Filomat 31:18 (2017), 5603–5616
https://doi.org/10.2298/FIL1718603Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Boundary Value Problems for Impulsive Fractional Differential Equations in Banach Spaces

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Abstract. This paper is devoted to studying the existence and uniqueness of solutions to the boundary value problems for a impulsive fractional differential equation in Banach spaces. The arguments are based upon the methods of noncompact measure, Banach fixed point theorem and Krasnoselskii's fixed point theorem. Some examples are given to demonstrate the application of our main results.

1. Introduction

It is now recognized that the states of many evolutionary processes are often subject to short-term perturbations and experience abrupt changes at certain moments of time. The duration of the changes is negligible in comparison with the duration of the process considered, and can be thought of as instantaneously changes or as impulses. In the modeling process, such systems are natural to be more accurately described by impulsive differential equations. Impulsive differential equations have become an active research topic in nonlinear science and have attracted further attention in many diverse fields. For instance, vaccination [1], population ecology [2], drug treatment [3], hematopoiesis [4], pest control [5], chemostat [6], tumor-normal cell interaction [7], plankton allelopathy [8], in communication security [9], neural networks [10], etc. The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. In general, the fundamental properties such as the concept of solutions may need a suitable new interpretation. More information about the impulsive differential equations, we refer to [15–21].

Moreover, it is known that many evolution processes exhibit the time dependence behaviors and memory effects. Consequently, many researchers devoted to describing such processes by fractional differential equations. The fractional derivative generalized the classical derivative of integer order to a differential operator of arbitrary order. Due to its non-local behavior which yields the memory effects and history dependence, in the last decades, fractional differential equations have gained remarkable applications in various areas of science and engineering, such as physics, control systems, electrochemistry, biology, viscoelasticity mechanics, signal processing, nuclear dynamics, etc. For details, one can see [11–14] and the references therein.

2010 *Mathematics Subject Classification*. Primary 26A33; Secondary 34A08; 34A37; 34G20; 34K10
Keywords. Boundary value problem, fractional differential equations, impulsive, Banach space, fixed point theorem
Received: 16 November 2016; Accepted: 28 October 2017
Communicated by Jelena Manojlović
Research supported by the National Natural Science Foundation of China .(Grant No.11371364)
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Impulsive fractional differential equations are a natural generalization of impulsive ordinary differential equations. Since, the tools of impulsive fractional differential equations are applicable to various fields of study, such as the mathematical simulation in chaos, fluid dynamics and many physical systems. At the present time, many results on the impulsive fractional differential equations have been obtained. For details, we refer to [22–30, 38, 39].

Although the qualitative theory of impulsive fractional equations undergoes rapid development, we should point out that the most investigations of such problems are based on the commonly real space. Resulting from the complexity of the real nonlinear phenomena, it is natural to generalize the theory about impulsive fractional differential equations to the more general Banach spaces. During the last ten years, fractional differential equations in Banach spaces have been attractive to many researchers, we refer to [31–37, 40, 41]. These existing results are very useful for the investigation of impulsive fractional differential equations in Banach spaces.

X.Dong et al. [37] considered a nonlocal problems of fractional differential equation

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), & t \in J = [0, T], 0 < r \leq 1 \\ u(0) + g(u) = u_0. \end{cases}$$

where ${}^cD^q$ is the Caputo fractional derivative, X is a Banach space, $f : J \times X \to X$ is a given function, $g : C(J, X) \to X$ is a given function that satisfies some assumptions. By employing Banach contraction principle and Krasnoselskii's fixed point theorem, the main results are gained.

Benchohra et al. [32] discussed a impulsive initial value problem for Caputo fractional differential equations

$$\begin{cases} {}^cD^r y(t) = f(t, y), & t \in J = [0, T], t \neq t_k, 0 < r \leq 1 \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, 2, \ldots, m, \\ y(0) = y_0. \end{cases}$$

where ${}^cD^r$ is the Caputo fractional derivative, X is a Banach space, $f : J \times X \to X$ is a given function, $I_k : X \to X, k = 1, 2, \ldots, m,$ and $y_0 \in X$, $\Delta y|_{t=t_k} = y(t_k^+) - u(t_k^-), y(t_k^+) = \lim_{h\to 0^+} y(t_k + h), y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limit of the function $y(t)$ at $t = t_k$ respectively. By using Mönch's fixed point theorem and the technique of measures of noncompactness, the existence of solutions is obtained.

Motivated by the papers mentioned previously, we will study the following boundary value problems for an impulsive fractional differential equation

$$\begin{cases} {}^cD^{\beta}_{t_k} u(t) = f(t, u(t)), & t \in J = [0, 1], t \neq t_k \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), & t_k \in (0, 1), k = 1, 2, \ldots, m, \\ u(0) = h(u), u(1) = g(u). \end{cases} \tag{1}$$

where ${}^cD^{\beta}_{0^+}$ is the Caputo fractional derivative, $\beta \in \mathbb{R}$, $1 < \beta \leq 2$, $f : [0, 1] \times X \to X$ is a continuous function, $I_k, \bar{I}_k : X \to X$ are continuous functions, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), \Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-), u(t_k^+), u(t_k^-)$ are the right limit and the left limit of the function $u(t)$ at $t = t_k$ respectively, $g, h \in PC(J, X)$ are any fixed continuous functionals defined on the Banach space $PC(J, X)$ which will be defined in Section 2. Furthermore, g, h may be given by

$$g(u) = \max_j \frac{|u(\xi_j)|}{\lambda + |u(\xi_j)|}, h(u) = \min_j \frac{|u(\zeta_j)|}{\kappa + |u(\zeta_j)|},$$

and the similar forms, where $0 < \xi_j, \zeta_j < 1$, $\xi_j, \zeta_j \neq t_i$, $j = 1, 2, \cdots, n, i = 1, 2, \cdots, m$, and λ, κ are given positive constants. We shall apply the Banach fixed point theorem, Kransnoselskii's fixed point theorem and the method of measures of noncompactness to prove the existence and uniqueness of solutions to the boundary value problem (1).

The rest of the paper is organized as follows. In Section 2, we will give some notations, recall some definitions, and introduce some lemmas which are essential to prove our main results. In Section 3, main results are given, and some examples are presented to demonstrate our main results.

2. Preliminaries

In this section, we introduce notations, definitions, lemmas, and preliminary facts that will be used in the reminder of this paper.

Throughout this paper, $(X, \|\cdot\|_X)$ will be a Banach space. We denote $J = [0,1], t_0 = 0, t_{m+1} = 1, J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_m = (t_m, 1]$, and the Banach space

$$PC(J,X) = \{u : J \to X; u(t) \in C\left((t_k, t_{k+1}], X\right), k = 0, 1, \ldots, m+1, \textit{and } u(t_k^+), u(t_k^-) \textit{ exist with } u(t_k^-) = u(t_k), k = 1, 2, \ldots, m\}$$

with the norm $\|u\|_{PC} := \sup\{\|u(t)\|_X : t \in J\}$.

Definition 2.1. *([11]) The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $f \in L^1[J,X]$ is given by*

$$I_{0+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)}{(t-s)^{1-\beta}} ds,$$

provided that the integral exists.

Definition 2.2. *([11]) The Caputo fractional derivative of order $\beta > 0$ of function $f \in L^1(J,X) \cap C(J,X)$ is given by*

$$^cD_{0+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\beta-n+1}} ds,$$

where $n = [\beta] + 1$, and the notation $[\beta]$ stands for the largest integer not greater than β, provided that the right side is pointwise defined on J.

Note that the integrals appearing in the two previous definitions are taken in Bochner's sense.

We have the following auxiliary lemmas which are useful in what follows.

Lemma 2.3. *([12]) For $\beta > 0, f(t) \in C(J,X) \cap L^1(J,X)$, the homogeneous fractional differential equation*

$$^cD_{0+}^{\beta} f(t) = 0$$

has a solution

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in X, i = 0, 1, \cdots, n-1$, and $n = [\beta] + 1$.

Lemma 2.4. *([12]) Assume that $f(t) \in C(J,X) \cap L^1(J,X)$, with derivative of order n that belongs to $C(J,X) \cap L_1(J,X)$, then*

$$I_{0+}^{\beta}\, {}^cD_{0+}^{\beta} f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in X, i = 0, 1, \cdots, n-1$, and $n = [\beta] + 1$.

We recall here some definitions and fundamental facts of Kuratowski measure of noncompactness.

Definition 2.5. *([15]) Kuratowski measure of noncompactness α of each bounded subset B in Banach space X is defined by*

$$\alpha(B) = \inf\left\{\varepsilon > 0 : B \subset \bigcup_{j=1}^{m} M_j \textit{ and } diam(M_j) \le \varepsilon\right\}.$$

Proposition 2.6. *([15]) The Kuratowski measure of noncompactness has the following properties*

(i) *Monotone* $\alpha(B_1) \le \alpha(B_2)$ *for all bounded subsets B_1, B_2 of X, and $B_1 \subseteq B_2$.*

(ii) *Nonsingular* $\alpha(\{x\} \cup B) = \alpha(B)$ *for every* $x \in X$ *and every nonempty subset* $B \subseteq X$.

(iii) *Regular* $\alpha(B) = 0$ *if and only if* B *is relatively compact in* X.

(iv) $\alpha(B_1 + B_2) = \alpha(B_1) + \alpha(B_2)$ *for all bounded subsets* B_1, B_2 *of* X, *where* $B_1 + B_2 = \{x + y : x \in B_1, y \in B_2\}$.

(v) $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$ *for all bounded subsets* B_1, B_2 *of* X.

(vi) $\alpha(\lambda B) = |\lambda|\alpha(B)$ *for any* $\lambda \in \mathbb{R}$ *and all bounded subsets* B *of* X.

For any $W \subset C(J, X)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : U \in W \right\}, \textit{for } t \in J,$$

where $W(s) = \{u(s) \in X : u \in W\}$.

We present the following lammas.

Lemma 2.7. *([16]) If* $W \in C(J, X)$ *is bounded and equicontinuous, then* $t \to \alpha(W(t))$ *is continuous on* J, *and*

$$\alpha(W) = \max_{t \in J} \alpha(W(t)), \qquad \alpha\left(\int_0^t W(s)ds\right) \le \int_0^t \alpha(W(s))ds, \textit{for } t \in J.$$

Lemma 2.8. *([17]) Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence of Bochner integrable functions from* J *into* X *with* $|u_n(t)| \le m(t)$ *for all* $t \in J$ *and every* $n \ge 1$, *where* $m \in L(J, \mathbb{R}^+)$, *then the function* $\psi(t) = \alpha(\{u_n(t)\}_{n=1}^{\infty})$ *belongs to* $L(J, \mathbb{R}^+)$ *and satisfies*

$$\alpha\left(\left\{ \int_0^t u_n(s)ds : n \ge 1 \right\}\right) \le 2 \int_0^t \psi(s)ds.$$

Lemma 2.9. *([13])(Arzelà-Ascoli's theorem) If a family* $F(t) = \{f(t)\}$ *in* $C(J, X)$ *is uniformly bounded and equicontinuous on* J, *and for any* $t^* \in J$, $\{f(t^*)\}$ *is relatively compact,then* F *has a uniformly convergence subsequence* $\{f_n(t)\}_{n=1}^{\infty}$.

Lemma 2.10. *([13])(Krasnoselskii's fixed point theorem) Let* X *be a Banach space, let* Ω *be a bounded closed subset of* X *and let* F, G *be mappings of* Ω *into* X *such that* $Fu + Gv \in \Omega$ *for every pair* $u, v \in \Omega$. *If* F *is a contraction and* G *is completely continuous, then the equation* $Fu + Gu = u$ *has a solution on* Ω.

3. Main Results

First of all, we present the following lemma.

Lemma 3.1. *Let* $1 < \beta \le 2$. *Assume that* $f \in C(J, X)$. *Then the problem*

$$\begin{cases} {}^cD_{t_k}^{\beta} u(t) = f(t), & t \in J = [0,1], t \ne t_k \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), & t_k \in (0,1), k = 1, 2, \ldots, m, \\ u(0) = h(u), u(1) = g(u). \end{cases} \tag{2}$$

has a solution of the following form

$$u(t) = \begin{cases} ct + h(u) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds, & t \in J_0 \\ ct + h(u) + \frac{1}{\Gamma(\beta)} \int_{t_k}^t (t-s)^{\beta-1} f(s)ds & \\ + \sum_{j=1}^k \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} f(s)ds + \sum_{j=1}^k (t - t_j)\bar{I}_j(u(t_j)) & \\ + \sum_{j=1}^k \frac{t - t_j}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-2} f(s)ds + \sum_{j=1}^k I_j(u(t_j)). & t \in J_k, k = 1, 2, \ldots, m \end{cases} \tag{3}$$

where

$$\begin{aligned} c =& g(u) - h(u) - \sum_{j=1}^{m+1} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} f(s)ds - \sum_{j=1}^{m} I_j(u(t_j)) \\ & - \sum_{j=1}^{m} \frac{1 - t_j}{\Gamma(\beta - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-2} f(s)ds - \sum_{j=1}^{m} (1 - t_j)\bar{I}_j(u(t_j)) \end{aligned} \tag{4}$$

Proof. Assume that $u(t)$ is a solution of impulsive boundary value problem (2), using lemma 2.3 and lemma 2.4, we have

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} f(s)ds - c_0 - c_1 t, \qquad t \in J_0$$

for some c_0, c_1. Then we find that

$$u'(t) = \frac{1}{\Gamma(\beta - 1)} \int_0^t (t - s)^{\beta-2} f(s)ds - c_1.$$

The boundary condition $u(0) = -h(u)$ implies that $c_0 = h(u)$ and thus

$$u(t) = ct + h(u) + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} f(s)ds, \qquad t \in J_0$$

with $c = -c_1$.

If $t \in J_1$, then also by lemma 2.3 and lemma 2.4, we have

$$u(t) = \frac{1}{\Gamma(\beta)} \int_{t_1}^t (t - s)^{\beta-1} f(s)ds - d_0 - d_1(t - t_1),$$

and

$$u'(t) = \frac{1}{\Gamma(\beta - 1)} \int_{t_1}^t (t - s)^{\beta-2} f(s)ds - d_1.$$

Considering the conditions $\Delta u|_{t=t_1} = I_1(u(t_1))$, $\Delta u'|_{t=t_1} = \bar{I}_1(u(t_1))$, we can gain

$$-d_0 = ct_1 + h(u) + I_1(u(t_1)) + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} f(s)ds,$$

$$-d_1 = c + \bar{I}_1(u(t_1)) + \frac{1}{\Gamma(\beta - 1)} \int_0^{t_1} (t_1 - s)^{\beta-2} f(s)ds.$$

Thus, for $t \in J_1$ we have

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\beta)} \int_{t_1}^t (t - s)^{\beta-1} f(s)ds + \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} f(s)ds + \frac{t - t_1}{\Gamma(\beta - 1)} \int_0^{t_1} (t_1 - s)^{\beta-2} f(s)ds \\ & + (t - t_1)\bar{I}_1(u(t_1)) + I_1(u(t_1)) + h(u) + ct. \end{aligned}$$

Repeating the same fashion, we obtain the expression of the solution $u(t)$ for $t \in J_k$ as follows

$$\begin{aligned} u(t) =& ct + h(u) + \frac{1}{\Gamma(\beta)} \int_{t_k}^t (t - s)^{\beta-1} f(s)ds + \sum_{j=1}^{k} \frac{1}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} f(s)ds + \sum_{j=1}^{k} (t - t_j)\bar{I}_j(u(t_j)) \\ & + \sum_{j=1}^{k} \frac{t - t_j}{\Gamma(\beta - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-2} f(s)ds + \sum_{j=1}^{k} I_j(u(t_j)). \end{aligned} \tag{5}$$

Applying the boundary condition $u(1) = g(u)$, we can imply the expression (4).

Conversely, we assume that $x(t)$ is a solution of (3). If $t \in J_0$, then, using the fact that ${}^cD^{\alpha}_{0^+}$ is the left inverse of I^{α}_{0+}, we get ${}^cD^{\alpha}_{0^+}x(t) = f(t)$. If $t \in J_k, k = 0, 1, \dots, m$, due to the fact that the Caputo fractional derivative of a constant is equal to zero, we can verify easily that $x(t)$ satisfies (2), therefore, $x(t)$ is a solution of (2). The lemma is proved. □

According to Lemma 3.1, we gain the integral representation of the impulsive boundary value problem. Now, all we need to show that the integral equation has a solution. Furthermore,the solution of the integral equation coincides with the fixed point of the operator $T : PC(J, X) \to PC(J, X)$ which is defined as follows

$$\begin{aligned} Tu(t) =& c_0 t + (1-t)h(u) + tg(u) + \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}f(u(s),s)ds \\ &+ \sum_{0<t_k<t}\frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}f(u(s),s)ds + \sum_{0<t_k<t}(t-t_k)\bar{I}_k(u(t_k)) \\ &+ \sum_{0<t_k<t}\frac{t-t_k}{\Gamma(\beta-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}f(u(s),s)ds + \sum_{0<t_k<t}I_k(u(t_k)). \end{aligned} \tag{6}$$

where

$$\begin{aligned} c_0 =& -\sum_{k=1}^{m+1}\frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}f(u(s),s)ds - \sum_{k=1}^{m}I_k(u(t_k)) \\ &-\sum_{k=1}^{m}\frac{1-t_k}{\Gamma(\beta-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}f(u(s),s)ds - \sum_{k=1}^{m}(1-t_k)\bar{I}_k(u(t_k)). \end{aligned} \tag{7}$$

Our first result is based on Banach fixed point theorem. Before stating and proving the main result, we introduce the following hypotheses.

(H1) $f : J \times X \to X$ is a continuous function and satisfies the Lipschitz condition

$$\left\|f(u,t) - f(v,t)\right\|_X \le L_1\|u-v\|_X$$

with a constant $L_1 > 0$ for any $u, v \in X$, and $t \in J$.

(H2)$I_k, \bar{I}_k$ are continuous functions and there exist $L_2, L_3 > 0$ such that

$$\|I_k(u_1) - I_k(u_2)\|_X \le L_2\|u_1-u_2\|_X, \|\bar{I}_k(u_1) - \bar{I}_k(u_2)\|_X \le L_3\|u_1-u_2\|_X,$$

for all $u_1, u_2 \in X, k = 1, 2, \cdots, m$.

(H3) g, h are continuous functionals and satisfy the Lipschitz conditions with Lipschitz constants $L_4, L_5 > 0$.

Theorem 3.2. *Assume that* $(H1) - (H3)$ *hold. If*

$$\frac{4m+2}{\Gamma(\beta)}L_1 + 2m(L_2+L_3) + L_4 + L_5 < 1,$$

then the impulsive boundary value problem (1) has a unique solution.

Proof. The proof is based on the Banach fixed point theorem. Let us denote

$$\sup_{t\in J}\|f(0,t)\|_X = M_1, \max_k\|I_k(0)\|_X = M_2, \max_k\|\bar{I}_k(0)\|_X = M_3, \|h(0)\|_X = M_4, \|g(0)\|_X = M_5.$$

Considering

$$U_0 := \{u(t) \in PC(J,X) : \|u\|_{PC} \le R_0\},$$

where

$$R_0 \geq \frac{(4m+2)M_1 + 2m\Gamma(\beta)(M_2+M_3) + \Gamma(\beta)(M_4+M_5)}{\Gamma(\beta) - (4m+2)L_1 - 2m\Gamma(\beta)(L_2+L_3) - \Gamma(\beta)(L_4+L_5)}.$$

Firstly, we show that T maps U_0 into U_0. It is clear that T is well defined on $PC(J,X)$. Moreover for any $u(t) \in U_0$ and $t \in J_k, k = 0,1,\ldots,m$, we have

$$\begin{aligned}
\|c_0\|_X \leq& \frac{1}{\Gamma(\beta)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-1} \|f(u(s),s) - f(0,s)\|_X ds + \sum_{k=1}^{m} \|I_k(u(t_k)) - I_k(0)\|_X \\
&+ \sum_{k=1}^{m} \|I_k(0)\|_X + \frac{1}{\Gamma(\beta-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-2} \|f(u(s),s) - f(0,s)\|_X ds \\
&+ \frac{1}{\Gamma(\beta)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-1} \|f(0,s)\|_X ds + \sum_{k=1}^{m} \|\bar{I}_k(u(t_k)) - \bar{I}_k(0)\|_X \\
&+ \sum_{k=1}^{m} \|\bar{I}_k(0)\|_X + \frac{1}{\Gamma(\beta-1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-2} \|f(0,s)\|_X ds \\
\leq& \frac{m+1}{\Gamma(\beta+1)} (L_1\|u\|_{PC} + M_1) + \frac{m}{\Gamma(\beta)} (L_1\|u\|_{PC} + M_1) + m(L_2\|u\|_{PC} + M_2) \\
&+ m(L_3\|u\|_{PC} + M_3) \\
\leq& \frac{2m+1}{\Gamma(\beta)} (L_1\|u\|_{PC} + M_1) + m(L_2\|u\|_{PC} + M_2) + m(L_3\|u\|_{PC} + M_3).
\end{aligned}$$

and then

$$\begin{aligned}
\|Tu(t)\|_X \leq& \|c_0\|_X + \|h(u)\|_X + \|g(u)\|_X + \frac{1}{\Gamma(\beta+1)} (L_1\|u\|_{PC} + M_1) + \frac{m}{\Gamma(\beta)} (L_1\|u\|_{PC} + M_1) \\
&+ \frac{m}{\Gamma(\beta+1)} (L_1\|u\|_{PC} + M_1) + m(L_2\|u\|_{PC} + M_2) + m(L_3\|u\|_{PC} + M_3) \\
\leq& \frac{4m+2}{\Gamma(\beta)} (L_1\|u\|_{PC} + M_1) + 2m(L_2\|u\|_{PC} + M_2) + 2m(L_3\|u\|_{PC} + M_3) \\
&+ (L_4\|u\|_{PC} + M_4) + (L_5\|u\|_{PC} + M_5) \\
\leq& R_0.
\end{aligned}$$

Consequently T maps U_0 into itself.

Next, we show that T is a contraction operator. Let $u, v \in PC(J,X)$, then for any $t \in J_k, k = 0,1,\cdots,m$, we can easy to know from (6) and (7) that

$$\begin{aligned}
\|Tu(t) - Tv(t)\|_X \leq& \left[\frac{2(m+1)L_1}{\Gamma(\beta+1)} + \frac{2m}{\Gamma(\beta)} L_1 + 2mL_2 + 2mL_3 + L_4 + L_5 \right] \|u - v\|_{PC} \\
\leq& \left(\frac{4m+2}{\Gamma(\beta)} L_1 + 2m(L_2 + L_3) + L_4 + L_5 \right) \|u - v\|_{PC} \\
\leq& \|u - v\|_{PC}.
\end{aligned}$$

Hence, T is a contraction operator and followed by Banach fixed point theorem that T has a unique fixed point on $PC(J,X)$ which is a unique solution to (1). □

The second result is based on the Krasnoselskii's fixed point theorem. We introduce the following assumptions.

(H4) There exists a nonnegative function $a(t) \in L^1(J, \mathbb{R}^+)$ such that

$$\|f(x,t)\|_X \leq a(t) + l_1\|u\|_X^{\rho},$$

and for any $u \in X, k = 1, 2, \cdots, m,$

$$\|I_k(u)\|_X \leq l_2\|u\|_X^{\mu}, \|\bar{I}_k(u)\|_X \leq l_3\|u\|_X^{\nu},$$

$$\|h(u)\|_X \leq l_4\|u\|_X^{\theta}, \|g(u)\|_X \leq l_5\|u\|_X^{\gamma},$$

where $l_1 \geq 0, l_i > 0, i = 2, \cdots, 5$, and $0 < \rho, \mu, \nu, \theta, \gamma \leq 1$.

Furthermore

$$\frac{2(2m+1)l_1}{\Gamma(\alpha)} + 2ml_2 + 2ml_3 + l_4 + l_5 < 1. \tag{8}$$

(H5)For each bounded subset $W \subset X$, there exist $d_1 > 0$ such that

$$\alpha(f(W,s)) \leq d_1\alpha(W).$$

(H6)For each bounded subset $W \subset X$, there exist $d_2, d_3 > 0$ such that

$$\alpha(I_k(W)) \leq d_2\alpha(W), \alpha(\bar{I}_k(W)) \leq d_3\alpha(W), k = 1, 2, \cdots, m.$$

Theorem 3.3. *Assume that (H3)-(H6) is satisfied, if*

$$L_4 + L_5 < 1, \quad \textit{and } \frac{2(2m+1)d_1}{\Gamma(\beta)} + md_2 + md_3 < 1,$$

then the impulsive boundary value problem (1) has at least one solution.

Proof. We subdivide the operator T defined by (6) into two parts F and G as follows

$$\begin{aligned}
Fu(t) =& c_0 t + (1-t)h(u) + tg(u)\\
Gu(t) =& \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}f(u(s),s)ds + \sum_{0<t_k<t}\frac{1}{\Gamma(\beta)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}f(u(s),s)ds\\
&+ \sum_{0<t_k<t}\frac{t-t_k}{\Gamma(\beta-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}f(u(s),s)ds + \sum_{0<t_k<t}(t-t_k)\bar{I}_k(u(t_k)) + \sum_{0<t_k<t}I_k(u(t_k)).
\end{aligned}$$

Therefore, the existence of a solution of the impulsive boundary value problem (1) is equivalent to that the operator $F + G$ has a fixed point.

We will use the Krasnoselskii's fixed point theorem to prove this result. The proof will be given in several steps.

Step 1. $Fu + Gv \in W$, whenever W is a closed convex subset of $PC(J, X)$ and $u, v \in W$.

We denote

$$W := \{u(t) \in PC(J, X) : \|u\|_{PC} \leq R\},$$

where

$$R \geq \max\left\{1, \frac{\frac{4}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds}{1-\left[\frac{2(2m+1)l_1}{\Gamma(\beta)} + 2ml_2 + 2ml_3 + l_4 + l_5\right]}\right\}. \tag{9}$$

Indeed, for any $u \in W$, we have

$$\|c_0\|_X \leq \frac{1}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}a(s)ds + \frac{l_1R^{\rho}}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}ds + ml_2R^{\mu} + ml_3R^{\nu}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds+\frac{l_1R^{\rho}}{\Gamma(\beta-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}ds\\
\le&\frac{1}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}a(s)ds+\frac{1}{\Gamma(\beta-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds\\
&+\frac{(m+1)l_1R^{\rho}}{\Gamma(\beta+1)}+\frac{ml_1R^{\rho}}{\Gamma(\beta)}+ml_2R^{\mu}+ml_3R^{\nu}\\
\le&\frac{1}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left[(t_k-s)^{\beta-1}+(t_k-s)^{\beta-2}\right]a(s)ds+\frac{(2m+1)l_1R^{\rho}}{\Gamma(\beta)}+ml_2R^{\mu}+ml_3R^{\nu}\\
\le&\frac{2}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds+\frac{(2m+1)l_1R^{\rho}}{\Gamma(\beta)}+ml_2R^{\mu}+ml_3R^{\nu}.
\end{aligned}
$$

and then for any pair $u, v \in W$,

$$
\begin{aligned}
\|Fu(t)+Gv(t)\|_X\le&\|c_0\|_X+\|g(u)\|_X+\|h(u)\|_X+\frac{l_1R^{\rho}}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}a(s)ds\\
&+\frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}a(s)ds+\frac{1}{\Gamma(\beta)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}a(s)ds\\
&+\frac{l_1R^{\rho}}{\Gamma(\beta)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}ds+\frac{1}{\Gamma(\beta-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds\\
&+\frac{l_1R^{\rho}}{\Gamma(\beta-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}ds+ml_2R^{\mu}+ml_3R^{\nu}\\
\le&\|c_0\|_X+\|g(u)\|_X+\|h(u)\|_X+\frac{1}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}a(s)ds\\
&+\frac{l_1R^{\rho}}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-1}ds+\frac{1}{\Gamma(\beta-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds\\
&+\frac{l_1R^{\rho}}{\Gamma(\beta-1)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}ds+ml_2R^{\mu}+ml_3R^{\nu}\\
\le&\|c_0\|_X+\frac{2}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds+\frac{(m+1)l_1R^{\rho}}{\Gamma(\beta+1)}+\frac{ml_1R^{\rho}}{\Gamma(\beta)}\\
&+ml_2R^{\mu}+ml_3R^{\nu}+l_4R^{\theta}+l_5R^{\gamma}\\
\le&\|c_0\|_X+\frac{2}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds+\frac{(2m+1)l_1R^{\rho}}{\Gamma(\beta)}\\
&+ml_2R^{\mu}+ml_3R^{\nu}+l_4R^{\theta}+l_5R^{\gamma}\\
\le&\frac{4}{\Gamma(\beta)}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\beta-2}a(s)ds+\frac{2(2m+1)l_1R^{\rho}}{\Gamma(\beta)}+2ml_2R^{\mu}\\
&+2ml_3R^{\nu}+l_4R^{\theta}+l_5R^{\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{4}{\Gamma(\beta)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-2} a(s) ds + \frac{2(2m+1) l_1 R}{\Gamma(\beta)} + 2m l_2 R \\
&\quad + 2m l_3 R + l_4 R + l_5 R \\
&\leq R.
\end{aligned}
$$

which implies that $Fu + Gv \in W$.

Step 2. F is a contraction operator.

In fact for any $u_1, u_2 \in W$,

$$
\begin{aligned}
\|Fu_1(t) - Fu_2(t)\|_X &\leq \|h(u_1) - h(u_2)\|_X + \|g(u_1) - g(u_2)\|_X \\
&\leq (L_4 + L_5)\|u_1 - u_2\|_{PC} \\
&\leq \|u_1 - u_2\|_{PC}.
\end{aligned}
$$

Thus F is a contraction operator.

Step 3. G is a completely continuous operator.

It is very easy to imply that G is continuous since $f, I_k, \bar{I}_k, k = 0, 1, \cdots, m$, are continuous functions. We omit the details.

G maps bounded sets into uniformly bounded sets in $PC(J, X)$.

For every $v \in W$, arguing as in the Step 1, we can show that

$$
\begin{aligned}
\|Gv(t)\|_X &\leq \frac{4}{\Gamma(\beta)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-2} a(s) ds + \frac{2(2m+1) l_1 R}{\Gamma(\beta)} + m l_2 R + m l_3 R \\
&=: r.
\end{aligned}
$$

Thus, for every $v \in W$,$G(W)$ is bounded in $B_r := \{v(t) \in PC(J, X) : \|v\|_{PC} \leq r\}$.

Now we prove that G maps bounded sets into equicontinuous sets of $PC(J, X)$.

We take $N_f = \max_{t \in J} \|f(v, t)\|_X + 1$, let $t, \tau \in J_k$ with $t < \tau$, W be a bounded set of $PC(J, X)$ as in Step 1. For any $v \in W$, we have

$$
\begin{aligned}
\|Gv(\tau) - Gv(t)\|_X &\leq \frac{1}{\Gamma(\beta)} \left\| \int_{t_k}^{\tau} (\tau - s)^{\beta-1} f(v, s) ds - \int_{t_k}^{t} (t - s)^{\beta-1} f(v, s) ds \right\|_X \\
&\quad + \frac{\tau - t}{\Gamma(\beta - 1)} \sum_{k=1}^{m} \left\| \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-2} f(v, s) ds \right\|_X + \sum_{k=1}^{m} (\tau - t) \|\bar{I}_k(v)\|_X \\
&\leq \frac{N_f}{\Gamma(\beta)} \left\| \int_{t_k}^{\tau} (\tau - s)^{\beta-1} ds - \int_{t_k}^{t} (t - s)^{\beta-1} ds \right\|_X + m l_3 R^{\nu} (\tau - t) \\
&\quad + \frac{N_f(\tau - t)}{\Gamma(\beta - 1)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-2} ds \\
&\leq \frac{N_f(\tau - t)}{\Gamma(\beta)} \sum_{k=1}^{m} (t_{k-1} - t_k)^{\beta-1} + \frac{N_f}{\Gamma(\beta)} \left\| \int_{t_k}^{t} (\tau - s)^{\beta-1} ds - \int_{t_k}^{t} (t - s)^{\beta-1} ds \right\|_X \\
&\quad + m l_3 R^{\nu} (\tau - t) \\
&\leq m l_3 R^{\nu} (\tau - t) + \frac{N_f(\tau - t)}{\Gamma(\beta)} \sum_{k=1}^{m} (t_{k-1} - t_k)^{\beta-1} + \frac{N_f}{\Gamma(\beta + 1)} \left[(\tau - t_k)^{\beta} - (t - t_k)^{\beta} \right].
\end{aligned}
$$

As $\tau \to t$, the right-hand side of the above inequality tends to zero, hence we conclude that $G(W)$ is equicontinuous.

Let us consider a bounded set

$$W(t) := \Big\{v_n(t) : \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} f(v_n, s)ds + \sum_{0<t_k<t} \frac{1}{\Gamma(\beta)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-1} f(v_n, s)ds$$
$$+ \sum_{0<t_k<t} \frac{t-t_k}{\Gamma(\beta-1)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-2} f(v_n, s)ds + \sum_{0<t_k<t} (t-t_k)\bar{I}_k(v_n(t_k)) + \sum_{0<t_k<t} I_k(v_n(t_k))\Big\} \subset B_r.$$

Applying proposition 2.6 and lemma 2.7, we know that $t \to \alpha(W(t))$ is continuous on J. Furthermore,

$$(t_k-s)^{\beta-1}\|f(v_n, s)\|_X \leq (t_k-s)^{\beta-1}(a(t) + l_1 r^{\rho}) \in L^1(J, \mathbb{R}^+),$$
$$(t_k-s)^{\beta-2}\|f(v_n, s)\|_X \leq (t_k-s)^{\beta-2}(a(t) + l_1 r^{\rho}) \in L^1(J, \mathbb{R}^+).$$

Using H5-H6 and lemma 2.8, we have

$$\begin{aligned}
\alpha(W(t)) \leq & \frac{2}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} \alpha(f(W(s), s))ds + \sum_{0<t_k<t} \frac{2}{\Gamma(\beta)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-1} \alpha(f(W(s), s))ds \\
& + \sum_{0<t_k<t} \frac{2(t-t_k)}{\Gamma(\beta-1)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-2} \alpha(f(W(s), s))ds + \sum_{0<t_k<t} \alpha(I_k(W(s))) + \sum_{0<t_k<t} (t-t_k)\alpha(\bar{I}_k(W(s))) \\
\leq & \frac{2d_1}{\Gamma(\beta)} \int_{t_k}^{t} (t-s)^{\beta-1} \alpha(W(s))ds + \sum_{0<t_k<t} \frac{2d_1}{\Gamma(\beta)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-1} \alpha(W(s))ds \\
& + \sum_{0<t_k<t} \frac{2d_1}{\Gamma(\beta-1)} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-2} \alpha(W(s))ds + \sum_{0<t_k<t} d_2\alpha(W(s)) + \sum_{0<t_k<t} d_3\alpha(W(s)),
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha(W) \leq & \frac{2d_1}{\Gamma(\beta+1)}\alpha(W) + \frac{2md_1}{\Gamma(\beta+1)}\alpha(W) + \frac{2md_1}{\Gamma(\beta)}\alpha(W) + md_2\alpha(W(s)) + md_3\alpha(W) \\
\leq & \left[\frac{2(2m+1)d_1}{\Gamma(\beta)} + md_2 + md_3\right]\alpha(W) \\
< & \alpha(W),
\end{aligned}$$

due to the condition

$$\frac{2(2m+1)d_1}{\Gamma(\beta)} + md_2 + md_3 < 1.$$

Then we can deduce that $\alpha(W) = 0$. Therefore, invoking to the regularity of the Kuratowski measure of noncompactness, we know that $G(W)$ is relatively compact,so that there exists a subsequence v_n which converge uniformly on J to some $v^* \in PC(J, X)$ together with the Arzelà-Ascoli's theorem. This proves that G is a completely continuous operator.

As a result of Steps 1-3, the Krasnoselskii's fixed point theorem implies that $F + G$ has at least one fixed point which is a solution of the impulsive boundary value problem (1) and the theorem is proved. □

Remark 3.4. *The condition (8) in the assumption (H4) can be removed, during the proof of this situation, we choose the bounded set* $U := \{u(t) \in PC(J, X) : \|u\|_{PC} \leq R\}$ *with*

$$R \geq \max\left\{6K, (12ml_3)^{\frac{1}{1-\nu}}, (6l_4)^{\frac{1}{1-\theta}}, (6l_5)^{\frac{1}{1-\gamma}}, \left[\frac{12(2m+1)l_1}{\Gamma(\beta)}\right]^{\frac{1}{1-\rho}}\right\} \tag{10}$$

where $K = \frac{4}{\Gamma(\beta)} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (t_k-s)^{\beta-2} a(s)ds$. *That is, transforms (9) into (10), the same results also can be obtained.*

Remark 3.5. *It is worth noting that the existing results on impulsive fractional differential equations are often found in the space $PC(J,R)$. Since the Arzelà-Ascoli's theorem is the key to the following results: A subset F in $C(J,R)$ is relatively compact if and only if it is uniformly bounded and equicontinuous on J, it would be relatively easy to prove the relative compactness of a set in $C(J,R)$. However, in the abstract Banach space $C(J,X)$ case, due to Lemma 2.10 (Arzelà-Ascoli's theorem), we should also verify that for any $t^* \in J$, $\{f(t^*)\}$ is relatively compact. Consequently, in the proof of Theorem 3.3, we used the methods of noncompact measure to deal with this problem. In addition, the integrals appearing in the present work are taken in Bochner's sense. The present work naturally generalized the theory about impulsive fractional differential equations in $C(J,R)$ to the abstract Banach spaces $C(J,X)$.*

In the end of this section, we will give some examples to illustrate our results. We note that our Banach space in the following examples is defined by

$$X = c_0 = \left\{ u = (u_1, u_2, u_3, \ldots) : 1 \leq \sup_{n\geq 1} |u_n| < \infty \right\},$$

with the norm $\|u\|_X = \sup_{n\geq 1} |u_n|$.

Example 3.6. *Consider the following problem*

$$\begin{cases} {}^cD_{t_k}^{\frac{7}{4}} u_n(t) = \frac{1}{10} t(t-\frac{1}{3}) \frac{u_n}{1+u_n^2}, & t \in J = [0,1], t \neq \frac{1}{3} \\ \Delta u_n(\frac{1}{3}) = \frac{1}{10} \ln\left(1 + u_n(\frac{1}{3})^2\right), \Delta u_n'(\frac{1}{3}) = \frac{1}{10+|u(\frac{1}{3})|} \\ u_n(0) = \min_j \frac{|u_n(\zeta_j)|}{15+|u_n(\zeta_j)|}, u_n(1) = \max_j \frac{1}{15+|u_n(\xi_j)|}. \end{cases}$$

where $\xi_j, \zeta_j \neq \frac{1}{3}$, $j = 1, 2, \cdots, 10$.

A simple computation shows that $L_1 = L_4 + L_5 = \frac{1}{15}$, $L_2 = L_3 = \frac{1}{10}$, then

$$\frac{4m+2}{\Gamma(\beta)} L_1 + 2m(L_2 + L_3) + L_4 + L_5 \approx 0.969 < 1,$$

Hence, by Theorem 3.2, this impulsive boundary value problem has a unique solution.

Example 3.7. *Consider the problem*

$$\begin{cases} {}^cD_{t_k}^{\frac{5}{3}} u_n(t) = (t-\frac{1}{2})^4 \frac{|u_n|^{\frac{1}{4}}}{1+|u_n|}, & t \in J = [0,1], t \neq \frac{1}{2} \\ \Delta u_n(\frac{1}{2}) = \frac{|u_n(\frac{1}{2})|^{\frac{1}{4}}}{12+|u_n(\frac{1}{2})|}, \Delta u_n'(\frac{1}{2}) = \frac{|u_n(\frac{1}{2})|^{\frac{1}{5}}}{10+|u_n(\frac{1}{2})|} \\ u_n(0) = \min_j \frac{|u_n(\zeta_j)|^{\frac{1}{3}}}{15+|u_n(\zeta_j)|}, u_n(1) = \max_j \frac{|u_n(\xi_j)|^{\frac{1}{2}}}{15+|u_n(\xi_j)|}. \end{cases}$$

where $\xi_j, \zeta_j \neq \frac{1}{2}$, $j = 1, 2, \cdots, 10$.

Since we can get $l_1 = d_1 = \frac{1}{16}$, $l_2 = l_5 = d_2 = L_5 = \frac{1}{12}$, $l_3 = l_4 = d_3 = L_4 = \frac{1}{15}$, we have

$$L_4 + L_5 = 0.15 < 1, \quad \frac{2(2m+1)d_1}{\Gamma(\beta)} + md_2 + md_3 \approx 0.565 < 1,$$

$$\frac{2(2m+1)l_1}{\Gamma(\beta)} + 2ml_2 + 2ml_3 + l_4 + l_5 \approx 0.865 < 1.$$

Therefore, by Theorem 3.3, we know that the above problem has at least one solution.

Example 3.8. *We can pay attention to a class of problems as follows*

$$\begin{cases} {}^cD_{t_k}^{\beta} u_n(t) = (t-\frac{1}{2})^4 \frac{|u_n|^{\frac{1}{\rho}}}{1+|u_n|}, & t \in J = [0,1], t \neq \frac{1}{2} \\ \Delta u_n(\frac{1}{2}) = \frac{|u_n(\frac{1}{2})|^{\frac{1}{\mu}}}{12+|u_n(\frac{1}{2})|}, \Delta u_n'(\frac{1}{2}) = \frac{|u_n(\frac{1}{2})|^{\frac{1}{\nu}}}{10+|u_n(\frac{1}{2})|} \\ u_n(0) = \min_j \frac{|u_n(\zeta_j)|^{\frac{1}{\theta}}}{15+|u_n(\zeta_j)|}, u_n(1) = \max_j \frac{|u_n(\xi_j)|^{\frac{1}{\gamma}}}{15+|u_n(\xi_j)|}. \end{cases}$$

where $\xi_j, \zeta_j \neq \frac{1}{2}, j = 1, 2, \cdots, 10, 1 < \beta \leq 2, 0 < \rho, \mu, \nu, \theta, \gamma \leq 1.$ *By invoking Theorem* 3.3 *and Remark* 3.4*, we can imply that the above problem has at least one solution.*

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11371364). The author is very grateful to the anonymous referee for his or her valuable suggestions.

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