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Berezin Number Inequality for Convex Function in Reproducing Kernel Hilbert Space

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Abstract. By using Hardy-Hilbert's inequality, some power inequalities for the Berezin number of a self-adjoint operators in Reproducing Kernel Hilbert Spaces (RKHSs) with applications for convex functions are given.

1. Introduction

If $p > 1$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, $a_m, b_n \geq 0$, such that $0 < \sum\limits_{m=0}^{\infty} a_m^p < \infty$ and $0 < \sum\limits_{n=0}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. The equivalent form of (1) is as follows:

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \sum_{n=1}^{\infty} a_n^p, \tag{2}$$

where the constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ is the best possible. The equivalent integral analogues of (1) and (2) are as follows:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)\,g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)\,dx\right)^{1/p}\left(\int_0^{\infty} g^q(x)\,dx\right)^{1/q}, \tag{3}$$

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$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right) dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x)\, dx.$$

Inequalities (1) and (3) are called the Hardy-Hilbert's inequality and Hardy-Hilbert's integral inequality, respectively (see [12, 21]).

The Hardy-Hilbert inequalities and their applications have been studied by many authors in operator theory. For more information about the Hardy-Hilbert inequalities and its consequences, see [5, 8, 10, 11, 16, 17] and references therein.

Recall that a reproducing kernel Hilbert space (shorty, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that:

(a) the evaluation functional $f \to f(\lambda)$ is continuos for each $\lambda \in \Omega$;

(b) for any $\lambda \in \Omega$ there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then by the classical Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_{\mathcal{H},\lambda} \in \mathcal{H}$ such that $f(\lambda) = \left\langle f, k_{\mathcal{H},\lambda} \right\rangle$ for all $f \in \mathcal{H}$. The function $k_{\mathcal{H},\lambda}$ is called the reproducing kernel of the space $\mathcal{H}$. It is well known that (see [3, 18])

$$k_{\mathcal{H},\lambda}(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of the space $\mathcal{H}(\Omega)$. Let $\widehat{k}_{\mathcal{H},\lambda} = \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}$ denote the normalized reproducing kernel of the space $\mathcal{H}$ (note that by (b), we surely have $k_\lambda \neq 0$). For a bounded linear operator A on the RKHS $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ is defined by the formula (see [4])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle_{\mathcal{H}} \ (\lambda \in \Omega).$$

The Berezin symbol is a function that is bounded by the numerical radius of the operator. On the most familiar RKHS, the Berezin symbol uniquely determines, that is, $\widetilde{A}(\lambda) = \widetilde{B}(\lambda)$ for all λ implies $A = B$. (For applications in the various questions of analysis of Berezin symbols see, for instance, [14, 15]).

Berezin set and Berezin number of operator A are defined by (see Karaev [13])

$$Ber(A) := Range\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\}$$

and

$$ber(A) := \sup\left\{\left|\widetilde{A}(\lambda)\right| : \lambda \in \Omega\right\},$$

respectively.

Recall that $W(A) := \left\{\langle Af, f\rangle : \|f\|_{\mathcal{H}} = 1\right\}$ is the numerical range of the operator A and

$$w(A) := \sup\left\{\left|\langle Af, f\rangle\right| : \|f\|_{\mathcal{H}} = 1\right\}$$

is the numerical radius of A. It is trivial that

$$Ber(A) \subset W(A) \text{ and } ber(A) \leq w(A) \leq \|A\|$$

for any $A \in \mathcal{B}(\mathcal{H})$. More information about the numerical radius and numerical range can be found, for example, in [1, 2, 6, 7, 9, 19, 20].

It is open question in the literature whether the inequalities

$$ber(A^n) \leq (ber(A))^n, \ (n \geq 2)$$

and

$$(ber\,(A))^n \leq C\,(ber\,(A^n))\,, \;\; (n < 1) \tag{4}$$

are hold. The questions are partially solved by Garayev et al. [8]. However, this inequalities are not known for convex functions. So, in this article, we partially solve (4) for convex functions and some special operators in RKHS.

2. The Main Results

In the following result, we prove an inequality similar to (1) for convex functions and self-adjoint operators acting on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$.

Theorem 2.1. *Let $f, g : J \to [0, \infty)$ be convex functions. If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{2} f(\widetilde{A}(\lambda)) g(\widetilde{A}(\lambda)) + \frac{1}{3} f(\widetilde{A}(\lambda)) g\left(\widetilde{B}\,(\eta)\right) + \frac{1}{3} f\left(\widetilde{B}\,(\eta)\right) g(\widetilde{A}(\lambda)) + \frac{1}{4} \widetilde{f\,(B)\,g}\,(B)\,(\eta)$$
$$< \frac{\pi}{\sin(\pi/p)} \left(\frac{1}{p} \left(\widetilde{f^p\,(A)}\,(\lambda) + \widetilde{f^p\,(B)}\,(\eta) \right) + \frac{1}{q} \left(\widetilde{g^q\,(A)}\,(\lambda) + \widetilde{g^q\,(B)}\,(\eta) \right) \right)$$

for any self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ with spectrum contained in J and all $\lambda, \eta \in \Omega$.

Proof. Let a_1, a_2, b_1, b_2 be positive scalars. Then using (1), we obtain

$$\frac{a_1 b_1}{2} + \frac{a_1 b_2}{3} + \frac{a_2 b_1}{3} + \frac{a_2 b_2}{4} < \frac{\pi}{\sin(\pi/p)} \left(a_1^p + a_2^p \right)^{1/p} \left(b_1^q + b_2^q \right)^{1/q}. \tag{5}$$

Let $x, y \in J$. By taking into consideration that $f, g \geq 0$ and placing $a_1 = f\,(x)$, $a_2 = f\,(y)$, $b_1 = g\,(x)$, $b_2 = g\,(y)$ in (5), we have

$$\frac{1}{2} f(x) g(x) + \frac{1}{3} f(x) g(y) + \frac{1}{3} f(y) g(x) + \frac{1}{4} f(y) g(y) \tag{6}$$
$$< \frac{\pi}{\sin(\pi/p)} \left(f^p\,(x) + f^p\,(y) \right)^{1/p} \left(g^q\,(x) + g^q\,(x) \right)^{1/q}$$

for all $x, y \in J$. Putting $x = \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle$ in (6), we have

$$\frac{1}{2} f(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle) g(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle) + \frac{1}{3} f(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle) g(y) + \frac{1}{3} f(y) g(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle) + \frac{1}{4} f(y) g(y)$$
$$< \frac{\pi}{\sin(\pi/p)} \left(f^p \left(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle \right) + f^p\,(y) \right)^{1/p} \left(g^q \left(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle \right) + g^q\,(y) \right)^{1/q}$$

for all $\lambda \in \Omega$ and any $y \in J$. Applying the functional calculus for B to the above inequality (since B is self-adjoint operator), we have

$$\frac{1}{2} f(\widetilde{A}(\lambda)) g(\widetilde{A}(\lambda)) + \frac{1}{3} f(\widetilde{A}(\lambda)) \left\langle g\,(B)\,\widehat{k}_{\mathcal{H},\eta}, \widehat{k}_{\mathcal{H},\eta} \right\rangle + \frac{1}{3} \left\langle f\,(B)\,\widehat{k}_{\mathcal{H},\eta}, \widehat{k}_{\mathcal{H},\eta} \right\rangle g(\widetilde{A}(\lambda)) + \frac{1}{4} \left\langle f\,(B)\,g\,(B)\,\widehat{k}_{\mathcal{H},\eta}, \widehat{k}_{\mathcal{H},\eta} \right\rangle$$
$$< \frac{\pi}{\sin(\pi/p)} \left\langle \left(f^p \left(\widetilde{A}(\lambda) \right) + f^p\,(B) \right)^{1/p} \left(g^q \left(\widetilde{A}(\lambda) \right) + g^q\,(B) \right)^{1/q} \widehat{k}_{\mathcal{H},\eta}, \widehat{k}_{\mathcal{H},\eta} \right\rangle$$

for all $\lambda, \eta \in \Omega$.

This shows that

$$\frac{1}{2}f(\widetilde{A}(\lambda))g(\widetilde{A}(\lambda))+\frac{1}{3}f(\widetilde{A}(\lambda))\widetilde{g(B)}(\eta)+\frac{1}{3}\widetilde{f(B)}(\eta)\,g(\widetilde{A}(\lambda))+\frac{1}{4}\widetilde{f(B)\,g(B)}(\eta) \tag{7}$$
$$<\frac{\pi}{\sin(\pi/p)}\widetilde{\left[\left(f^{p}\left(\widetilde{A}(\lambda)\right)+f^{p}(B)\right)^{1/p}\left(g^{q}\left(\widetilde{A}(\lambda)\right)+g^{q}(B)\right)^{1/q}\right]}(\eta)$$

From the convexity of f and g, we obtain that

$$f\left(\left\langle B\widehat{k}_{\mathcal{H},\eta},\widehat{k}_{\mathcal{H},\eta}\right\rangle\right)\leq\left\langle f(B)\,\widehat{k}_{\mathcal{H},\eta},\widehat{k}_{\mathcal{H},\eta}\right\rangle\left(\text{ or } f\left(\widetilde{B}(\eta)\right)\leq\widetilde{f(B)}(\eta)\right)$$

and

$$g\left(\left\langle B\widehat{k}_{\mathcal{H},\eta},\widehat{k}_{\mathcal{H},\eta}\right\rangle\right)\leq\left\langle g(B)\,\widehat{k}_{\mathcal{H},\eta},\widehat{k}_{\mathcal{H},\eta}\right\rangle\left(\text{ or } g\left(\widetilde{B}(\eta)\right)\leq\widetilde{g(B)}(\eta)\right).$$

Thus,

$$\frac{1}{2}f(\widetilde{A}(\lambda))g(\widetilde{A}(\lambda))+\frac{1}{3}f(\widetilde{A}(\lambda))\widetilde{g(B)}(\eta)+\frac{1}{3}\widetilde{f(B)}(\eta)\,g(\widetilde{A}(\lambda))+\frac{1}{4}\widetilde{f(B)\,g(B)}(\eta) \tag{8}$$
$$\geq\frac{1}{2}f(\widetilde{A}(\lambda))g(\widetilde{A}(\lambda))+\frac{1}{3}f(\widetilde{A}(\lambda))g\left(\widetilde{B}(\eta)\right)+\frac{1}{3}f\left(\widetilde{B}(\eta)\right)g(\widetilde{A}(\lambda))+\frac{1}{4}\widetilde{f(B)\,g(B)}(\eta).$$

The convexity of f and g and the power functions x^r $(r\geq 1)$ follow that

$$f^{p}\left(\widetilde{A}(\lambda)\right)\leq\widetilde{f^{p}(A)}(\lambda),$$
$$g^{q}\left(\widetilde{A}(\lambda)\right)\leq\widetilde{g^{q}(A)}(\lambda).$$

Hence

$$\left(f^{p}\left(\widetilde{A}(\lambda)\right)+f^{p}(B)\right)^{1/p}\left(g^{q}\left(\widetilde{A}(\lambda)\right)+g^{q}(B)\right)^{1/q} \tag{9}$$
$$\leq\left(\widetilde{f^{p}(A)}(\lambda)+f^{p}(B)\right)^{1/p}\left(\widetilde{g^{q}(A)}(\lambda)+g^{q}(B)\right)^{1/q}.$$

Since the operators $\widetilde{f^{p}(A)}(\lambda)+f^{p}(B)$ and $\widetilde{g^{q}(A)}(\lambda)+g^{q}(B)$ commute, we get from the arithmetic-geometric mean inequality that

$$\left(\widetilde{f^{p}(A)}(\lambda)+f^{p}(B)\right)^{1/p}\left(\widetilde{g^{q}(A)}(\lambda)+g^{q}(B)\right)^{1/q} \tag{10}$$
$$\leq\frac{1}{p}\left(\widetilde{f^{p}(A)}(\lambda)+f^{p}(B)\right)+\frac{1}{q}\left(\widetilde{g^{q}(A)}(\lambda)+g^{q}(B)\right).$$

Combining the above (9) and (10) we have

$$\widetilde{\left[\left(f^{p}\left(\widetilde{A}(\lambda)\right)+f^{p}(B)\right)^{1/p}\left(g^{q}\left(\widetilde{A}(\lambda)\right)+g^{q}(B)\right)^{1/q}\right]}(\eta)$$
$$\leq\frac{1}{p}\left(\widetilde{f^{p}(A)}(\lambda)+\widetilde{f^{p}(B)}(\eta)\right)+\frac{1}{q}\left(\widetilde{g^{q}(A)}(\lambda)+\widetilde{g^{q}(B)}(\eta)\right).$$

So, we have desired result from (7) , (8) and (10). □

Corollary 2.2. *Let $f:J\to[0,\infty)$ be a convex function. Then we have*

$$[f(ber(A))]^{2}<\left[\frac{12}{7}\pi-\frac{3}{14}\right]ber\left(f^{2}(A)\right)$$

for any self-adjoint operators $A\in\mathcal{B}(\mathcal{H})$ with spectrum contained in J.

Proof. In particular, for $B = A$, $g = f$, $\mu = \eta$ and $p = q$ in Theorem 2.1, we obtain

$$\frac{7}{6}\left[f(\widetilde{A}(\lambda))\right]^2 < \left[2\pi - \frac{1}{4}\right]\widetilde{f^2(A)}(\lambda)$$

and hence

$$\left[f(\widetilde{A}(\lambda))\right]^2 < \left[\frac{12}{7}\pi - \frac{3}{14}\right]\widetilde{f^2(A)}(\lambda)$$

for all $\lambda \in \Omega$. Since $\left[f(\widetilde{A}(\lambda))\right]^2 \geq 0$ and $\widetilde{f^2(A)}(\lambda) \geq 0$, we get

$$\left[f(\widetilde{A}(\lambda))\right]^2 < \left[\frac{12}{7}\pi - \frac{3}{14}\right]\sup_{\lambda\in\Omega}\widetilde{f^2(A)}(\lambda) = \left[\frac{12}{7}\pi - \frac{3}{14}\right]ber\left(f^2(A)\right)$$

for all $\lambda \in \Omega$. This implies that

$$[f(ber(A))]^2 < \left[\frac{12}{7}\pi - \frac{3}{14}\right]ber\left(f^2(A)\right)$$

for any self-adjoint operators $A \in \mathcal{B}(\mathcal{H})$ with spectrum contained in J. □

Theorem 2.3. *Let $f : J \to [0,\infty)$ be a convex function and $A : \mathcal{H}(\Omega)\to\mathcal{H}(\Omega)$ be self-adjoint operator on a RKHS $\mathcal{H}(\Omega)$ with spectrum contained in J. Then we have*

$$[f(ber(A))]^p < Cber(f^p(A)).$$

Proof. By using (3), we obtain for $p = 2$ that

$$\left(\frac{a_1}{2} + \frac{a_2}{3}\right)^p + \left(\frac{a_1}{3} + \frac{a_2}{4}\right)^p < \left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(a_1^p + a_2^p\right). \tag{11}$$

Let $x, y \in J$. Since $f(x) \geq 0$ for all $x \in J$, by placing $a_1 = f(x), a_2 = f(y)$ in (11), we obtain

$$\left(\frac{f(x)}{2} + \frac{f(y)}{3}\right)^p + \left(\frac{f(x)}{3} + \frac{f(y)}{4}\right)^p < \left(\frac{\pi}{\sin(\pi/p)}\right)^p(f^p(x) + f^p(y)). \tag{12}$$

By putting $x = \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle$ in (12), we get

$$\left(\frac{f(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle)}{2} + \frac{f(y)}{3}\right)^p + \left(\frac{f(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle)}{3} + \frac{f(y)}{4}\right)^p$$
$$< \left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(f^p(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle) + f^p(y)\right)$$

for all $\lambda \in \Omega$ and any $y \in J$.

Applying the functional calculus to the self-adjoint operator B, we get

$$\left(\frac{f(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle)}{2} + \frac{\left\langle f(B)\widehat{k}_{\mathcal{H},\mu}, \widehat{k}_{\mathcal{H},\mu}\right\rangle}{3}\right)^p + \left(\frac{f(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle)}{3} + \frac{\left\langle f(B)\widehat{k}_{\mathcal{H},\mu}, \widehat{k}_{\mathcal{H},\mu}\right\rangle}{4}\right)^p$$
$$< \left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(f^p(\left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle) + \left\langle f^p(B)\widehat{k}_{\mathcal{H},\mu}, \widehat{k}_{\mathcal{H},\mu}\right\rangle\right)$$

and hence

$$\left(\frac{f(\widetilde{A}(\lambda))}{2}+\frac{\widetilde{f(B)}(\mu)}{3}\right)^p+\left(\frac{f(\widetilde{A}(\lambda))}{3}+\frac{\widetilde{f(B)}(\mu)}{4}\right)^p \tag{13}$$
$$<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(f^p(\widetilde{A}(\lambda))+\widetilde{f^p(B)}(\mu)\right)$$

for all self-adjoint operator B and $\lambda,\mu\in\Omega$. Since f is convex function, $f(\widetilde{B}(\mu))\le\widetilde{f(B)}(\mu)$ $\left(\text{or} f\left(\left\langle B\widehat{k}_{\mathcal{H},\mu},\widehat{k}_{\mathcal{H},\mu}\right\rangle\right)\le\left\langle f(B)\widehat{k}_{\mathcal{H},\mu},\widehat{k}_{\mathcal{H},\mu}\right\rangle\right)$. So,

$$\left(\frac{f(\widetilde{A}(\lambda))}{2}+\frac{f(\widetilde{B}(\mu))}{3}\right)^p+\left(\frac{f(\widetilde{A}(\lambda))}{3}+\frac{f(\widetilde{B}(\mu))}{4}\right)^p \tag{14}$$
$$\le\left(\frac{f(\widetilde{A}(\lambda))}{2}+\frac{\widetilde{f(B)}(\mu)}{3}\right)^p+\left(\frac{f(\widetilde{A}(\lambda))}{3}+\frac{\widetilde{f(B)}(\mu)}{4}\right)^p.$$

From the properties of convex function f and power function x^r $(r\ge 1)$, we obtain

$$\left(\frac{\pi}{\sin(\pi/p)}\right)\left(f^p(\widetilde{A}(\lambda))+\widetilde{f^p(B)}(\mu)\right) \tag{15}$$
$$\le\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\widetilde{f^p(A)}(\lambda)+\widetilde{f^p(B)}(\mu)\right).$$

Together with (13), (14) and (15), we have

$$\left(\frac{f(\widetilde{A}(\lambda))}{2}+\frac{f(\widetilde{B}(\mu))}{3}\right)^p+\left(\frac{f(\widetilde{A}(\lambda))}{3}+\frac{f(\widetilde{B}(\mu))}{4}\right)^p$$
$$<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left(\widetilde{f^p(A)}(\lambda)+\widetilde{f^p(B)}(\mu)\right).$$

Now by replacing $A=B$, $\lambda=\mu$ above the inequality

$$\left[\left(\frac{5}{6}\right)^p+\left(\frac{7}{12}\right)^p\right]\left[f(\widetilde{A}(\lambda))\right]^p\le 2\left(\frac{\pi}{\sin(\pi/p)}\right)^p\widetilde{f^p(A)}(\lambda)$$

and therefore

$$\left[f(\widetilde{A}(\lambda))\right]^p\le 2\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left[\left(\frac{5}{6}\right)^p+\left(\frac{7}{12}\right)^p\right]^{-1}\widetilde{f^p(A)}(\lambda)$$

for all $\lambda\in\Omega$. Since $\left[f(\widetilde{A}(\lambda))\right]^p\ge 0$ and $\widetilde{f^p(A)}(\lambda)\ge 0$, this inequality implies that

$$[f(ber(A))]^p\le 2\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left[\left(\frac{5}{6}\right)^p+\left(\frac{7}{12}\right)^p\right]^{-1}ber\left(f^p(A)\right)$$

for all self-adjoint operator A and $\lambda\in\Omega$. This proves the theorem. □

References

[1] A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for $n\times n$ operator matrices, Linear Algebra Appl., 468 (2015), 18-26.

[2] A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for products and commutators of operators, Houston J. Math., 41 (4) (2015), 1163-1173.
[3] N. Aronzajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68 (1950) , 337-404.
[4] F.A. Berezin, Covariant and contravariant symbols for operators, Math. USSR-Izv., 6 (1972) , 1117-1151.
[5] N. Das, S. Sahoo, New inequalities similar to Hardy- Hilbert's inequality, Turk J. Math., 34(2010), 153-165.
[6] S.S. Dragomir, Reverse inequalities for the numerical radius of linear operators in Hilbert spaces, Bull. Aust. Math. Soc., 73 (2) (2006), 255-262.
[7] S.S. Dragomir, A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces, Banach J. Math. Anal., 1 (2) (2007), 154-175.
[8] M.T. Garayev, M. Gürdal, A. Okudan, Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators, Math. Inequal. Appl., (3) (19) (2016), 883-891.
[9] P.R. Halmos, A Hilbert Space Problem Book, Springer-Verlag, New York, 1982.
[10] F. Hansen, Non-commutative Hardy inequalities, Bull. Lond. Math. Soc., 41(6)(2009), 1009-1016.
[11] F. Hansen, K. Krulić, J. Pečarić, L.-E. Persson, Generalized noncommutative Hardy and Hardy-Hilbert type inequalities, Internat. J. Math., 21 (10) (2010), 1283-1295.
[12] G. Hardy, J.E. Littlewood, G. Polya, Inequalities, 2 nd ed. Cambridge University Press, Cambridge, 1967.
[13] M.T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, J. Funct. Anal., 238 (2006) , 181-192.
[14] M.T. Karaev, Reproducing Kernels and Berezin symbols Techniques in Various Questions of Operator Theory, Complex Anal. Oper. Theory, 7 (2013), 983-1018.
[15] M.T. Karaev, M. Gürdal, U. Yamancı, Some results related with Berezin symbols and Toeplitz operators, Math. Inequal. Appl., 17(3)(2014), 1031-1045.
[16] M. Kian, Hardy-Hilbert type inequalities for Hilbert space operators, Ann. Funct. Anal., 3(2)(2012), 128-134.
[17] A. Kufner, L. Maligranda, L.E. Persson, The prehistory of the Hardy inequality, Amer. Math. Monthly, 113 (8) (2006), 715-732.
[18] S. Saitoh, Theory of reproducing kernels and its applications, Pitman Research Notes in Mathematics Series, v.189, 1988.
[19] M. Sattari, M.S. Moslehian, T. Yamazaki, Some generalized numerical radius inequalities for Hilbert space operators, Linear Algebra Appl., 470 (2015) , 216-227.
[20] T. Yamazaki, On upper and lower bounds of the numerical radius and equality condition, Studia Math., 178 (2007) , 83-89.
[21] B. Yang, Discrete Hilbert-type inequalities, Bentham Science Publishers Ltd., 2011.