Filomat 31:18 (2017), 5581–5590
https://doi.org/10.2298/FIL1718581M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Almost Sure Central Limit Theorems for m-Dependent Random Variables

Yu Miao[a], Xiaoyan Xu[a]

[a]*College of Mathematics and Information Science, Henan Normal University, Henan Province, 453007, China.*

Abstract. In the present paper, the almost sure central limit theorem for the m-dependent random sequence is established, which weakens the moment conditions of Giuliano [10] for the stationary m-dependent sequence and gets the same results with different methods.

1. Introduction

Almost sure central limit theorems were first established simultaneously by Brosamler [5] and Schatte [19] for real independent identically distributed (i.i.d.) random variables $\{X_k, k \geq 1\}$. Assume that $\mathbb{E}X_k = \mu$ and $Var(X_k) = \sigma^2$, they proved that the classical central limit theorem

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

has an almost sure version

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{(\sqrt{k})^{-1} \sum_{j=1}^{k} (X_j - \mu)} \Longrightarrow \mathcal{N}(0, \sigma^2) \ \ a.s. \tag{1}$$

Here $\mathcal{N}$, δ_x denote the Gaussian distribution and the point mass at x. The symbols " $\xrightarrow{\mathcal{D}}$ ", " $\Longrightarrow$ " mean the convergence in distribution and the weak convergence. Specifically, the relation (1) has the following simple version:

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I \left\{ \frac{1}{\sigma \sqrt{k}} \sum_{j=1}^{k} (X_j - \mu) \leq x \right\} \to \Phi(x) \ \ a.s. \tag{2}$$

where I denotes indicator function and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

2010 *Mathematics Subject Classification*. Primary 60F10
Keywords. Almost sure central limit theorem, m-dependent random variables.
Received: 11 November 2016; Accepted: 07 March 2017
Communicated by Miljana Jovanović
Research supported by IRTSTHN (14IRTSTHN023), NSFC (11471104).
Email addresses: yumiao728@gmail.com (Yu Miao), xiaoyanxu15@163.com (Xiaoyan Xu)

For i.i.d. sequence, the property (1) was proved by Brosamler [5] and Schatte [19] provided $\mathbb{E}|X_1|^{2+\gamma}$ exists for some $\gamma > 0$ ($\gamma = 1$ was assumed in the later paper). Lacey and Philipp [12] obtained the property (1) under assuming only finite variance. Csáki et al. [7] studied the properties of some logarithmic average. Based on invariance principles for integrals of an Ornstein-Uhlenbeck process and on strong approximations of normalized partial sums by Ornstein-Uhlenbeck processes, Csörgö and Horváth [8] proved the invariance principles for logarithmic averages.

For independent not necessarily identically distributed random variables, Rodzik and Rychlik [18] studied the almost sure central limit theorem. Berkes and Dehling [3] gave a general result and found a necessary and sufficient condition under mild technical conditions. Berkes and Csáki [2] proved that every weak limit theorem for independent random variables, subject to minor technical conditions, has an analogous almost sure version.

For some dependent sequences, Peligrad and Shao [17] gave an almost sure central limit theorem for associated sequences, strongly mixing and ρ-mixing sequences. Gonchigdanzan [11] proved an almost sure central limit theorem for a strongly mixing sequence of random variables with a slightly slow mixing rate and also showed that the almost sure central limit theorem holds for an associated sequence of random variables without a stationarity assumption. Wang and Liang [21] obtained the almost sure central limit theorem for negatively associated fields that assures the usual central limit theorem. Bercu et al. [1] established the almost sure asymptotic properties of vector martingale transforms. Cénac [6] considered the problem of the convergence of stochastic approximation algorithms and obtained the convergence of moments in the almost sure central limit theorem.

Based on the above works, in the present paper, we shall consider the almost sure central limit theorem of the m-dependent sequence. Recall that for a given non-negative integer m, a sequence $\{X_n\}_{n\geq 1}$ of strictly stationary random variables is called m-dependent if for every $k \geq 1$, the following two collections

$$\{X_1, \cdots, X_k\} \text{ and } \{X_{k+m+1}, X_{k+m+2}, \cdots\}$$

are independent. Hence, an i.i.d. random variables sequence is 0-dependent and for any non-negative integers $m_1 < m_2$, m_1-dependence implies m_2-dependence. The study in m-dependence context has its own interest in the applications. For example, the additive functional of a recurrent Markov chain with the general state space can be approximated in a certain way by the sums of an 1-dependent sequence which may not be independent (cf. [16]). Some nonlinear functionals of moving average processes can be approximated by m-dependent sequence (cf. [9, 14, 15]). Tómács [20] studied the m-dependent random fields and got the almost sure central limit theorem. Giuliano [10] obtained the almost sure central limit theorem of the m-dependent sequence by proving a new bound for the Rosenblatt coefficient of the normalized partial sums of a sequence of m-dependent random variables.

2. Main Results

The following theorem is our main results.

Theorem 2.1. *Let $\{X_n\}_{n\geq 1}$ be a sequence of strictly stationary m-dependent random variables with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 < \infty$. Then for any $x \in \mathbb{R}$, we have*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{S_k}{\sigma\sqrt{k}} \leq x\right\} \to \Phi(x), \quad a.s.$$

where $S_n = X_1 + \cdots + X_n$ denotes the partial sums and

$$\sigma^2 := \mathbb{E}X_1^2 + 2\sum_{k=2}^{m+1}\mathbb{E}X_1X_k.$$

Remark 2.2. *Giuliano [10] also studied the almost sure central limit theorem of the m-dependent random variables with the condition*

$$\sup_n \mathbb{E}X_n^{2+\delta} < \infty$$

for a suitable $\delta \in (0, 1]$. His method is firstly to prove the following Rosenblatt coefficient of m-dependent random variables: There exists an absolute constant H such that, for every pair of integers p, q with $p \leq q$, the following bound holds:

$$\sup_{A,x} |\mathbb{P}(U_p \in A, U_q \leq x) - \mathbb{P}(U_p \in A)\mathbb{P}(U_q \leq x)| \leq H\left(\sqrt[4]{\frac{\nu_p}{\nu_q}} + \frac{1}{q^\alpha}\right)$$

where the sup is taken over $A \in \mathbb{B}(\mathbb{R})$ and $x \in \mathbb{R}$. Here

$$S_n = \sum_{i=1}^n X_i, \;\; \nu_n = Var(S_n), \;\; U_n = \frac{S_n}{\sqrt{\nu_n}}, \;\; \liminf_{n\to\infty} \frac{\nu_n}{n} > 0, \;\; \alpha = \delta(6\delta + 8)^{-1}.$$

The second step is to prove that

$$\frac{1}{n}\sum_{i=1}^n 1_A(U_{2^i}) \;\; and \;\; \frac{1}{\log n}\sum_{i=1}^n \frac{1}{i} 1_A(U_i), \;\; n \geq 1$$

have the same limit point as $n \to \infty$. From this result, the almost sure central limit theorem for the m-dependent sequence can be established.

Remark 2.3. *Comparing Theorem 2.1 with the works of Giuliano [10], we weaken the moment condition of Giuliano [10] from $(2+\delta)$-order moment to 2-order moment, but add the stationarity of the m-dependent sequence. In addition, the approach in the paper which is from Berkes and Csáki [2], is more direct.*

Proof. Given the integer $p > 1$ and for any $n \geq 1$, we can write

$$n = ml_n + r_n, \quad l_n = pk_n + q_n$$

where l_n, k_n, r_n, q_n are nonnegative integers with $0 \leq r_n \leq m-1$, $0 \leq q_n \leq p-1$. For $i \geq 1$, define

$$Y_i = \sum_{j=1}^{m-1} X_{(i-1)m+j} \quad \text{and} \quad Z_i = \sum_{j=1}^{p-1} Y_{(i-1)p+j},$$

then it is easy to see that $\{Y_i\}$ is a 1-dependent random sequence and $\{Z_i\}$ is an i.i.d. sequence. Hence we have

$$\begin{aligned}\sum_{i=1}^n X_i &= \sum_{i=1}^{k_n} Z_i + \sum_{i=1}^{k_n} Y_{ip} + \sum_{i=pk_n+1}^{l_n} Y_i + \sum_{i=ml_n+1}^{n} X_i \\ &=: S_{1,n} + S_{2,n} + S_{3,n} + S_{4,n}.\end{aligned}$$

Next we shall show that for any $x \in \mathbb{R}$ and any $\varepsilon > 0$, the following claims hold:

$$\frac{1}{\log N}\sum_{n=1}^N \frac{1}{n} I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}} \leq x\right\} \to \Phi(x), \;\; a.s. \tag{3}$$

$$\limsup_{N\to\infty} \frac{1}{\log N}\sum_{n=1}^N \frac{1}{n} I\left\{\left|\frac{S_{2,n}}{\sigma\sqrt{n}}\right| \geq \varepsilon\right\} \leq \frac{C}{p}, \;\; a.s. \tag{4}$$

$$\limsup_{N\to\infty} \frac{1}{\log N}\sum_{n=1}^N \frac{1}{n} I\left\{\left|\frac{S_{3,n}}{\sigma\sqrt{n}}\right| \geq \varepsilon\right\} = 0, \;\; a.s. \tag{5}$$

$$\limsup_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\left|\frac{S_{4,n}}{\sigma\sqrt{n}}\right| \geq \varepsilon\right\} = 0, \quad a.s. \tag{6}$$

where $\sigma_Z^2 = Var(Z_1)$.

Throughout the following proof, C denotes a positive constant, which may take different values whenever it appears in different expressions.

Step 1. Proof of (3). From the definition of Z_1, we have $\mathbb{E}Z_1 = 0$ and

$$\begin{aligned}\sigma_Z^2 :=& Var(Z_1) = (p-1)Var(Y_1) + 2\sum_{i=1}^{p-2} Cov(Y_i, Y_{i+1})\\ =& m(p-1)Var(X_1) + 2(p-1)\sum_{i=1}^{m-1}(m-i)Cov(X_1^\tau, X_{i+1})\\ &+ 2(p-2)\sum_{i=1}^{m} iCov(X_1^\tau, X_{i+1})\\ =& m(p-1)\left[Var(X_1) + 2\sum_{i=1}^{m} Cov(X_1, X_{i+1})\right]\\ &- 2\sum_{i=1}^{m} iCov(X_1, X_{i+1}) - 2mCov(X_1, X_{m+1}).\end{aligned}$$

Since $\{Z_i\}$ is an i.i.d. random sequence, we have

$$\frac{1}{\sigma_Z\sqrt{k_n}} S_{1,n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{7}$$

Let f be a bounded Lipschitz function. From (7), we know

$$\mathbb{E}f\left(\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}}\right) \to \mathbb{E}f(\mathcal{N}(0,1)) \text{ as } n\to\infty. \tag{8}$$

Note that (3) is equivalent to (cf. [4, Theorem 7.1], [12, Section 2] and [17, Section 2])

$$\lim_{N\to\infty} \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n} f\left(\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}}\right) = \mathbb{E}f\left(\mathcal{N}(0,1)\right) \quad a.s.$$

Hence it suffices to prove as $N\to\infty$,

$$\begin{aligned}G_{1,N} :=& \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(f\left(\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}}\right) - \mathbb{E}f\left(\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}}\right)\right)\\ =:& \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}T_n \to 0 \ \ a.s.\end{aligned}$$

It is easy to see

$$\mathbb{E}G_{1,N}^2 = \frac{1}{\log^2 N}\left(\sum_{n=1}^{N}\frac{1}{n^2}\mathbb{E}T_n^2 + 2\sum_{n<j}\frac{1}{jn}\mathbb{E}T_nT_j\right) =: \frac{1}{\log^2 N}[\Delta_1 + \Delta_2].$$

Since f is a bounded Lipschitz function, we have

$$\frac{1}{\log^2 N}|\Delta_1| \leq \frac{C}{\log^2 N}\sum_{n=1}^{N}\frac{1}{n^2} \leq \frac{C}{\log^2 N}. \tag{9}$$

Moreover, for $1 \le n < j \le N$, we have

$$\begin{aligned} |\mathbb{E}T_n T_j| &= \left| Cov\left(f\left(\frac{S_{1,n}}{\sigma_Z \sqrt{k_n}}\right), f\left(\frac{S_{1,j}}{\sigma_Z \sqrt{k_j}}\right)\right)\right| \\ &= \left| Cov\left(f\left(\frac{S_{1,n}}{\sigma_Z \sqrt{k_n}}\right), f\left(\frac{S_{1,j}}{\sigma_Z \sqrt{k_j}}\right) - f\left(\frac{S_{1,j}-S_{1,n}}{\sigma_Z \sqrt{k_j}}\right)\right)\right| \\ &\le \frac{C}{\sigma_Z \sqrt{k_j}} \mathbb{E}|S_{1,n}| \le \frac{C}{\sigma_Z \sqrt{k_j}} \left(\mathbb{E}|S_{1,n}|^2\right)^{1/2} \le C \frac{\sqrt{k_n}}{\sqrt{k_j}}. \end{aligned}$$

From the definition of k_n, it follows that $mpk_n \sim n$, so we have

$$\frac{1}{\log^2 N}|\Delta_2| \le \frac{C}{\log^2 N} \sum_{n<j} \frac{1}{jn} |\mathbb{E}T_n T_j| \le \frac{C}{\log^2 N} \sum_{n<j} \frac{1}{j^{3/2}\sqrt{n}} \le \frac{C}{\log N}. \tag{10}$$

From (9) and (10), we have

$$\mathbb{E}G_{1,N}^2 \le \frac{C}{\log N}.$$

Let $\rho > 1$ and $N_k = e^{k^\rho}$, then we have

$$G_{1,N_k} \to 0 \quad a.s.$$

For $N_k < N \le N_{k+1}$, we obtain

$$\begin{aligned} |G_{1,N}| &\le \frac{1}{\log N_k} \left| \sum_{n=1}^{N_k} \frac{1}{n} T_n \right| + \frac{1}{\log N_k} \left| \sum_{n=N_k+1}^{N_{k+1}} \frac{1}{n} T_n \right| \\ &\le |G_{1,N_k}| + \frac{C}{\log N_k} \sum_{n=N_k+1}^{N_{k+1}} \frac{1}{n} \to 0 \quad a.s. \end{aligned}$$

Step 2. Proof of (4). From the definition of Y_1, we have $\mathbb{E}Y_1 = 0$ and

$$\sigma_Y^2 := Var(Y_1) = m\mathbb{E}X_1^2 + 2\sum_{i=1}^{m-1}(m-i)\mathbb{E}X_1 X_{i+1}.$$

For any $\varepsilon > 0$, let g be a real valued function such that

$$I\{x \ge \varepsilon\} \le g(x) \le I\{x \ge \varepsilon/2\} \quad \text{and} \quad \sup_x |g'(x)| < \infty, \tag{11}$$

then we have

$$I\left\{\frac{S_{2,j}}{\sigma\sqrt{j}} \ge \varepsilon\right\} \le g\left(\frac{S_{2,j}}{\sigma\sqrt{j}}\right) = g\left(\frac{S_{2,j}}{\sigma\sqrt{j}}\right) - \mathbb{E}g\left(\frac{S_{2,j}}{\sigma\sqrt{j}}\right) + \mathbb{E}g\left(\frac{S_{2,j}}{\sigma\sqrt{j}}\right).$$

First we shall prove

$$G_{2,N} := \frac{1}{\log N}\sum_{n=1}^{N} \frac{1}{n}\left(g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right) - \mathbb{E}g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right)\right) \to 0, \quad a.s. \tag{12}$$

It is easy to see

$$\begin{aligned} \mathbb{E}G_{2,N}^2 &= \frac{1}{\log^2 N}\left[\sum_{n=1}^{N} \frac{1}{n^2}\mathbb{E}g^2\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right) + 2\sum_{n<l} \frac{1}{nl} Cov\left(g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right), g\left(\frac{S_{2,l}}{\sigma\sqrt{l}}\right)\right)\right] \\ &\le \frac{C}{\log^2 N} + \frac{2}{\log^2 N}\sum_{n=1}^{N-1}\sum_{l=n+1}^{N} \frac{1}{nl} Cov\left(g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right), g\left(\frac{S_{2,l}}{\sigma\sqrt{l}}\right)\right). \end{aligned}$$

Since for $p > 1$, $\{Y_{ip}\}_{i\geq 1}$ is an i.i.d. random sequence, we have

$$\begin{aligned}
&\left|Cov\left(g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right), g\left(\frac{S_{2,l}}{\sigma\sqrt{l}}\right)\right)\right| \\
=&\left|Cov\left(g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right), g\left(\frac{S_{2,l}}{\sigma\sqrt{l}}\right) - g\left(\frac{S_{2,l}-S_{2,n}}{\sigma\sqrt{l}}\right)\right)\right| \\
\leq& C\mathbb{E}\left|g\left(\frac{S_{2,l}}{\sigma\sqrt{l}}\right) - g\left(\frac{S_{2,l}-S_{2,n}}{\sigma\sqrt{l}}\right)\right| \\
\leq& \frac{C}{\sigma\sqrt{l}}\mathbb{E}\left|S_{2,n}\right| \leq \frac{C}{\sigma\sqrt{l}}\left(\mathbb{E}\left|S_{2,n}\right|^2\right)^{1/2} \leq \frac{C\sigma_Y\sqrt{k_n}}{\sigma\sqrt{l}}.
\end{aligned}$$

From the definition of k_n, we have

$$\begin{aligned}
&\frac{1}{\log^2 N}\sum_{n=1}^{N-1}\sum_{l=n+1}^{N}\frac{1}{nl}\left|Cov\left(g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right), g\left(\frac{S_{2,l}}{\sigma\sqrt{l}}\right)\right)\right| \\
\leq&\frac{C}{\log^2 N}\sum_{n=1}^{N-1}\sum_{l=n+1}^{N}\frac{\sqrt{k_n}}{nl^{3/2}} \leq \frac{C}{\sqrt{p}\log^2 N}\sum_{n<j}\frac{1}{j^{3/2}\sqrt{n}} \leq \frac{C}{\sqrt{p}\log N}.
\end{aligned} \tag{13}$$

Similar to the proof of $G_{1,N} \to 0$ *a.s.*, we obtain

$$G_{2,N} \to 0, \quad a.s.$$

From (11), we get

$$\mathbb{E}g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right) \leq \mathbb{P}\left(\frac{S_{2,n}}{\sigma\sqrt{n}} \geq \varepsilon/2\right) \leq C\frac{\sigma_Y^2 k_n}{\sigma^2 n} \leq \frac{C}{p}$$

which yields

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbb{E}g\left(\frac{S_{2,n}}{\sigma\sqrt{n}}\right) \leq \frac{C}{p}.$$

Step 3. Proofs of (5) and (6). Here we only give the proof of (5) and the proof of (6) is similar. Let

$$G_{3,N} := \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\left|\frac{S_{3,n}}{\sigma\sqrt{n}}\right| \geq \varepsilon\right\},$$

then we have

$$\begin{aligned}
\mathbb{E}G_{3,N}^2 \leq& \frac{C}{\log^2 N} + \frac{2}{\log^2 N}\sum_{n<l}\frac{1}{nl}\mathbb{P}\left(\left|\frac{S_{3,n}}{\sigma\sqrt{n}}\right| \geq \varepsilon, \left|\frac{S_{3,l}}{\sigma\sqrt{l}}\right| \geq \varepsilon\right) \\
\leq& \frac{C}{\log^2 N} + \frac{2}{\log^2 N}\sum_{n<l}\frac{1}{nl}\mathbb{P}\left(\left|\frac{S_{3,n}}{\sigma\sqrt{n}}\right| \geq \varepsilon\right).
\end{aligned} \tag{14}$$

Since $\mathbb{E}|S_{3,n}| \leq pm\mathbb{E}|X_1|$, it follows

$$\mathbb{P}\left(\left|\frac{S_{3,n}}{\sigma\sqrt{n}}\right| \geq \varepsilon\right) \leq \frac{Cp}{\sqrt{n}}.$$

By (14), we get

$$\mathbb{E}G_{3,N}^2 \leq \frac{C}{\log N}.$$

Similar to the proof of $G_{1,N} \to 0$ *a.s.*, we obtain

$$G_{3,N} \to 0, \quad a.s.$$

Step 4. Continuation of proving Theorem 2.1. Putting $V_n = S_{2,n} + S_{3,n} + S_{4,n}$, it is not difficult to check the following relations: for $\varepsilon > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} & I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\} \\ \le & I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x, \frac{|V_n|}{\sigma\sqrt{n}} < \varepsilon\right\} + I\left\{\frac{|V_n|}{\sigma\sqrt{n}} \ge \varepsilon\right\} \\ \le & I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x+\varepsilon\right\} + I\left\{\frac{|V_n|}{\sigma\sqrt{n}} \ge \varepsilon\right\} \\ \le & I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x+\varepsilon\right\} + I\left\{\frac{|S_{2,n}|}{\sigma\sqrt{n}} \ge \varepsilon/2\right\} + I\left\{\frac{|S_{3,n}+S_{4,n}|}{\sigma\sqrt{n}} \ge \varepsilon/2\right\} \end{aligned} \tag{15}$$

and

$$\begin{aligned} & I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\} \\ \ge & I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x, \frac{|V_n|}{\sigma\sqrt{n}} < \varepsilon\right\} \\ = & I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x-\varepsilon\right\} - I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x, \frac{|V_n|}{\sigma\sqrt{n}} \ge \varepsilon\right\} \\ \ge & I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x-\varepsilon\right\} - I\left\{\frac{|V_n|}{\sigma\sqrt{n}} \ge \varepsilon\right\} \\ \ge & I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x-\varepsilon\right\} - I\left\{\frac{|S_{2,n}|}{\sigma\sqrt{n}} \ge \varepsilon/2\right\} - I\left\{\frac{|S_{3,n}+S_{4,n}|}{\sigma\sqrt{n}} \ge \varepsilon/2\right\}. \end{aligned} \tag{16}$$

From (4), (5) and (6), we get

$$\begin{aligned} & \limsup_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\} \\ & \qquad \le \limsup_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x+\varepsilon\right\} + \frac{C}{p} \end{aligned} \tag{17}$$

and

$$\begin{aligned} & \liminf_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\} \\ & \qquad \ge \liminf_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le x-\varepsilon\right\} - \frac{C}{p}. \end{aligned} \tag{18}$$

Furthermore, since

$$\lim_{p\to\infty} \lim_{n\to\infty} \frac{\sigma^2 n}{\sigma_Z^2 k_n} = \lim_{p\to\infty} \frac{\sigma^2 mp}{\sigma_Z^2} = 1,$$

for any $0 < r < 1$, for all n and p large enough, we have

$$(1-r)^2 \le \frac{\sigma^2 n}{\sigma_Z^2 k_n} \le (1+r)^2.$$

Hence for all n and p large enough, it is easy to check

$$I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}} \le y(1-r)\right\} \le I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le y\right\} \le I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}} \le y(1+r)\right\},\ \ y \ge 0 \tag{19}$$

and

$$I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}} \le y(1+r)\right\} \le I\left\{\frac{S_{1,n}}{\sigma\sqrt{n}} \le y\right\} \le I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}} \le y(1-r)\right\},\ \ y \le 0. \tag{20}$$

Next we shall divide the proof into three cases: $x > 0$, $x < 0$, $x = 0$.

For the case $x > 0$, we can choose $\varepsilon > 0$ enough small, such that $x+\varepsilon > 0$ and $x-\varepsilon > 0$. So from (17) and (19), we can obtain

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\}\\
&\le \limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}} \le (x+\varepsilon)(1+r)\right\} + \frac{C}{p}\\
&=\Phi\left((x+\varepsilon)(1+r)\right) + \frac{C}{p}
\end{aligned}$$

and from (18) and (19), we have

$$\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\}\\
&\ge \liminf_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{n}} \le (x-\varepsilon)(1-r)\right\} - \frac{C}{p}\\
&= \Phi\left((x-\varepsilon)(1-r)\right) - \frac{C}{p}.
\end{aligned}$$

For the case $x < 0$, we can choose $\varepsilon > 0$ enough small, such that $x+\varepsilon < 0$ and $x-\varepsilon < 0$. So from (17) and (20), we can obtain

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\}\\
&\le \limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}} \le (x+\varepsilon)(1-r)\right\} + \frac{C}{p}\\
&=\Phi\left((x+\varepsilon)(1-r)\right) + \frac{C}{p}
\end{aligned}$$

and from (18) and (20), we have

$$\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_n}{\sigma\sqrt{n}} \le x\right\}\\
&\ge \liminf_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{n}} \le (x-\varepsilon)(1+r)\right\} - \frac{C}{p}\\
&= \Phi\left((x-\varepsilon)(1+r)\right) - \frac{C}{p}.
\end{aligned}$$

For the case $x = 0$, from (17) and (19), we can obtain

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_n}{\sigma\sqrt{n}}\le 0\right\}\\
&\le\limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{k_n}}\le\varepsilon(1+r)\right\}+\frac{C}{p}\\
&=\Phi\left(\varepsilon(1+r)\right)+\frac{C}{p}
\end{aligned}$$

and from (18) and (20), we have

$$\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_n}{\sigma\sqrt{n}}\le 0\right\}\\
&\ge\liminf_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{\frac{S_{1,n}}{\sigma_Z\sqrt{n}}\le-\varepsilon(1+r)\right\}-\frac{C}{p}\\
&=\Phi\left(-\varepsilon(1+r)\right)-\frac{C}{p}.
\end{aligned}$$

At last, for the above three cases, letting $r\to 0$, $\varepsilon\to 0$ and $p\to\infty$, we can obtain the desired result. □

Acknowledgments

The authors are very grateful to the referees and the editor for their valuable reports, which improved the presentation of this article.

References

[1] B. Bercu, P. Cénac, G. Fayolle, On the almost sure central limit theorem for vector martingales: convergence of moments and statistical applications, Journal of Applied Probability 46 (2009) 151–169.

[2] I. Berkes, E. Csáki, A universal result in almost sure central limit theory, Stochastic Processes and their Applications 94 (2001) 105–134.

[3] I. Berkes, H. Dehling, Some limit theorems in log density, Annals of Probability 21 (1993) 1640–1670.

[4] P. Billingsley, Convergence of Probability Measures, John Wiley & Sons, Inc., New York-London-Sydney 1968.

[5] G. A. Brosamler, An almost everywhere central limit theorem, Mathematical Proceedings of the Cambridge Philosophical Society 104 (1988) 561–574.

[6] P. Cénac, On the convergence of moments in the almost sure central limit theorem for stochastic approximation algorithms, ESAIM. Probability and Statistics 17 (2013) 179–194.

[7] E. Csáki, A. Földes, P. Révész, On almost sure local and global central limit theorems, Probability Theory and Related Fields 97 (1993) 321–337.

[8] M. Csörgö, L. Horváth, Invariance principles for logarithmic averages, Mathematical Proceedings of the Cambridge Philosophical Society 112 (1992) 195–205.

[9] H. Djellout, A. Guillin, L. Wu, Moderate deviations of empirical periodogram and non-linear functionals of moving average processes, Annales de l'Institut Henri Poincaré. Probabilités et Statistiques 42 (2006) 393–416.

[10] R. Giuliano, The Rosenblatt coefficient of dependence for m-dependent random sequences with applications to the ASCLT, Theory of Stochastic Processes 14 (2008) 30–38.

[11] K. Gonchigdanzan, Almost sure central limit theorems for strongly mixing and associated random variables. International Journal of Mathematics and Mathematical Sciences 29 (2002) 125–131.

[12] M. T. Lacey, W. Philipp, A note on the almost sure central limit theorem, Statistics & Probability Letters 9 (1990) 201–205.

[13] P. Matuła, On the almost sure central limit theorem for associated random variables, Probability and Mathematical Statistics 18 (1998) 411–416.

[14] Y. Miao, S. Shen, Moderate deviation principle for autoregressive processes, Journal of Multivariate Analysis 100 (2009) 1952–1961.

[15] Y. Miao, Y. L. Wang, G. Y. Yang, Moderate deviation principles for empirical covariance in the neighbourhood of the unit root, Scandinavian Journal of Statistics 42 (2015) 234–255.

[16] E. Nummelin, General Irreducible Markov Chains and Nonnegative Operators, Cambridge University Press, Cambridge, 1984.

[17] M. Peligrad, Q. M. Shao, A note on the almost sure central limit theorem for weakly dependent random variables, Statistics & Probability Letters 22 (1995) 131–136.
[18] B. Rodzik, Z. Rychlik, An almost sure central limit theorem for independent random variables, Annales de l'Institut Henri Poincaré. Probabilités et Statistiques 30 (1994) 1–11.
[19] P. Schatte, On strong versions of the central limit theorem, Mathematische Nachrichten 137 (1988) 249–256.
[20] T. Tómács, Almost sure central limit theorems for m-dependent random fields. Acta Acad. Paedagog. Acta Academiae Paedagogicae Agriensis. Nova Series. Sectio Matematicae 29 (2002) 89–94.
[21] J. F. Wang, H. Y. Liang, A note on the almost sure central limit theorem for negatively associated fields. Statistics & Probability Letters 78 (2008) 1964–1970.