perp \oplus < \xi >$ is parallel. In other words $D^\perp \oplus < \xi >$ is an involutive distribution whose leaves are totally geodesic in M_1. Further, by virtue of(62)

$$g((\bar{\nabla}_X T)Z, W) = 0, \tag{68}$$

and for any $Y \in D^\theta$,

$$g((\bar{\nabla}_X T)Z, Y) = -g(Z, (\bar{\nabla}_X T)Y) = -(Zlnf)g(TX, Y) \tag{69}$$

It follows from (68) and (69) that

$$(\bar{\nabla}_X T)Z = -2g(\nabla\mu, Z)TX$$

which implies that

$$\nabla_X Z = 2g(\nabla\mu, Z)X.$$

Taking product with $Y \in D^\theta$, gives

$$g(\nabla_X Y, Z) = -2g(X, Y)g(\nabla\mu, Z).$$

As $X\mu = 0$ for all $X \in D^\theta$, the above equation implies that $g([X, Y], Z) = 0$. That means D^θ is involutive. If σ_2 is the second fundamental form of the immersion of the leaves of D^θ into M_1, then

$$\sigma_2(X, Y) = -2g(X, Y)\nabla\mu.$$

That means each leaf N_θ of D^θ is totally umbilical in M with mean curvature $\nabla\mu$ and as $X\mu = 0$, N_θ is an extrinsic sphere. Hence, by Theorem 5.6, M is a warped product $N_\perp \times_f N_\theta$, where $f = e^\mu$. □

Example 5.8. [21] Consider the complex space $\mathbb{C}^5$ with the usual Kaehler structure and real global coordinates $(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4, x^5, y^5)$. Let $\bar{M} = \mathbb{R} \times_f \mathbb{C}^5$ be the warped product between the real line $\mathbb{R}$ and $\mathbb{C}^5$, where the warping function is $f = e^t$, t being the global coordinates on $\mathbb{R}$. Then $\bar{M}$ is a Kenmotsu manifold. Now, consider a 7-dimensional submanifold M of $\bar{M}$ with an orthonormal frame of tangent vectors $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ as :

$$e_1 = cos\theta\frac{\partial}{\partial x^1} + sin\theta\frac{\partial}{\partial y^3},\ e_2 = -sin\theta\frac{\partial}{\partial x^3} + cos\theta\frac{\partial}{\partial y^1}$$

$$e_3 = cos\theta\frac{\partial}{\partial x^2} - sin\theta\frac{\partial}{\partial y^4},\ e_4 = sin\theta\frac{\partial}{\partial x^4} + cos\theta\frac{\partial}{\partial y^2}$$

$$e_5 = \frac{\partial}{\partial x^3},\ e_6 = \frac{\partial}{\partial x^4},\ e_7 = \frac{\partial}{\partial t},$$

for any $\theta \in (0, \frac{\pi}{2})$. Then the distribution $D_\theta = span\{e_1, e_2, e_3, e_4\}$ and $D^\perp = span\{e_5, e_6, e_7\}$ are obviously integrable. Let us denote by N_θ and $N_\perp$ their, integral submanifolds, respectively. The metrics on N_θ and $N_\perp$ are respectively given by $g_{N_\theta} = \sum_{i=1}^{2}((dx^i)^2 + (dy^i)^2)$ and $g_{N_\perp} = dt^2 + e^{2t}\sum_{\alpha=3}^{4}(dx^\alpha)^2$. Then $M = N_\perp \times_f N_\theta$ is a warped product submanifold, isometrically immersed in $\bar{M}$. The warping function is given by $f(t) = e^t$.

With regard to pseudo-slant warped product submanifold of the type $N_\theta \times_f N_\perp$ in a Kenmotsu manifold $\bar{M}$, we observed that the structure vector field ξ can not lie in the slant distribution on M (c.f. Note 3.1), whereas by Proposition 5.1, ξ can not be tangential to the second factor of a warped product submanifold of $\bar{M}$. This rules out the existence of a non-trivial warped product submanifolds of the type $N_\theta \times_f N_\perp$ in a Kenmotsu manifold.

However, if M is a pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ tangent to the structure vector field ξ such that the distribution $D^\theta \oplus < \xi >$ is involutive then one may think of a warped product submanifold of the type $N_\theta \times_f N_\perp$ of $\bar{M}$, where N_θ is a leaf of $D^\theta \oplus < \xi >$ (which infact is not a slant distribution in view of the formal definition) and $N_\perp$ is a leaf of ϕ-anti invariant distribution on M (which is involutive by Theorem 3.8). Such warped product submanifolds will be denoted by the symbol M_2.

First, we prove the following:

Proposition 5.9. *Let M be a proper pointwise pseudo-slant submanifold of a Kenmotsu manifold $\bar{M}$ tangent to the structure vector field ξ. Then $[X, \xi] \in D^\theta \oplus < \xi >$ for any $X \in D^\theta$.*

Proof. By using (2.5), and the fact that in this case $\eta(Z) = 0$, it can be seen that

$$g(\nabla_X \xi, Z) = 0 \tag{70}$$

for any $X \in D^\theta$ and $Z \in D^\perp$. On the other hand

$$g(\nabla_\xi X, Z) = g(\bar{\nabla}_\xi X, Z) = g(\phi\bar{\nabla}_\xi X, \phi Z) + \eta(\bar{\nabla}_\xi X)\eta(Z).$$

Making use of formulae (3),(4) and the fact that $\eta(Z) = 0$ the right hand side reduces to

$$g(\bar{\nabla}_\xi TX, \phi Z) + g(\bar{\nabla}_\xi FX, \phi Z)$$

The first term is zero by (6) and (9), whereas the second term in view of the fact that FD^θ and $\phi D^\perp$ are orthogonal, is written as $-g(FX, \bar{\nabla}_\xi \phi Z)$, which by virtue of (4) reduces to $-g(FX, \phi\bar{\nabla}_\xi Z)$. Thus, we have

$$g(\nabla_\xi X, Z) = -g(FX, F\nabla_\xi Z) = -sin^2\ \theta g(X, \nabla_\xi Z)$$

That gives

$$cos^2\ \theta g(\nabla_\xi X, Z) = 0 \tag{71}$$

As M is a proper pointwise pseudo-slant submanifold, we obtain from (70) and (71) that

$$g([X, \xi], Z) = 0$$

for each $X \in D^\theta \oplus < \xi >$ and $Z \in D^\perp$, proving the assertion. □

Thus, if the pointwise slant distribution D^θ is involutive on M, then the above Proposition guarantees the foliation of M by the leaves of $D^\theta \oplus < \xi >$. In this case, we will be denoting the leaves of $D^\theta \oplus < \xi >$ by N_θ itself. Further, if the orthogonal complement of $D^\theta \oplus < \xi >$ is a ϕ-anti invariant distribution , then $N_\theta \times_f N_\perp$ is a pseudo-slant submanifold of $\bar{M}$ (with $N_\perp$ a leaf of ϕ-anti invariant distribution $D^\perp$).

Theorem 5.10. *A proper pointwise pseudo-slant submanifold M of a Kenmotsu manifold tangent to the structure vector field ξ is a pseudo-slant warped product of the type $N_\theta \times_f N_\perp$ if and only if there is a function μ on M with $Z\mu = 0$ for each $Z \in D^\perp$ such that*

$$A_{FZ}TX - A_{FTX}Z = -cos^2(\theta)(X\mu)Z \tag{72}$$

for each $X \in D^\theta \oplus < \xi >$.

Proof. If $M_2 = N_\theta \times_f N_\perp$ is a warped product submanifold of a Kenmotsu manifold $\bar{M}$ tangent to the structure vector field ξ, then formula (72) holds on M by virtue of corollary 5.4.

Conversely, suppose formula (72) holds on a pointwise pseudo-slant submanifold M of a Kenmotsu manifold $\bar{M}$, then for any $X, Y \in D^\theta \oplus < \xi >$ and $Z \in D^\perp$,

$$g(A_{FZ}X - A_{FX}Z, Y) = 0$$

That is,

$$g(h(X, Y), FZ) = g(h(Y, Z), FX) \tag{73}$$

On the other hand, by (13),(8) and the facts that $\eta(Z) = 0$, and $g(TX, Z) = 0$, we have

$$g((\bar{\nabla}_X T)Y, Z) = g(h(X, Z), FY) - g(h(X, Y), FZ) \tag{74}$$

From (73) and (74)

$$g((\bar{\nabla}_X T)Y, Z) = 0,$$

which in view of formula (11) and the fact that

$$g(T\nabla_X Y, Z) = -g(\nabla_X Y, TZ) = 0,$$

implies that

$$g(\nabla_X TY, Z) = 0$$

As D^θ is proper, the above equation shows that $D^\theta \oplus < \xi >$ is involutive on M and its leaves are totally geodesic in M.

Now, by (13) and (72), we have

$$(\bar{\nabla}_X T)Z - (\bar{\nabla}_Z T)X = A_{FZ}X - A_{FX}Z = -(TX\mu)Z$$

That gives

$$g((\bar{\nabla}_X T)Z) - ((\bar{\nabla}_Z T)X, W) = -(TX\mu)g(Z, W)$$

for each $W \in D^{\perp}$. Taking account of (11) and the fact that $TW = 0$ for each $W \in D^{\perp}$ in the above equation , we get

$$g(\nabla_Z W, TX) = -g(Z, W)g(\nabla\mu, TX)$$

which shows that $D^{\perp}$ is totally umbilical in M with mean curvature vector $\nabla\mu$. Further as $Z\mu = 0$ and $D^{\perp}$ is involutive, the leaves $M_{\perp}$ of $D^{\perp}$ are extrinsic spheres in M. Hence, by virtue of Theorem 5.6, M is a warped product manifold $N_\theta \times_f N_\perp$ where N_θ and $N_\perp$ denote the leaves of $D^\theta \oplus <\xi>$ and $D^\perp$ respectively. □

Example 5.11. Let $\bar{M}$ be as in example 5.8. Then the distributions $D^\theta = span\{e_1, e_2, e_3, e_4, e_7\}$ and $D^\perp = span\{e_5, e_6\}$ are integrable on a 7-dimensional submanifold M of $\bar{M}$. The Riemannian metric g on the leaves of D^θ and $D^\perp$ are respectively given by

$$g_{N_\theta} = dt^2 + e^{2t}\sum_{i=1}^{2}\{(dx^i)^2 + (dy^i)^2\} \ \ and \ \ g_{N_\perp} = \sum_{j=3}^{4}(dx^j)^2$$

Then M is a pseudo-slant warped product submanifold of the type $N_\theta \times_f N_\perp$, where N_θ and $N_\perp$ denote the leaves of D^θ and $D^\perp$ respectively and warping function $f(t) = e^t$.

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