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The Global Dynamics of Stochastic Holling Type II Predator-Prey Models with Non Constant Mortality Rate

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Abstract. In this paper we study the global dynamics of stochastic predator-prey models with non constant mortality rate and Holling type II response. Concretely, we establish sufficient conditions for the extinction and persistence in the mean of autonomous stochastic model and obtain a critical value between them. Then by constructing appropriate Lyapunov functions, we prove that there is a nontrivial positive periodic solution to the non-autonomous stochastic model. Finally, numerical examples are introduced to illustrate the results developed.

1. Introduction

In the ecological sciences, dynamic of predator-prey system is one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. In [2], Cavani and Farkas introduced the following predator-prey model

$$\begin{cases} \dot{N}(t) = \varepsilon N(t)\left(1 - \frac{N(t)}{K}\right) - \frac{aP(t)N(t)}{\beta + N(t)}, \\ \dot{P}(t) = P(t)\left(-M(P(t)) + \frac{bN(t)}{\beta + N(t)}\right), \end{cases} \tag{1}$$

where $N(t)$ and $P(t)$ are the quantities of prey and predator, respectively. From [2] it follows that ε is specific growth rate of prey in the absence of predation and without environment limitation; K is the carrying capacity of the prey in the absence of predators; the functional response of the predator is of Holling type II; a, β and b are satiation coefficients or conversion rates; and function $M(P)$ is the mortality rate of predators in the absence of prey. If $M(P) = n$, model (1) is exactly the classic predator-prey model with Holling type-II response. Here the mortality rate of predators

$$M(P) = \frac{\gamma + \delta P}{1 + P} = \delta + \frac{\gamma - \delta}{1 + P}, \quad 0 < \gamma < \delta$$

is non constant and depends on the quantity of predator, γ is the mortality at low density, and δ is the maximal mortality with the natural assumption $\gamma < \delta$. All the parameters are assumed to be positive. Many results on this model and its deformations have been reported, see [3–5].

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However, in the real life situations, population systems are always affected by environmental noise, and stochastic population systems have been studied by many authors [6–14]. Up to now, few papers have considered Holling type II predator-prey model with non constant mortality rate in random environments, namely

$$\begin{cases} \mathrm{d}N(t) = \left(\varepsilon N(t)\left(1 - \frac{N(t)}{K}\right) - \frac{aP(t)N(t)}{\beta+N(t)}\right)\mathrm{d}t + \sigma_1 N(t)\mathrm{d}B_1(t), \\ \mathrm{d}P(t) = P(t)\left(-\frac{\gamma+\delta P(t)}{1+P(t)} + \frac{bN(t)}{\beta+N(t)}\right)\mathrm{d}t + \sigma_2 P(t)\mathrm{d}B_2(t), \end{cases} \tag{2}$$

where $B_1(t)$, $B_2(t)$ are mutually independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ with a σ−field filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and positive constants σ_1^2, σ_2^2 are their intensities. In this paper, we aim to study persistence and extinction of stochastic model (2), and analyze the effect of environmental noise on the dynamics of the system (2).

On the other hand, due to the seasonal variation, food supplies and harvesting and so on, ecological environments change significantly through the year. So it is reasonable and important to consider the non-autonomous population systems. In particular, many authors addressed the effect of periodic fluctuations because, as mentioned by Vance and Coddington [15], "periodic time variation holds considerable promise as a means to explore time-varying ecological processes"[16]. Recently, according to the theory of Has'minskii [17], progress has been made in stochastic population systems with periodic parameters. For example, papers [18–20] obtained the existence of periodic solutions to stochastic non-autonomous population systems. Motivated by above analysis, in this paper, we also consider the following stochastic periodic system

$$\begin{cases} \mathrm{d}N(t) = \left(\varepsilon(t) N(t)\left(1 - \frac{N(t)}{K(t)}\right) - \frac{a(t)P(t)N(t)}{\beta(t)+N(t)}\right)\mathrm{d}t + \sigma_1(t) N(t)\mathrm{d}B_1(t), \\ \mathrm{d}P(t) = P(t)\left(-\frac{\gamma(t)+\delta(t) P(t)}{1+P(t)} + \frac{b(t)N(t)}{\beta(t)+N(t)}\right)\mathrm{d}t + \sigma_2(t) P(t)\mathrm{d}B_2(t), \end{cases} \tag{3}$$

where $\varepsilon(t)$, $K(t)$, $a(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$, $b(t)$ and $\sigma_i^2(t)$ are all positive continuous θ-periodic functions, $i = 1, 2$. We also assume that $\gamma(t) < \delta(t)$ holds for all $t > 0$. The existence of periodic solution to stochastic model (3) will be discussed.

The remainder of the paper is organized as follows. In Section 2, we mainly prove the existence and uniqueness of the global positive solution to model (2). In Section 3, we investigate persistence in the mean and extinction of model (2) and furthermore, we try to obtain the critical value between them. The existence of nontrivial positive periodic solution to non-autonomous model (3) is obtained in Section 4 and the existence of ergodic stationary distribution of autonomous model (2) is also deduced. Finally, numerical simulations illustrate our theoretical results in Section 5.

2. Existence and Uniqueness of the Global Positive Solution

For simplicity, we introduce the following notations.

$\mathbb{R}_+^2 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_i > 0, i = 1, 2\}$.

$\langle f \rangle_t = \frac{1}{t}\int_0^t f(s)\mathrm{d}s$.

If $f(t)$ is a continuous bounded function, define $f^l = \inf_{t\in[0,\infty)} f(t)$, $f^u = \sup_{t\in[0,\infty)} f(t)$.

The following theorem is fundamental in this paper.

Theorem 2.1. *For any initial value $(N(0), P(0)) \in \mathbb{R}_+^2$, there is a unique positive solution $(N(t), P(t))$ of system (2) on $t \geq 0$, and the solution will remain in $\mathbb{R}_+^2$ with probability 1.*

Proof. Obviously, the coefficients of model (2) are locally Lipschitz continuous, so there is a unique local solution $(N(t), P(t))$ on $t \in [0, \rho)$ for any given initial value $(N(0), P(0)) \in \mathbb{R}_+^2$, where ρ is the explosion time. If $\rho = \infty$ a.s., then this local solution is global. Let k_0 be sufficiently large for every component of $(N(0), P(0))$ lying within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \rho) | N(t) \notin (1/k, k) \text{or } P(t) \notin (1/k, k)\},$$

where throughout this paper we set $\inf \emptyset = \infty$. Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, which implies $\tau_\infty \le \rho$ a.s. If we show that $\tau_\infty = \infty$ a.s., then $\rho = \infty$ a.s. This means that $(N(t), P(t)) \in \mathbb{R}^2_+$ a.s. for all $t \ge 0$. If $\tau_\infty < \infty$ a.s., then there is a pair of constants $T \ge 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \le T\} > \epsilon.$$

Hence there is an integer $k_1 \ge k_0$ such that

$$\mathbb{P}\{\tau_k \le T\} \ge \epsilon \text{ for all } k \ge k_1. \tag{4}$$

Define a C^2-function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ as follows:

$$V(N,P) = \frac{4b}{\beta\delta}\left(N - \frac{\beta\delta}{4b} - \frac{\beta\delta}{4b}\log\frac{4bN}{\beta\delta}\right) + \frac{2a}{\beta\delta}\left(P - \frac{\beta\delta}{2a} - \frac{\beta\delta}{2a}\log\frac{2aP}{\beta\delta}\right).$$

Applying Itô's formula we have

$$\mathrm{d}V(N,P) = \mathcal{L}V(N,P)\mathrm{d}t + \frac{4b\sigma_1}{\beta\delta}\left(N - \frac{\beta\delta}{4b}\right)\mathrm{d}B_1(t) + \frac{2a\sigma_2}{\beta\delta}\left(P - \frac{\beta\delta}{2a}\right)\mathrm{d}B_2(t),$$

in which

$$\begin{aligned}
\mathcal{L}V(N,P) =& \frac{4b}{\beta\delta}\left(N - \frac{\beta\delta}{4b}\right)\left(\varepsilon - \frac{\varepsilon}{K}N - \frac{aP}{\beta+N}\right) + \frac{\sigma_1^2}{2} \\
&+ \frac{2a}{\beta\delta}\left(P - \frac{\beta\delta}{2a}\right)\left(-\delta - \frac{\gamma-\delta}{1+P} + \frac{bN}{\beta+N}\right) + \frac{\sigma_2^2}{2} \\
=& -\frac{4b\varepsilon}{\beta\delta K}N^2 + \left(\frac{4b\varepsilon}{\beta\delta} + \frac{\varepsilon}{K}\right)N - \frac{4ab}{\beta\delta}\frac{PN}{\beta+N} + \frac{aP}{\beta+N} - \varepsilon + \frac{\sigma_1^2}{2} \\
&- \frac{2a}{\beta}P + \frac{2a(\delta-\gamma)}{\beta\delta}\frac{P}{1+P} + \frac{2ab}{\beta\delta}\frac{PN}{\beta+N} - \frac{\delta-\gamma}{1+P} - \frac{bN}{\beta+N} + \delta + \frac{\sigma_2^2}{2} \\
\le& -\frac{4b\varepsilon}{\beta\delta K}N^2 + \left(\frac{4b\varepsilon}{\beta\delta} + \frac{\varepsilon}{K}\right)N - \frac{2ab}{\beta\delta}\frac{PN}{\beta+N} \\
&+ \frac{a}{\beta}P - \frac{2a}{\beta}P - \varepsilon + \frac{\sigma_1^2}{2} + \delta + \frac{\sigma_2^2}{2} + \frac{2a}{\beta} \\
\le& -\frac{4b\varepsilon}{\beta\delta K}N^2 + \left(\frac{4b\varepsilon}{\beta\delta} + \frac{\varepsilon}{K}\right)N - \varepsilon + \frac{\sigma_1^2}{2} + \delta + \frac{\sigma_2^2}{2} + \frac{2a}{\beta} \\
\le& M,
\end{aligned}$$

where M is a positive constant. We therefore obtain

$$\mathbb{E}V(N(\tau_k \wedge T), P(\tau_k \wedge T)) \le V(N(0), P(0)) + M\mathbb{E}(\tau_k \wedge T) \le V(N(0), P(0)) + MT. \tag{5}$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and by (4), $\mathbb{P}(\Omega_k) \ge \epsilon$. Note that for every $\omega \in \Omega_k$, there is at least one of $N(\tau_k, \omega), P(\tau_k, \omega)$ equals either k or $1/k$, therefore

$$\begin{aligned}
V(N(\tau_k,\omega), P(\tau_k,\omega)) \ge& \frac{4b}{\beta\delta}\left(k - \frac{\beta\delta}{4b} - \frac{\beta\delta}{4b}\log\frac{4bk}{\beta\delta}\right) \wedge \frac{2a}{\beta\delta}\left(k - \frac{\beta\delta}{2a} - \frac{\beta\delta}{2a}\log\frac{2ak}{\beta\delta}\right) \\
&\wedge \frac{4b}{\beta\delta}\left(\frac{1}{k} - \frac{\beta\delta}{4b} - \frac{\beta\delta}{4b}\log\frac{4b}{\beta\delta k}\right) \wedge \frac{2a}{\beta\delta}\left(\frac{1}{k} - \frac{\beta\delta}{2a} - \frac{\beta\delta}{2a}\log\frac{2a}{\beta\delta k}\right).
\end{aligned}$$

It then follows from (5) that

$$\begin{aligned} V(N(0),P(0)) + MT \geq & \mathbb{E}\left(I_{\Omega_k} V(N(\tau_k,\omega),P(\tau_k,\omega))\right) \\ \geq & \epsilon \left\{ \frac{4b}{\beta\delta}\left(k - \frac{\beta\delta}{4b} - \frac{\beta\delta}{4b}\log\frac{4bk}{\beta\delta}\right) \wedge \frac{2a}{\beta\delta}\left(k - \frac{\beta\delta}{2a} - \frac{\beta\delta}{2a}\log\frac{2ak}{\beta\delta}\right) \right. \\ & \left. \wedge \frac{4b}{\beta\delta}\left(\frac{1}{k} - \frac{\beta\delta}{4b} - \frac{\beta\delta}{4b}\log\frac{4b}{\beta\delta k}\right) \wedge \frac{2a}{\beta\delta}\left(\frac{1}{k} - \frac{\beta\delta}{2a} - \frac{\beta\delta}{2a}\log\frac{2a}{\beta\delta k}\right) \right\}. \end{aligned}$$

Letting $k \to \infty$ leads to the contradiction

$$\infty > V(N(0),P(0)) + MT = \infty,$$

so we must have $\tau_\infty = \infty$ a.s. The proof is complete. □

3. Discussion on the Persistence and Extinction

In this section, we investigate the persistence and extinction of autonomous stochastic predator-prey model (2) under certain conditions. Furthermore, by using the ergodic property of stochastic Logistic model, we try to give the critical value which determines the extinction and persistence of model (2). To this end, we quote some concepts and lemmas.

Definition 3.1. [9]

(1) If $\lim_{t\to\infty} P(t) = 0$ a.s., then model (2) is said to be extinctive almost surely.
(2) If $\liminf_{t\to\infty}\langle P\rangle_t > 0$ a.s.,then model (2) is said to be persistent in the mean.

Lemma 3.2. [9] Suppose that $Z(t) \in C(\Omega \times [0,\infty), \mathbb{R}_+)$.

(I) If there are two positive constants T and δ_0 such that

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + \sum_{i=1}^n \alpha_i B(t) \ a.s.$$

for all $t > T$, where α_i, δ are constants, then

$$\begin{cases} \limsup_{t\to\infty}\langle Z\rangle_t \leq \frac{\delta}{\delta_0} \ a.s., & \text{if } \delta \geq 0; \\ \lim_{t\to\infty} Z(t) = 0 \ a.s., & \text{if } \delta < 0. \end{cases}$$

(II) If there exist three positive constants T, δ, δ_0 such that

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + \sum_{i=1}^n \alpha_i B(t) \ a.s.$$

for all $t > T$, then $\liminf_{t\to\infty}\langle Z\rangle_t \geq \frac{\delta}{\delta_0}$ *a.s.*.

Lemma 3.3. [10] Consider the following one-dimensional stochastic Logistic model

$$\mathrm{d}X(t) = \varepsilon X(t)\left(1 - \frac{X(t)}{K}\right)\mathrm{d}t + \sigma_1 X(t)\mathrm{d}B_1(t), \tag{6}$$

with $X(0) = N(0)$. If $\varepsilon - \sigma_1^2/2 > 0$, model (6) has a unique ergodic stationary distribution $\nu(\cdot)$ with stationary density $\mu(x) = Cx^{\frac{2-\sigma_1^2}{\sigma_1^2}-1} e^{-\frac{2}{\sigma_1^2}x}$, where $C = (2/\sigma_1^2)^{(2-\sigma_1^2)/\sigma_1^2}/\Gamma((2-\sigma_1^2)/\sigma_1^2)$, and

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s))\mathrm{d}s = \int_{\mathbb{R}_+} f(x)\mu(x)\mathrm{d}x\right\} = 1,$$

where f is a function integrable with respect to the measure ν.

Remark 3.4. *From stochastic comparison theory it follows that $N(t) \le X(t)$ a.s. and*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{bX(s)}{\beta + X(s)}\mathrm{d}s = \int_0^\infty \frac{bx}{\beta + x}\mu(x)\mathrm{d}x, \quad a.s. \tag{7}$$

Lemma 3.5. *Let $X(t)$ be the solution of stochastic Logistic model (6) with initial value $X(0) = N(0)$ and $\varepsilon - \sigma_1^2/2 > 0$. Denote $Y(t) = \frac{X(t)}{\beta + X(t)}$, then the following properties hold:*

$$\liminf_{t\to\infty}\langle Y\rangle_t \ge \frac{\varepsilon - \frac{\sigma_1^2}{2}}{\varepsilon + \frac{\beta\varepsilon}{K}}, \quad a.s. \tag{8}$$

and

$$\limsup_{t\to\infty}\langle Y\rangle_t \le \frac{\varepsilon - \frac{\sigma_1^2}{2}}{\varepsilon - \frac{\sigma_1^2}{2} + \frac{\beta\varepsilon}{K}}, \quad a.s. \tag{9}$$

Proof. An application of Itô's formula yields

$$\begin{aligned}\mathrm{d}Y(t) &= \left(\frac{\beta}{(\beta+X)^2}\varepsilon X\left(1 - \frac{X}{K}\right) - \frac{\beta}{(\beta+X)^3}\sigma_1^2 X^2\right)\mathrm{d}t + \frac{\beta\sigma_1 X}{(\beta+X)^2}\mathrm{d}B_1(t)\\ &= \left(\varepsilon Y(1-Y) - \frac{\beta\varepsilon}{K}Y^2 - \sigma_1^2 Y^2(1-Y)\right)\mathrm{d}t + \sigma_1 Y(1-Y)\mathrm{d}B_1(t),\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\log Y(t) &= \left(\varepsilon(1-Y) - \frac{\beta\varepsilon}{K}Y - \sigma_1^2 Y(1-Y) - \frac{\sigma_1^2}{2}(1-Y)^2\right)\mathrm{d}t + \sigma_1(1-Y)\mathrm{d}B_1(t)\\ &= \left(\varepsilon - \frac{\sigma_1^2}{2} - \left(\varepsilon + \frac{\beta\varepsilon}{K}\right)Y + \frac{\sigma_1^2}{2}Y^2\right)\mathrm{d}t + \sigma_1(1-Y)\mathrm{d}B_1(t).\end{aligned} \tag{10}$$

Since $Y(t) = X(t)/(\beta + X(t))$, so $0 < Y(t) < 1$. On the one hand, from (10) it follows that

$$\frac{\log Y(t) - \log Y(0)}{t} \ge \varepsilon - \frac{\sigma_1^2}{2} - \left(\varepsilon + \frac{\beta\varepsilon}{K}\right)\langle Y\rangle_t + \frac{M(t)}{t},$$

where $M(t) = \int_0^t \sigma_1(1 - Y(s))\mathrm{d}B_1(s)$ is a real-valued continuous local martingale and $\langle M, M\rangle_t = \int_0^t \sigma_1^2(1 - Y(s))^2\mathrm{d}s \le \sigma_1^2 t$. By strong law of large numbers [21], we have $\lim_{t\to\infty}\frac{M(t)}{t} = 0$ *a.s.*. Applying (II) in Lemma 3.2, one can derive that the assertion (8) holds.

On the other hand, from $Y^2 \le Y$ and (10) it follows that

$$\mathrm{d}\log Y(t) \le \left(\varepsilon - \frac{\sigma_1^2}{2} - \left(\varepsilon - \frac{\sigma_1^2}{2} + \frac{\beta\varepsilon}{K}\right)Y\right)\mathrm{d}t + \sigma_1(1-Y)\mathrm{d}B_1(t),$$

which implies that

$$\frac{\log Y(t) - \log Y(0)}{t} \le \varepsilon - \frac{\sigma_1^2}{2} - \left(\varepsilon - \frac{\sigma_1^2}{2} + \frac{\beta\varepsilon}{K}\right)\langle Y\rangle_t + \frac{M(t)}{t}.$$

Applying (I) in Lemma 3.2 we obtain

$$\limsup_{t\to\infty}\langle Y\rangle_t \le \frac{\varepsilon - \frac{\sigma_1^2}{2}}{\varepsilon - \frac{\sigma_1^2}{2} + \frac{\beta\varepsilon}{K}}, \quad a.s.$$

This completes the proof. □

Theorem 3.6. *Assume that $\varepsilon - \sigma_1^2/2 > 0$. Let $(N(t), P(t))$ be a positive solution of model (2) with initial value $(N(0), P(0)) \in \mathbb{R}_+^2$.*

(i) If $\lambda_1 := -\gamma - \frac{\sigma_2^2}{2} + \frac{b\left(\varepsilon - \frac{\sigma_1^2}{2}\right)}{\varepsilon - \frac{\sigma_1^2}{2} + \frac{\beta\varepsilon}{K}} < 0$, then the predator populations go to extinction a.s..

(ii) If $\lambda_2 := -\gamma - \frac{\sigma_2^2}{2} + \frac{b\left(\varepsilon - \frac{\sigma_1^2}{2}\right)}{\varepsilon + \frac{\beta\varepsilon}{K}} > 0$, then system (2) will be persistent in the mean.

Proof. (i). An application of Itô's formula to the second equation of (2) shows that

$$\begin{aligned} \mathrm{d}\log P(t) &= \left(-\frac{\gamma + \delta P(t)}{1 + P(t)} + \frac{bN(t)}{\beta + N(t)} - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t) \\ &= \left(-\gamma - \frac{\sigma_2^2}{2} - \frac{(\delta - \gamma)P(t)}{1 + P(t)} + \frac{bN(t)}{\beta + N(t)}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t). \end{aligned} \tag{11}$$

Integrating above inequality from 0 to t and dividing t on both sides, we get

$$\begin{aligned} \frac{\log P(t) - \log P(0)}{t} &\le -\gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{bN(s)}{\beta + N(s)}\mathrm{d}s + \frac{M_2(t)}{t} \\ &\le -\gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{bX(s)}{\beta + X(s)}\mathrm{d}s + \frac{M_2(t)}{t}, \end{aligned}$$

where $M_i(t) = \int_0^t \sigma_i \mathrm{d}B_i(t)$,$i = 1, 2$ are real-valued continuous local martingales. By strong law of large numbers [21], we have $\lim_{t\to\infty} \frac{M_i(t)}{t} = 0$ *a.s.*, $i = 1, 2$. From (9) it follows that

$$\limsup_{t\to\infty} \frac{\log P(t)}{t} \le -\gamma - \frac{\sigma_2^2}{2} + \frac{\varepsilon - \frac{\sigma_1^2}{2}}{\varepsilon - \frac{\sigma_1^2}{2} + \frac{\beta\varepsilon}{K}}.$$

Obviously, the predator populations $P(t)$ tends to zero a.s. when $\lambda_1 < 0$.

(ii). Applying Itô's formula to the first equation of (2) and (6) respectively, we have

$$\frac{\log N(t) - \log N(0)}{t} = \varepsilon - \frac{\sigma_1^2}{2} - \frac{1}{t}\int_0^t \frac{\varepsilon}{K}N(s)\mathrm{d}s - \frac{1}{t}\int_0^t \frac{aP(s)}{\beta + N(s)}\mathrm{d}s + \frac{M_1(t)}{t},$$

and

$$\frac{\log X(t) - \log X(0)}{t} = \varepsilon - \frac{\sigma_1^2}{2} - \frac{1}{t}\int_0^t \frac{\varepsilon}{K}X(s)\mathrm{d}s + \frac{M_1(t)}{t}.$$

These imply that

$$\begin{aligned} 0 \ge \frac{\log N(t) - \log X(t)}{t} &= -\frac{1}{t}\int_0^t \frac{\varepsilon}{K}(N(s) - X(s)\mathrm{d}s - \frac{1}{t}\int_0^t \frac{aP(s)}{\beta + N(s)}\mathrm{d}s \\ &\ge -\frac{\varepsilon}{K}\langle N - X\rangle_t - \frac{a}{\beta}\langle P\rangle_t, \end{aligned}$$

that is to say,

$$\frac{\varepsilon}{K}\langle X - N\rangle_t \le \frac{a}{\beta}\langle P\rangle_t. \tag{12}$$

From (11) we obtain

$$
\begin{aligned}
\mathrm{d}\log P(t) =&\left(-\gamma-\frac{\sigma_2^2}{2}+\frac{bN(t)}{\beta+N(t)}-\frac{(\delta-\gamma)P(t)}{1+P(t)}\right)\mathrm{d}t+\sigma_2\mathrm{d}B_2(t)\\
=&\left(-\gamma-\frac{\sigma_2^2}{2}+\frac{bX(t)}{\beta+X(t)}-\left(\frac{bX(t)}{\beta+X(t)}-\frac{bN(t)}{\beta+N(t)}\right)-\frac{(\delta-\gamma)P(t)}{1+P(t)}\right)\mathrm{d}t+\sigma_2\mathrm{d}B_2(t)\\
\geq&\left(-\gamma-\frac{\sigma_2^2}{2}+\frac{bX(t)}{\beta+X(t)}-\frac{b\beta(X(t)-N(t))}{(\beta+X(t))(\beta+N(t))}-(\delta-\gamma)P(t)\right)\mathrm{d}t+\sigma_2\mathrm{d}B_2(t)\\
\geq&\left(-\gamma-\frac{\sigma_2^2}{2}+\frac{bX(t)}{\beta+X(t)}-\frac{b}{\beta}(X(t)-N(t))-(\delta-\gamma)P(t)\right)\mathrm{d}t+\sigma_2\mathrm{d}B_2(t).
\end{aligned}
\tag{13}
$$

Integrating (13) from 0 to t, combining (12) and (8), one can derive that

$$
\begin{aligned}
\frac{\log P(t)-\log P(0)}{t}\geq&-\gamma-\frac{\sigma_2^2}{2}+b\langle Y\rangle_t-\left(\frac{abK}{\varepsilon\beta^2}+\delta-\gamma\right)\langle P\rangle_t+\frac{M_2(t)}{t}\\
\geq&-\gamma-\frac{\sigma_2^2}{2}-\epsilon+\frac{b\left(\varepsilon-\frac{\sigma_1^2}{2}\right)}{\varepsilon+\frac{\beta\varepsilon}{K}}-\left(\frac{abK}{\varepsilon\beta^2}+\delta-\gamma\right)\langle P\rangle_t+\frac{M_2(t)}{t}\\
=&\lambda_2-\epsilon-\left(\frac{abK}{\varepsilon\beta^2}+\delta-\gamma\right)\langle P\rangle_t+\frac{M_2(t)}{t}
\end{aligned}
$$

for sufficiently large t. By virtue of the arbitrariness of ϵ and (II) in Lemma 3.2, we derive that

$$\liminf_{t\to\infty}\langle P\rangle_t\geq\frac{\lambda_2}{\frac{abK}{\varepsilon\beta^2}+\delta-\gamma}>0,\quad a.s.$$

That is to say model (2) will be persistent in the mean when $\lambda_2>0$. The proof is complete. □

Remark 3.7. *From [4] it follows that if $b-\gamma<0$ or $\frac{\beta\gamma}{b-\gamma}>K$, point $(K,0)$ of deterministic system (1) is global asymptotically stable; while system (1) is uniformly persistent if and only if $b-\gamma>0$ and $\frac{\beta\gamma}{b-\gamma}<K$. Theorem 3.6 shows that if $\lambda_1=-\gamma-\frac{\sigma_2^2}{2}+\frac{b\left(\varepsilon-\frac{\sigma_1^2}{2}\right)}{\varepsilon-\frac{\sigma_1^2}{2}+\frac{\beta\varepsilon}{K}}<0$, the predator populations of stochastic system (2) will be extinctive and the prey population satisfies $\lim_{t\to\infty}\langle N\rangle_t=\frac{K(\varepsilon-\sigma_1^2/2)}{\varepsilon}$ a.s.. If $\lambda_2:=-\gamma-\frac{\sigma_2^2}{2}+\frac{b\left(\varepsilon-\frac{\sigma_1^2}{2}\right)}{\varepsilon+\frac{\beta\varepsilon}{K}}>0$, then system (2) will be persistent in the mean. Obviously, the conditions which guarantee the persistence and extinction of deterministic system (1) coincide with those in stochastic system (2) if there is no white noise.*

Remark 3.8. *Expressions of λ_1 and λ_2 show that $\lambda_2<\lambda_1$. Note that there is a gap between λ_1 and λ_2, hence Theorem 3.6 only gives the sufficient conditions for the persistence and extinction of model (2).*

From the proof of Theorem 3.6, we observe that $\lambda_1=\lambda_2=\lambda:=-\gamma-\frac{\sigma_2^2}{2}+\int_0^\infty\frac{bx}{\beta+x}\mu(x)\mathrm{d}x$ if we use (7) to estimate $\lim_{t\to\infty}\int_0^t\frac{bX(s)}{\beta+X(s)}\mathrm{d}s$. In other words, the following theorem gives the threshold between persistence in the mean and extinction of model (2).

Theorem 3.9. *Assume that $\varepsilon-\sigma_1^2/2>0$. Then, for any initial value $(N(0),P(0))\in\mathbb{R}_+^2$, we have*

(i) if $\lambda<0$, then the predator populations go to extinction a.s.;
(ii) if $\lambda>0$, then system (2) will be persistent in the mean.

4. Existence of Periodic Solution of Non-Autonomous Model

In this section, we mainly give sufficient conditions for the existence of periodic solution to non-autonomous stochastic model (3) according to the theory of Has'minskii. For the sake of convenience, we introduce some results concerning the periodic Markov processes in Appendix.

Theorem 4.1. *If* $\rho := \frac{b^l}{\varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)}\langle \varepsilon - \frac{\sigma_1^2}{2}\rangle_\theta - \langle \gamma + \frac{\sigma_2^2}{2}\rangle_\theta > 0$ $(i = 1, 2)$, *then model (3) admits a nontrivial positive* θ*-periodic solution.*

Proof. By similar proof of Theorem 2.1 we obtain that non-autonomous stochastic model (3) has a unique global positive solution $(N(t), P(t))$ for any initial value $(N(0), P(0)) \in \mathbb{R}_+^2$. In order to prove Theorem 4.1, it suffices to find a C^2-function $V(t, N, P)$ which is θ-periodic in t and a close set $U \subset \mathbb{R}_+^2$ such that (27) and (28) hold.

Define a C^2-function

$$\begin{aligned} V(t,N,P) =& M\left(-\log P - \frac{b^l}{\varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)}(\log N - \log(\beta^u + N)) + w(t) + HP\right) + \frac{(N + \frac{a^l}{b^u}P)^{\vartheta+1}}{\vartheta+1} \\ =& M(V_1(N,P) + w(t) + HP) + V_2(N,P), \end{aligned}$$

here $\vartheta \in (0, 1)$, H and M are positive constants satisfying the following conditions respectively

$$\frac{\vartheta}{2}\sigma_2^{2u} < (\delta-\gamma)^l, \tag{14}$$

$$H(\delta-\gamma)^l - \frac{a^u b^l}{\varepsilon^u\beta^l\left(1+\frac{\beta^u}{K^l}\right)} > 0,$$

$$\left(\frac{a^u b^l}{\varepsilon^u\beta^l\left(1+\frac{\beta^u}{K^l}\right)} + (\delta-\gamma)^u - \rho\right)^2 < 4\rho\left(H(\delta-\gamma)^l - \frac{a^u b^l}{\varepsilon^u\beta^l\left(1+\frac{\beta^u}{K^l}\right)}\right), \tag{15}$$

$$-AM + f^u + g^u \le -2, \tag{16}$$

and positive constant A, functions $f(x)$, $g(x)$ and θ-periodic function $w(t) \in C^1(\mathbb{R}_+; \mathbb{R})$ will be determined later. It is obvious that condition (27) is satisfied. Hence we only confirm condition (28) in Lemma 5.4. Applying Itô' formula, we obtain

$$\begin{aligned} \mathcal{L}(-\log P) =& \frac{\gamma(t)+\delta(t)P}{1+P} - \frac{b(t)N}{\beta(t)+N} + \frac{\sigma_2^2(t)}{2} \\ \le& \gamma(t) + \frac{\sigma_2^2(t)}{2} + \frac{(\delta-\gamma)^u P}{1+P} - \frac{b^l N}{\beta^u + N}, \\ \mathcal{L}(-\log N) =& -\varepsilon(t) + \frac{\sigma_1^2(t)}{2} + \frac{\varepsilon(t)}{K(t)}N + \frac{a(t)P}{\beta(t)+N} \\ \le& -\left(\varepsilon(t) - \frac{\sigma_1^2(t)}{2}\right) + \frac{\varepsilon(t)}{K(t)}N + \frac{a^u}{\beta^l}P, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\log(\beta^u+N)) =& \frac{N}{\beta^u+N}\left(\varepsilon(t) - \frac{\varepsilon(t)}{K(t)}N\right) - \frac{a(t)PN}{(\beta^u+N)(\beta(t)+N)} - \frac{\sigma_1^2(t)}{2}\frac{N^2}{(\beta^u+N)^2} \\ \le& \frac{\varepsilon(t)N}{\beta^u+N} - \frac{\varepsilon(t)}{K(t)}\frac{N^2}{\beta^u+N} = \left(\varepsilon(t)\left(1+\frac{\beta^u}{K(t)}\right)\right)\frac{N}{\beta^u+N} - \frac{\varepsilon(t)}{K(t)}N \\ \le& \varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)\frac{N}{\beta^u+N} - \frac{\varepsilon(t)}{K(t)}N. \end{aligned}$$

Therefore we have

$$\mathcal{L}(V_1(N,P)) \le \gamma(t) + \frac{\sigma_2^2(t)}{2} - \frac{b^l}{\varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)}\left(\varepsilon(t) - \frac{\sigma_1^2(t)}{2}\right) + \frac{(\delta-\gamma)^u P}{1+P} + \frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)}P. \tag{17}$$

Let

$$\begin{aligned} w'(t) &= -\langle \gamma + \frac{\sigma_2^2}{2}\rangle_\theta + \frac{b^l}{\varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)}\langle \varepsilon - \frac{\sigma_1^2}{2}\rangle_\theta + \left(\frac{b^l}{\varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)}\left(\varepsilon(t) - \frac{\sigma_1^2(t)}{2}\right) - \left(\gamma(t) + \frac{\sigma_2^2(t)}{2}\right)\right) \\ &= -\rho + \left(\frac{b^l}{\varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)}\left(\varepsilon(t) - \frac{\sigma_1^2(t)}{2}\right) - \left(\gamma(t) + \frac{\sigma_2^2(t)}{2}\right)\right). \end{aligned}$$

Then $w(t)$ is a θ-periodic function. This, together with (17), implies that

$$\mathcal{L}(V_1(N,P) + w(t)) \le -\rho + \frac{(\delta-\gamma)^u P}{1+P} + \frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)}P. \tag{18}$$

Furthermore

$$\begin{aligned} \mathcal{L}(V_1(N,P) + w(t) + HP) &\le -\rho + \frac{(\delta-\gamma)^u P}{1+P} + \frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)}P + HP\left(-\gamma(t) - \frac{(\delta(t)-\gamma(t))P}{1+P} + \frac{b(t)N}{\beta(t)+N}\right) \\ &\le -\rho + \frac{(\delta-\gamma)^u P}{1+P} + \frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)}P - \frac{H(\delta-\gamma)^l P^2}{1+P} + \frac{Hb^u PN}{\beta^l + N} \\ &= -\frac{F(P)}{1+P} + \frac{Hb^u PN}{\beta^l+N}, \end{aligned}$$

where

$$F(P) = \left(H(\delta-\gamma)^l - \frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)}\right)P^2 - \left(\frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)} + (\delta-\gamma)^u - \rho\right)P + \rho.$$

Note that $\left(\frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)} + (\delta-\gamma)^u - \rho\right)^2 - 4\rho\left(H(\delta-\gamma)^l - \frac{a^u b^l}{\varepsilon^u \beta^l\left(1+\frac{\beta^u}{K^l}\right)}\right) < 0$ when condition (15) holds. This implies that $F(P) > 0$ for all $P \in (0,\infty)$. Therefore, define a positive constant $A = \inf_{P\in(0,\infty)} \frac{F(P)}{1+P}$, then one derives

$$\mathcal{L}(V_1(N,P) + w(t) + HP) \le -A + \frac{Hb^u PN}{\beta^l + N}. \tag{19}$$

Also

$$\begin{aligned} &\mathcal{L}V_2(N,P) \\ &= \left(N + \frac{a^l}{b^u}P\right)^{\vartheta}\left(\varepsilon(t)N - \frac{\varepsilon(t)}{K(t)}N^2 - \frac{a(t)PN}{\beta(t)+N} - \frac{a^l}{b^u}P\frac{\gamma(t)+\delta(t)P}{1+P} + \frac{a^l b(t)PN}{b^u(\beta(t)+N)}\right) \\ &\quad + \frac{\vartheta}{2}\left(N + \frac{a^l}{b^u}P\right)^{\vartheta-1}\left(\sigma_1^2(t)N^2 + \left(\frac{a^l}{b^u}\right)^2\sigma_2^2(t)P^2\right) \end{aligned} \tag{20}$$

$$\begin{aligned}
&\leq \left(N + \frac{a^l}{b^u}P\right)^{\vartheta} \left(\varepsilon^u N - \frac{\varepsilon^l}{K^u}N^2 - \frac{a^l}{b^u}\frac{(\delta-\gamma)^l P^2}{1+P} - \frac{a^l PN}{\beta(t)+N} + \frac{a^l b^u PN}{b^u(\beta(t)+N)}\right) \\
&\quad + \frac{\vartheta}{2}\left(N + \frac{a^l}{b^u}P\right)^{\vartheta-1}\left(\sigma_1^{2u}N^2 + \left(\frac{a^l}{b^u}\right)^2 \sigma_2^{2u}P^2\right) \\
&\leq \left(N + \frac{a^l}{b^u}P\right)^{\vartheta} \left(\varepsilon^u N - \frac{\varepsilon^l}{K^u}N^2 - \frac{a^l}{b^u}\frac{(\delta-\gamma)^l P^2}{1+P}\right) + \frac{\vartheta}{2}\left(N + \frac{a^l}{b^u}P\right)^{\vartheta-1}\left(\sigma_1^{2u}N^2 + \left(\frac{a^l}{b^u}\right)^2 \sigma_2^{2u}P^2\right) \\
&\leq \left(N + \frac{a^l}{b^u}P\right)^{\vartheta} \varepsilon^u N - \left(N + \frac{a^l}{b^u}P\right)^{\vartheta}\frac{\varepsilon^l}{K^u}N^2 - \left(N + \frac{a^l}{b^u}P\right)^{\vartheta}\frac{a^l}{b^u}\frac{(\delta-\gamma)^l P^2}{1+P} \\
&\quad + \frac{\vartheta}{2}\left(N + \frac{a^l}{b^u}P\right)^{\vartheta-1}\left(\sigma_1^{2u}N^2 + \left(\frac{a^l}{b^u}\right)^2 \sigma_2^{2u}P^2\right) \\
&\leq 2^{\vartheta}\varepsilon^u N\left(N^{\vartheta} + \left(\frac{a^l}{b^u}P\right)^{\vartheta}\right) - \frac{\varepsilon^l}{K^u}N^{2+\vartheta} - \left(\frac{a^l}{b^u}\right)^{\vartheta+1}(\delta-\gamma)^l\frac{P^{2+\vartheta}}{1+P} + \frac{\vartheta}{2}\sigma_1^{2u}N^{1+\vartheta} + \frac{\vartheta}{2}\left(\frac{a^l}{b^u}\right)^{\vartheta+1}\sigma_2^{2u}P^{\vartheta+1} \\
&\leq 2^{\vartheta}\varepsilon^u N^{1+\vartheta} + 2^{\vartheta-1}\varepsilon^u\left(\frac{a^l}{b^u}\right)^{\vartheta}N^2 - \frac{\varepsilon^l}{K^u}N^{2+\vartheta} + \frac{\vartheta}{2}\sigma_1^{2u}N^{1+\vartheta} \\
&\quad + 2^{\vartheta-1}\varepsilon^u\left(\frac{a^l}{b^u}\right)^{\vartheta}P^{2\vartheta} + \frac{\vartheta}{2}\left(\frac{a^l}{b^u}\right)^{\vartheta+1}\sigma_2^{2u}P^{1+\vartheta} - \left(\frac{a^l}{b^u}\right)^{\vartheta+1}(\delta-\gamma)^l\frac{P^{2+\vartheta}}{1+P} \\
&=: f(N) + g(P).
\end{aligned} \tag{21}$$

Clearly

$$f(N) \to -\infty, \quad as\ N \to +\infty.$$

Applying inequalities $0 < \vartheta < 1$ and (14) yields

$$g(P) \to -\infty, \quad as\ P \to +\infty.$$

From (19) and (20), we obtain

$$\mathcal{L}V(N,P) \leq M\left(-A + \frac{Hb^u PN}{\beta^l + N}\right) + f(N) + g(P),$$

where M satisfy

$$-AM + f^u + g^u \leq -2.$$

To confirm condition (28) in Lemma 5.4, we consider the following bounded subset

$$U = \left\{\varepsilon_1 \leq N \leq \frac{1}{\varepsilon_1},\ \ \varepsilon_2 \leq P \leq \frac{1}{\varepsilon_2}\right\},$$

where ε_1, $\varepsilon_2 \in (0,1)$ are sufficiently small positive constants satisfying the following inequalities

$$-MA + MHb^u\varepsilon_2 + f^u + g^u \leq -1, \tag{22}$$

$$-MA + \frac{MHb^u}{\beta^l}\frac{\varepsilon_1}{\varepsilon_2} + f^u + g^u \leq -1, \tag{23}$$

$$-MA + f^u + B - \left(\frac{a^l}{b^u}\right)^{\vartheta+1}\frac{\eta}{2}\frac{1}{\varepsilon_2^{\vartheta+1}} \leq -1, \tag{24}$$

$$-MA + (MHb^uP + g(P))^u + C - \frac{\varepsilon^l}{2K^u}\frac{1}{\varepsilon_1^{2+\vartheta}} \le -1, \tag{25}$$

$$\varepsilon_1 = \varepsilon_2^2,$$

where inequality (22) can be derived from (16), the constants η, B and C will be determined later. Then

$$\mathbb{R}_+^2 \setminus U = U_1^c \cup U_2^c \cup U_3^c \cup U_4^c,$$

with

$$U_1^c = \left\{(N,P) \in \mathbb{R}_+^2 |\ 0 < P < \varepsilon_2\right\},\ \ U_2^c = \left\{(N,P) \in \mathbb{R}_+^2 |\ 0 < N < \varepsilon_1,\ \ \varepsilon_2 < P < \frac{1}{\varepsilon_2}\right\},$$

$$U_3^c = \left\{((N,P)) \in \mathbb{R}_+^2 |\ P > \frac{1}{\varepsilon_2}\right\},\ \ U_4^c = \left\{(N,P) \in \mathbb{R}_+^2 |\ N > \frac{1}{\varepsilon_1}\right\}.$$

Case 1. If $(N,P) \in U_1^c$, (22) implies that

$$\mathcal{L}V \le -MA + MHb^uP + f(N) + g(P) \le -MA + MHb^u\varepsilon_2 + f^u + g^u \le -1.$$

Case 2. If $(N,P) \in U_2^c$, we obtain that

$$\mathcal{L}V \le -MA + \frac{MHb^u}{\beta^l}\frac{\varepsilon_1}{\varepsilon_2} + f^u + g^u,$$

Choosing $\varepsilon_1 = \varepsilon_2^2$, combining (23), we have

$$\mathcal{L}V \le -MA + \frac{MHb^u}{\beta^l}\varepsilon_2 + f^u + g^u \le -1.$$

Case 3. If $(N,P) \in U_3^c$, we have

$$\mathcal{L}V \le -MA + f^u + B - \eta\left(\frac{a^l}{b^u}\right)^{\vartheta+1}\frac{P^{2+\vartheta}}{1+P} \le -MA + f^u + B - \left(\frac{a^l}{b^u}\right)^{\vartheta+1}\frac{\eta}{2}\frac{1}{\varepsilon_2^{1+\vartheta}} \le -1,$$

which follows from (24), where η and B satisfy $\frac{\vartheta}{2}\sigma_2^{2u} < (\delta-\gamma)^l - \eta$ and

$$B = \sup_{P\in(0,\infty)}\left\{MHb^uP + 2^{\vartheta-1}\varepsilon^u\left(\frac{a^l}{b^u}\right)^{\vartheta}P^{2\vartheta} + \frac{\vartheta}{2}\left(\frac{a^l}{b^u}\right)^{\vartheta+1}\sigma_2^{2u}P^{1+\vartheta} - \left(\frac{a^l}{b^u}\right)^{\vartheta+1}\left((\delta-\gamma)^l - \eta\right)\frac{P^{2+\vartheta}}{1+P}\right\} < \infty.$$

Case 4. If $(N,P) \in U_4^c$, we have by (25)

$$\mathcal{L}V \le -MA + MHb^uP + g(P) + C - \frac{\varepsilon^l}{2K^u}N^{2+\vartheta} \le -MA + (MHb^uP + g(P))^u + C - \frac{\varepsilon^l}{2K^u}\frac{1}{\varepsilon_1^{2+\vartheta}} \le -1,$$

where

$$C = \sup_{N\in(0,\infty)}\left\{2^{\vartheta}\varepsilon^uN^{1+\vartheta} + 2^{\vartheta-1}\varepsilon^u\left(\frac{a^l}{b^u}\right)^{\vartheta}N^2 - \frac{\varepsilon^l}{K^u}\frac{N^{2+\vartheta}}{2} + \frac{\theta}{2}\sigma_1^{2u}N^{1+\vartheta}\right\} < \infty,$$

From the above discussion it follows that

$$\mathcal{L}V \le -1,\ \ (N,P) \in \mathbb{R}_+^2 \setminus U.$$

Thus, condition (28) is verified. From Lemma 5.4 it follows that stochastic model (3) has a nontrivial positive periodic solution. The proof is complete. □

Remark 4.2. *Theorem 4.1 shows that stochastic periodic model (3) admits a nontrivial positive periodic solution under condition* $\frac{b^l}{\varepsilon^u\left(1+\frac{\beta^u}{K^l}\right)}\langle\varepsilon - \frac{\sigma_1^2}{2}\rangle_\theta - \langle\gamma + \frac{\sigma_2^2}{2}\rangle_\theta > 0$ *by using theory of Has'minskii. According to the proof of Theorem 4.1, we can similarly derive that the autonomous stochastic model (2) has an ergodic stationary distribution when* $\frac{b}{\varepsilon\left(1+\frac{\beta}{K}\right)}\left(\varepsilon - \frac{\sigma_1^2}{2}\right) - \left(\gamma + \frac{\sigma_2^2}{2}\right) > 0$*, that is,* $\lambda_2 > 0$.

5. Numerical Examples

In this section, we will introduce some numerical simulations to illustrate our main results by using the method developed in [22].

Example 5.1. *In autonomous stochastic model (2), let* $\varepsilon = 0.08$, $K = 100$, $a = 1$, $\beta = 2$, $\gamma = 0.1$, $\delta = 0.3$, $b = 0.9$ *and the initial value* $(N(0), P(0)) = (0.9, 0.7)$.

Case 1. Let the environmental noise intensities be $\sigma_1 = \sigma_2 = 0.1$. *Then* $\varepsilon > \sigma_1^2/2$ *and*

$$\lambda_2 = -\gamma - \frac{\sigma_2^2}{2} + \frac{b\left(\varepsilon - \frac{\sigma_1^2}{2}\right)}{\varepsilon + \frac{\beta\varepsilon}{K}} = 0.7222 > 0.$$

From Theorem 3.6 it follows that stochastic model (2) is persistent in the mean. See Fig.1.

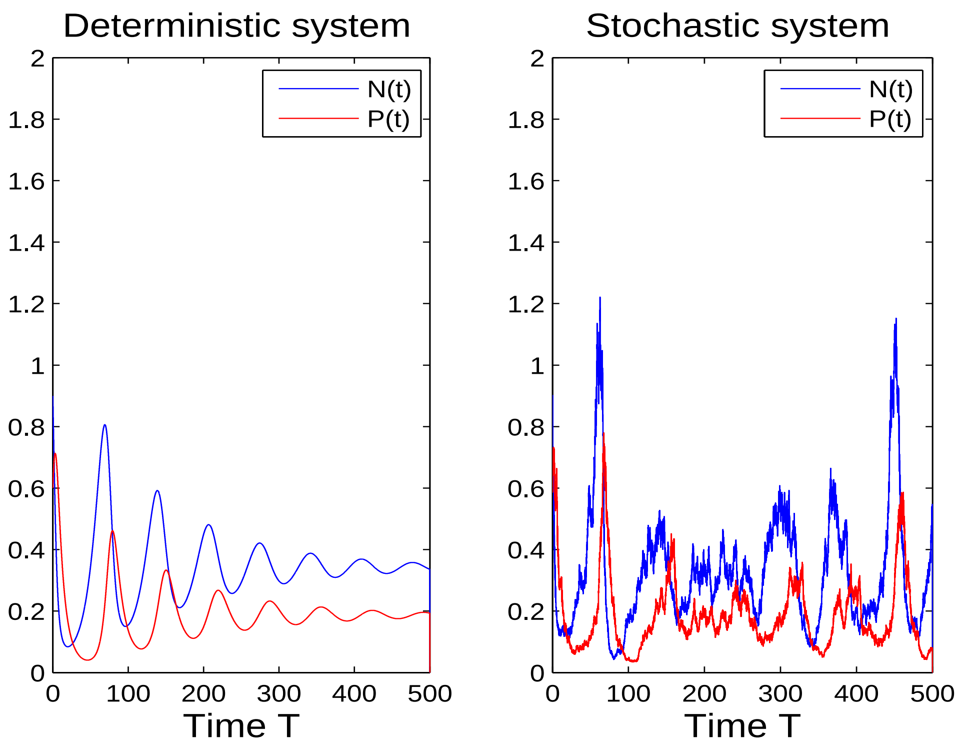

Figure 1: The left figure is the solution $(N(t), P(t))$ of deterministic model (1). The right figure is the solution of autonomous stochastic model (2) with $\sigma_1 = \sigma_2 = 0.1$.

Case 2. We choose environment noise $\sigma_1 = 0.1$, $\sigma_2 = 1.3$. Then $\varepsilon > \sigma_1^2/2$ and $\lambda_1 = -\gamma - \frac{\sigma_2^2}{2} + \frac{b\left(\varepsilon - \frac{\sigma_1^2}{2}\right)}{\varepsilon - \frac{\sigma_1^2}{2} + \frac{\beta\varepsilon}{K}} = -0.034 < 0$. Theorem 3.6 implies that the predator populations go to extinction and the prey is persistent in the mean. Fig. 2 confirms this. This also shows that large environmental noise can make population species extinct.

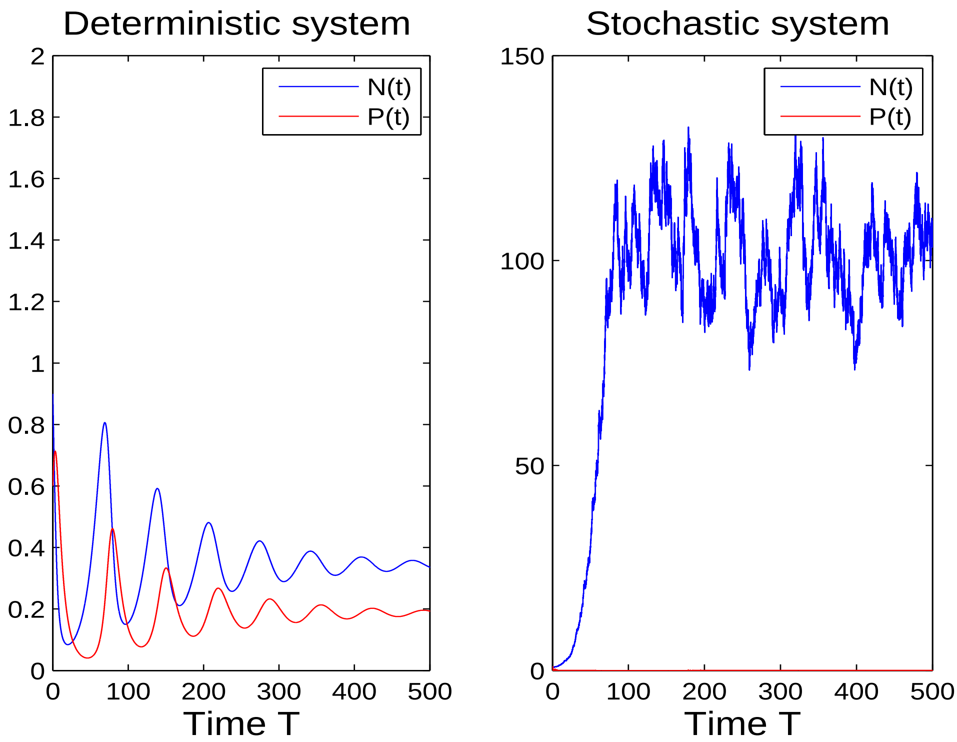

Figure 2: The left figure is the solution $(N(t), P(t))$ of deterministic model (1). The right figure is the solution of autonomous stochastic model (2) with $\sigma_1 = 0.1$ and $\sigma_2 = 1.3$. Hence large environmental noise can make population species extinct.

Example 5.2. *In non-autonomous stochastic model (3), let the parameters be $\varepsilon(t) = 0.08 + 0.06\sin t$, $K(t) = 100 + 90\sin t$, $a(t) = 1 + 0.5\sin t$, $\beta(t) = 2 + 0.8\sin t$, $\gamma(t) = 0.1 + 0.05\sin t$, $\delta(t) = 0.3 + 0.1\sin t$ and $b(t) = 0.9 + 0.6\sin t$. We choose $\sigma_1(t) = \sigma_2(t) = 0.03 + 0.01\sin t$, and then*

$$\frac{b^l}{\varepsilon^u\left(1 + \frac{\beta^u}{K^l}\right)}\langle \varepsilon - \frac{\sigma_1^2}{2}\rangle_\theta - \langle \gamma + \frac{\sigma_2^2}{2}\rangle_\theta > 0.$$

From Theorem 4.1 it follows that model (3) has a positive nontrivial periodic solution. Fig.3 confirms this.

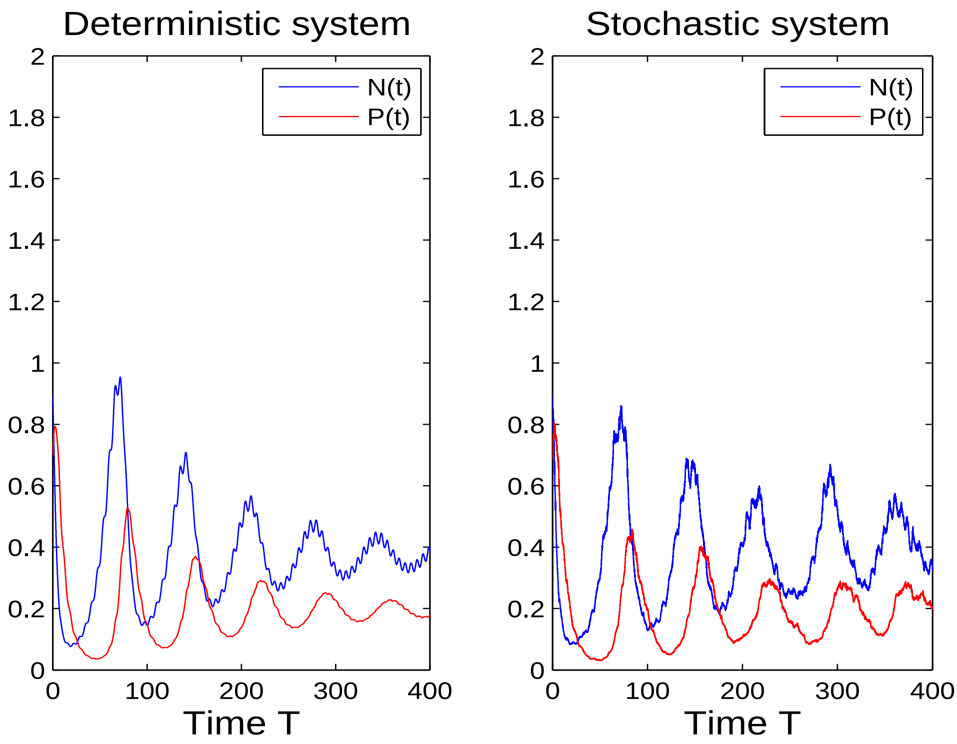

Figure 3: The right figure is the solution of stochastic model (3) and the left figure is corresponding solution of deterministic system.

Appendix

In this section, we will summarize some facts contained in [17].

Definition 5.3. *A stochastic process $x(t,\omega)$ is said to be periodic with period θ if its finite dimensional distributions are periodic with period θ, i.e., for any positive integer m and any moments of time $t_1,\ldots,t_m$, the joint distributions of the random variables $x(t_{1+k\theta},\omega),\ldots,x(t_{m+k\theta},\omega)$ are independent of k, $(k=\pm1,\pm2,\ldots)$.*

The transition function of a Markov process, $p\left(v,x(v),t,A\right) = \mathbb{P}\left(x(t)\in A\,|x(v)\right)$, *a.s.*, is called periodic if $p\left(v,x(v),t+v,A\right)$ is periodic in v.

Consider the following periodic stochastic equation

$$dx(t) = f(t,x(t))dt + g(t,x(t))dB(t),\quad x\in\mathbb{R}^n, \tag{26}$$

where functions f and g are θ-periodic in t.

Lemma 5.4. *Assume that system (26) admits a unique global solution. Suppose further that there exists a function $V(t,x)\in C^2$ in $\mathbb{R}^n$ which is θ-periodic in t, and satisfies the following conditions*

$$\inf_{|x|>R} V(t,x)\to\infty \ \text{ as } \ \mathrm{R}\to\infty, \tag{27}$$

and

$$\mathcal{L}V(t,x)\le -1 \ \text{ outside some compact set}, \tag{28}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V(t,x) = V_t(t,x) + V_x(t,x)f(t,x) + \frac{1}{2}\mathrm{trace}(\mathrm{g^T(t,x)V_{xx}(t,x)g(t,x)}).$$

Then system (26) has a θ-periodic solution.

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