Filomat 31:18 (2017), 5783–5790
https://doi.org/10.2298/FIL1718783A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Strongly Generalized Convex Functions

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Abstract. The main objective of this article is to introduce the notion of strongly generalized convex functions which is called as strongly η-convex functions. Some related integral inequalities of Hermite-Hadamard and Hermite-Hadamard-Fejér type are also obtained. Special cases are also investigated.

1. Introduction and Preliminaries

A function $F : \Omega = [x, y] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex function in the classical sense, if

$$F(tx + (1 - t)y) \leq tF(x) + (1 - t)F(y), \quad \forall x, y \in \Omega, t \in [0, 1].$$

One cannot deny the importance of convex functions. It can be viewed as one of the most natural and simple notion in mathematics. These functions play a key role in different fields of pure and applied sciences. Many researchers have been attracted to study different aspects of convex functions. And as gradually this concept has been generalized in various directions, see [4, 5, 8, 9, 13, 16]. Polyak [16] introduced the class of strongly convex functions. This class is defined as
A function $F : \Omega = [x, y] \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly convex function in the classical sense with modulus $\mu \geq 0$, if

$$F(tx + (1 - t)y) \leq tF(x) + (1 - t)F(y) - \mu t(1 - t)(y - x)^2, \quad \forall x, y \in \Omega, t \in [0, 1].$$

Strongly convex functions are useful in optimization, mathematical economics, nonlinear programming etc. Strong convexity is a strengthening of the notion of convexity. Some properties of strongly convex functions are just stronger versions of properties of convex functions. For more details, see [12, 13, 16] and for some recent studies on strongly convexity, see [2]. Recently Gordji et al. [9] has introduced the notion of η-convex function. This class generalizes the class of convex functions. It is defined as:

Definition 1.1. *A function $F : \Omega \subset \mathbb{R} \to \mathbb{R}$ is said to be η-convex function with respect to $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$F(tx + (1 - t)y) \leq F(y) + t\eta(F(x), F(y)), \quad \forall x, y \in \Omega, t \in [0, 1]. \tag{1.1}$$

2010 *Mathematics Subject Classification*. 26D15; 26A51
Keywords. Convex; η-convex; function; Hermite-Hadamard; Fejér; inequalities.
Received: 28 April 2017; Accepted: 25 August 2017
Communicated by Dragan S. Djordjevic
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They have also introduced the class of η-quasiconvex functions. A new refinement of Hermite-Hadamard's inequality was also proved by the authors. The study of relating theory of convex functions with theory of inequalities has been remained a constant inspiration for many researchers. This is one of the reason which makes theory of convex functions more attractive. For some recent investigations on this topic interested readers are referred to [7, 11, 14, 15, 17].

The main motivation of writing this article is to introduce the notion of strongly η-convex functions. We also derive some new integral inequalities of Hermite-Hadamard type for the class of strongly η-convex functions. Some special cases are also discussed.

Now we are in a position to introduce the class of strongly η-convex function.

Definition 1.2. *A function $F : \Omega \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly η-convex function with respect to $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and modulus $\mu \geq 0$, if*

$$F(tx + (1-t)y) \leq F(y) + t\eta(F(x), F(y)) - \mu t(1-t)(x-y)^2, \quad \forall x, y \in \Omega, t \in [0,1]. \tag{1.2}$$

Definition 1.3. *A function $F : \Omega \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly η-quasiconvex function with respect to $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and modulus $\mu \geq 0$, if*

$$F(tx + (1-t)y) \leq \max\{F(y), F(y) + \eta(F(x), F(y))\} - \mu t(1-t)(x-y)^2, \quad \forall x, y \in \Omega, t \in [0,1]. \tag{1.3}$$

Remark 1.4. *If $x = y$ in (1.2), then*

$$0 \leq \eta(F(x), F(x)).$$

If $t = 1$ in (1.2), then

$$F(x) - F(y) \leq \eta(F(x), F(y)).$$

Example 1.5. *It is not hard to verify by definition that if $F(x) = x^2$ and $\eta(x, y) = 2x + y$, then F is strongly η-convex function with modulus 1.*

Theorem 1.6. *Let $F : \Omega \to \mathbb{R}$ be a differentiable strongly η-convex function. If $x \in \Omega$ is the minimum of strongly η-convex function, then*

$$\langle F'(x), y - x \rangle \geq 0 \quad \Longrightarrow \quad \eta(F(y), F(x)) \geq \mu(y-x)^2. \tag{1.4}$$

Proof. Let $x \in \Omega$ be the minimum of strongly η-convex function F on Ω, then

$$F(x) \leq F(y).$$

Since Ω is a convex set, so $\forall x, y \in \Omega$, $t \in [0,1]$, we have $y_t = x + t(y-x) \in \Omega$. Setting $v = v_t$, we have

$$0 \leq F(x + t(y-x)) - F(x).$$

Dividing above inequality by t and taking limit as $t \to 0$, we have

$$\langle F'(x), y - x \rangle \geq 0. \tag{1.5}$$

Since it is given that F is strongly η-convex function, then

$$F(x + t(y-x)) \leq F(x) + t\eta(F(y), F(x)) - \mu t(1-t)(y-x)^2.$$

This implies

$$\frac{F(x + t(y-x)) - F(x)}{t} \leq \eta(F(y), F(x)) - \mu(1-t)(y-x)^2.$$

Taking limit on both sides as $t \to 0$, we have

$$\langle F'(x), y - x\rangle \leq \eta(F(y), F(x)) - \mu(y-x)^2.$$

Using (1.5) implies

$$\eta(F(y), F(x)) - \mu(y-x)^2 \geq 0.$$

This completes the proof. □

Lemma 1.7 ([10]). *If $F^{(n)}$ for $n \in \mathbb{N}$ exists and is integrable on $[c, d]$, then*

$$\frac{F(c)+F(d)}{2} - \frac{1}{d-c}\int_c^d F(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(d-c)^k}{2(k+1)!}F^{(k)}(c) = \frac{(d-c)^n}{2n!}\int_0^1 t^{n-1}(n-2t)F^{(n)}(tc+(1-t)d)\mathrm{d}t.$$

2. Main Results

In this section, we derive our main results. Our first result is a new refinement of Hermite-Hadamard type inequality via strongly η-convex function.

Theorem 2.1. *Let $F : [c, d] \to \mathbb{R}$ be strongly η-convex function with modulus $\mu \geq 0$. If $\eta(.,.)$ is bounded from above on $F([c, d]) \times F([c, d])$, then*

$$\begin{aligned} F\Big(\frac{c+d}{2}\Big) - \frac{M_\eta}{2} + \frac{\mu}{12}(d-c)^2 &\leq \frac{1}{d-c}\int_c^d F(x)\mathrm{d}x \\ &\leq \frac{F(c)+F(d)}{2} + \frac{\eta(F(c),F(d)) + \eta(F(d),F(c))}{4} - \frac{\mu}{6}(d-c)^2 \\ &\leq \frac{F(c)+F(d)}{2} + \frac{M_\eta}{2} - \frac{\mu}{6}(d-c)^2. \end{aligned} \tag{2.1}$$

Proof. Since it is known that F is a strongly η-convex function, that is

$$\begin{aligned} F\Big(\frac{c+d}{2}\Big) &= F\Big(\frac{c+d}{4} - \frac{t(d-c)}{4} + \frac{c+d}{4} + \frac{t(d-c)}{4}\Big) \\ &= F\bigg(\frac{1}{2}\Big(\frac{c+d-t(d-c)}{2}\Big) + \frac{1}{2}\Big(\frac{c+d+t(d-c)}{2}\Big)\bigg) \\ &\leq F\Big(\frac{c+d-t(d-c)}{2}\Big) + \frac{1}{2}\eta\bigg(F\Big(\frac{c+d-t(d-c)}{2}\Big), F\Big(\frac{c+d+t(d-c)}{2}\Big)\bigg) - \frac{\mu}{4}(2t-1)^2(d-c)^2 \\ &\leq F\Big(\frac{c+d-t(d-c)}{2}\Big) + \frac{M_\eta}{2} - \frac{\mu}{4}(2t-1)^2(d-c)^2. \end{aligned}$$

This implies

$$F\Big(\frac{c+d-t(d-c)}{2}\Big) \geq F\Big(\frac{c+d}{2}\Big) - \frac{M_\eta}{2} + \frac{\mu}{4}(2t-1)^2(d-c)^2,$$

and

$$F\Big(\frac{c+d+t(d-c)}{2}\Big) \geq F\Big(\frac{c+d}{2}\Big) - \frac{M_\eta}{2} + \frac{\mu}{4}(2t-1)^2(d-c)^2,$$

Now using the change of variable technique, we have

$$\begin{aligned}
\frac{1}{d-c}\int_c^d F(x)\mathrm{d}x &= \frac{1}{d-c}\Big[\int_c^{\frac{c+d}{2}} F(x)\mathrm{d}x + \int_{\frac{c+d}{2}}^d F(x)\mathrm{d}x\Big] \\
&= \frac{1}{2}\int_0^1 \Big[F\Big(\frac{c+d-t(d-c)}{2}\Big) + F\Big(\frac{c+d+t(d-c)}{2}\Big)\Big]\mathrm{d}t \\
&\geq \frac{1}{2}\int_0^1 \Big[2F\Big(\frac{c+d}{2}\Big) - M_\eta + \frac{\mu}{2}(2t-1)^2(d-c)^2\Big]\mathrm{d}t \\
&= F\Big(\frac{c+d}{2}\Big) - \frac{M_\eta}{2} + \frac{\mu}{12}(d-c)^2. \qquad (2.2)
\end{aligned}$$

We now move towards the right hand side of the theorem. For this utilizing the hypothesis of the theorem that F is strongly η-convex function, we have

$$F(tc + (1-t)d) \leq F(d) + t\eta(F(c), F(d)) - \mu t(1-t)(d-c)^2.$$

Integrating above inequality with respect to t on $[0,1]$ yields

$$\frac{1}{d-c}\int_c^d F(x)\mathrm{d}x \leq F(d) + \frac{1}{2}\eta(F(c), F(d)) - \frac{\mu}{6}(d-c)^2 = A.$$

Also

$$\frac{1}{d-c}\int_c^d F(x)\mathrm{d}x \leq F(c) + \frac{1}{2}\eta(F(d), F(c)) - \frac{\mu}{6}(d-c)^2 = B.$$

Therefore, we have

$$\begin{aligned}
\frac{1}{d-c}\int_c^d F(x)\mathrm{d}x &\leq \min\{A, B\} \\
&\leq \frac{F(c)+F(d)}{2} + \frac{\eta(F(c),F(d)) + \eta(F(d),F(c))}{4} - \frac{\mu}{6}(d-c)^2 \\
&\leq \frac{F(c)+F(d)}{2} - \frac{\mu}{6}(d-c)^2 + \frac{M_\eta}{2}. \qquad (2.3)
\end{aligned}$$

This completes the proof. □

Remark 2.2. *It is worth to mention here that if $\mu = 0$, then Theorem 2.1 reduces to Theorem 5 [8]. If $\eta(x,y) = x - y$, then Theorem 2.1 reduces to the result for strongly convex functions in the classical sense [13]. If $\mu = 0$ and $\eta(x,y) = x - y$, then Theorem 2.1 reduces to classical Hermite-Hadamard inequality.*

Our next result is a Fejér type inequality via strongly η-convex function.

Theorem 2.3. *Let $F : [c,d] \to \mathbb{R}$ be a strongly η-convex function such that $\eta(.,.)$ is bounded above on $F([c,d]) \times F([c,d])$. Also suppose that $W : [c,d] \to \mathbb{R}^+$ is integrable and symmetric with respect to $\frac{c+d}{2}$, then*

$$F\Big(\frac{c+d}{2}\Big)\int_a^b W(x)\mathrm{d}x - \frac{\mu}{4}\int_a^b (a+b-2x)^2 W(x)\mathrm{d}x + L_\eta(c,d)$$
$$\leq \int_a^b F(x)W(x)\mathrm{d}x$$
$$\leq \frac{F(c)+F(d)}{2}\int_c^d W(x)\mathrm{d}x - \frac{\mu}{d-c}\int_c^d (d-c)(x-c)W(x)\mathrm{d}x + R_\eta(c,d),$$

where

$$L_\eta(a,b) := \frac{1}{2}\int_a^b \eta(F(a+b-x), F(x))W(x)\mathrm{d}x,$$

and

$$R_\eta(c,d) := \frac{\eta(F(c),F(d)) + \eta(F(d),F(c))}{2(d-c)}\int_c^d (d-x)W(x)\mathrm{d}x,$$

respectively.

Proof. Since from the hypothesis of theorem it is given that F is strongly η-convex function, then

$$F\Big(\frac{c+d}{2}\Big)$$
$$\leq F((1-t)c+td) + \frac{1}{2}\eta(F((1-t)d+tc), F((1-t)c+td)) - \frac{\mu}{4}[((1-t)c+td)-(tc+(1-t)d)]^2.$$

Using the integrability and symmetry of $W : [c,d] \to \mathbb{R}^+$ with respect to $\frac{c+d}{2}$ implies

$$\begin{aligned}
F\Big(\frac{c+d}{2}\Big)\int_c^d W(x)\mathrm{d}x &= F\Big(\frac{c+d}{2}\Big)\int_0^1 W((1-t)c+td)(d-c)\mathrm{d}t \\
&\leq \int_0^1 F((1-t)c+td)W((1-t)c+td)(d-c)\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^1 \eta(F((1-t)d+tc), F((1-t)c+td))W((1-t)c+td)(d-c)\mathrm{d}t \\
&\quad - \frac{\mu}{4}\int_0^1 [((1-t)c+td)-(tc+(1-t)d)]^2 W((1-t)c+td)(d-c)\mathrm{d}t \\
&= \int_c^d F(x)W(x)\mathrm{d}x + \frac{1}{2}\int_c^d \eta(F(c+d-x), F(x))W(x)\mathrm{d}x - \frac{\mu}{4}\int_c^d (c+d-2x)^2 W(x)\mathrm{d}x.
\end{aligned}$$

We now prove the right hand side of the theorem. Again using the fact that F is strongly η-convex function and the fact that $W(x)$ is symmetric about $\frac{c+d}{2}$, we have

$$\begin{aligned}
\int_c^d F(x)W(x)\mathrm{d}x &\leq (d-c)\int_0^1 [F(d)+t\eta(F(c),F(d))-\mu t(1-t)(d-c)^2]W(tc+(1-t)d)\mathrm{d}t \\
&= (d-c)\left[\int_0^1 F(d)W(tc+(1-t)d)\mathrm{d}t+\eta(F(c),F(d))\int_0^1 tg(tc+(1-t)d)\mathrm{d}t\right. \\
&\qquad\left. -\mu(d-c)^2\int_0^1 t(1-t)W(tc+(1-t)d)\mathrm{d}t\right]. \qquad (2.4)
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_c^d F(x)W(x)\mathrm{d}x &\leq (d-c)\left[\int_0^1 F(c)W(tc+(1-t)d)\mathrm{d}t+\eta(F(d),F(c))\int_0^1 tg(tc+(1-t)d)\mathrm{d}t\right. \\
&\qquad\left. -\mu(d-c)^2\int_0^1 t(1-t)W(tc+(1-t)d)\mathrm{d}t\right]. \qquad (2.5)
\end{aligned}$$

Now adding (2.4) and (2.5), we have

$$\begin{aligned}
&2\int_c^d F(x)W(x)\mathrm{d}x \\
&\leq (d-c)[F(c)+F(d)]\int_0^1 W(tc+(1-t)d)\mathrm{d}t+(d-c)[\eta(F(c),F(d))+\eta(F(d),F(c))]\int_0^1 tg(tc+(1-t)d)\mathrm{d}t \\
&\qquad -2\mu(d-c)^2\int_0^1 t(1-t)W(tc+(1-t)d)\mathrm{d}t.
\end{aligned}$$

Using change of variable technique, we have

$$\begin{aligned}
\int_c^d F(x)W(x)\mathrm{d}x &\leq \frac{F(c)+F(d)}{2}\int_c^d W(x)\mathrm{d}x+\frac{\eta(F(c),F(d))+\eta(F(d),F(c))}{2(d-c)}\int_c^d (d-x)W(x)\mathrm{d}x \\
&\qquad -\frac{\mu}{d-c}\int_c^d (d-c)(x-c)W(x)\mathrm{d}x.
\end{aligned}$$

This completes the proof. □

Remark 2.4. *If $\mu = 0$ then Theorem 2.3 coincides with Theorem 6 and Theorem 7 of [8]. Observe that if $W(x) = 1$, then Theorem 2.3 reduces to Theorem 2.1. If $\mu = 0$ and $\eta(x, y) = x - y$, then Theorem 2.3 reduces to classical Hermite-Hadamard-Fejér inequality. Under the assumptions of Theorem 2.3, if we take $\eta(x, y) = x - y$, then we have Theorem 1 [3].*

Now using Lemma 1.7, we derive a new variant of Hermite-Hadamard inequality via differentiable strongly η-convex functions.

Theorem 2.5. *Let $F : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a n-times differentiable strongly η-convex function on I° where $c, d \in I^\circ$ with $c < d$ and $F' \in L_1[c, d]$. If $|F'|^p$ is strongly η-convex function with $\mu \geq 1$, then for $n \geq 2$ and $p \geq 1$, we have*

$$\left|\frac{F(c)+F(d)}{2} - \frac{1}{d-c}\int_c^d F(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(d-c)^k}{2(k+1)!}F^{(k)}(c)\right|$$
$$\leq \frac{(d-c)^n}{2n!}\psi_1^{1-\frac{1}{p}}(n)\left[\psi_1(n)|F^{(n)}(d)|^p + \psi_2(n)\eta(|F^{(n)}(c)|^p, |F^{(n)}(d)|^p) - \psi_3(n)\mu(x-y)^2\right],$$

where

$$\psi_1(n) := \frac{n-1}{n+1},$$

$$\psi_2(n) := \frac{n^2-2}{(n+1)(n+2)},$$

and

$$\psi_3(n) := \frac{n-1}{(n+1)(n+3)}.$$

Proof. For $p = 1$, since it is known that $|F'|$ is strongly η-convex function, then using the property of modulus, we have

$$\left|\frac{F(c)+F(d)}{2} - \frac{1}{d-c}\int_c^d F(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(d-c)^k}{2(k+1)!}F^{(k)}(c)\right|$$
$$\leq \frac{(d-c)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left|F^{(n)}(tc+(1-t)d)\right|\mathrm{d}t$$
$$\leq \frac{(d-c)^n}{2n!}\left[|F^{(n)}(d)|\int_0^1 t^{n-1}(n-2t)\mathrm{d}t + \eta(|F^{(n)}(c)|, |F^{(n)}(d)|)\int_0^1 t^n(n-2t)\mathrm{d}t \right.$$
$$\left. -\mu(x-y)^2\int_0^1 t^n(1-t)(n-2t)\mathrm{d}t\right]$$
$$= \frac{(d-c)^n}{2n!}\left[\frac{n-1}{n+1}|F^{(n)}(d)| + \Big(\frac{n^2-2}{(n+1)(n+2)}\Big)\eta(|F^{(n)}(c)|, |F^{(n)}(d)|) - \Big(\frac{\mu(n-1)}{(n+1)(n+3)}\Big)(x-y)^2\right].$$

The proof for the case $p = 1$ is complete. We now proceed towards the case when $p1$, for this, using the Holder's inequality, we have

$$\left|\frac{F(c)+F(d)}{2} - \frac{1}{d-c}\int_c^d F(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(d-c)^k}{2(k+1)!}F^{(k)}(c)\right|$$
$$\leq \frac{(d-c)^n}{2n!}\left[\int_0^1 t^{n-1}(n-2t)\mathrm{d}t\right]^{1-\frac{1}{p}}\left[\int_0^1 t^{n-1}(n-2t)\left|F^{(n)}(tc+(1-t)d)\right|^p\mathrm{d}t\right]^{\frac{1}{p}}$$

$$\leq \frac{(d-c)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{p}}\left[|F^{(n)}(d)|^p\int_0^1 t^{n-1}(n-2t)\mathrm{d}t\right.$$

$$\left.+\eta(F^{(n)}(c),F^{(n)}(d))\int_0^1 t^n(n-2t)\mathrm{d}t-\mu(x-y)^2\int_0^1 t^n(n-2t)(1-t)\mathrm{d}t\right]$$

$$=\frac{(d-c)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{p}}\left[\frac{n-1}{n+1}|F^{(n)}(d)|^p+\Big(\frac{n^2-2}{(n+1)(n+2)}\Big)\eta(|F^{(n)}(c)|^p,|F^{(n)}(d)|^p)-\Big(\frac{\mu(n-1)}{(n+1)(n+3)}\Big)(x-y)^2\right].$$

This completes the proof. □

Conclusion

Some new joint generalizations of strong convexity and η-convexity of functions have been introduced. Some new counterparts of Hermite-Hadamard and Fejér inequalities via strongly η-convex functions are obtained. It has been noticed that these new results contain previously known results. It is worth to mention here that one can easily obtain counterparts of Iynger type of inequalities via Definition 1.3. We would like to emphasize here that the one can extend the main results of this paper for strongly generalized beta preinvex functions, [1]. This will be an interesting problem for future research.

Acknowledgements

Authors are thankful to the editor and anonymous referee for their valuable comments and suggestions. Authors are pleased to acknowledge the "support of Distinguished Scientist Fellowship Program (DSFP), King Saud University, Riyadh, Saudi Arabia".

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