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A New Approach to Jacobsthal Quaternions

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Abstract. The Jacobsthal quaternions defined by Szynal-Liana and Wloch [35]. In this paper, we defined some properties of Jacobsthal quaternions. Also, we investigated the relations between the Jacobsthal quaternions which connected with Jacobsthal and Jacobsthal-Lucas numbers. Furthermore, we gave the Binet formulas and Cassini identities for these quaternions.

1. Introduction

In 1973, the first use of this numbers appears "A Handbook of Integer Sequences" in a paper by Sloane by the title applications of Jacobsthal sequences to curves [1].
Furter, in 1988, Horadam [3] introduced the Jacobsthal and Jacobsthal-Lucas sequences recurrence relation $\{J_n\}$ and $\{j_n\}$ are defined by the recurrence relations

$$J_0 = 0, \;\; J_1 = 1, \;\; J_n = J_{n-1} + 2J_{n-2}, \;\; for\ n \geq 2, \tag{1}$$

$$j_0 = 2, \;\; j_1 = 1, \;\; j_n = j_{n-1} + 2j_{n-2}, \;\; for\ n \geq 2 \tag{2}$$

respectively.

In 1996, Horadam studied on the Jacobsthal and Jacobsthal-Lucas sequences and he gave Cassini-like formulas as follows ([3],[4])

$$J_{n+1}J_{n-1} - {J_n}^2 = (-1)^n.\, 2^{n-1}, \tag{3}$$

$$j_{n+1}j_{n-1} - {j_n}^2 = 3^2.\, (-1)^{n+1}.\, 2^{n-1}. \tag{4}$$

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The first eleven terms of Jacobsthal sequence $\{J_n\}$ are $0, 1, 1, 3, 5, 11, 21, 43, 85, 171$ and 341. This sequence is given by the formula

$$J_n = \frac{2^n - (-1)^n}{3}\,. \tag{5}$$

The first eleven terms of Jacobsthal-Lucas sequence $\{j_n\}$ are $2, 1, 5, 7, 17, 31, 65, 127, 257, 511$ and 1025. This sequence is given by the formula

$$j_n = 2^n + (-1)^n. \tag{6}$$

Also, for Jacobsthal and Jacobsthal-Lucas numbers the following properties hold [3]:

$$J_n + j_n = 2J_{n+1}, \tag{7}$$

$$j_n = J_{n+1} + 2J_{n-1}, \tag{8}$$

$$3J_n + j_n = 2^{n+1}, \tag{9}$$

$$j_n J_n = J_{2n}, \tag{10}$$

$$J_m j_n + J_n j_m = 2J_{m+n}, \tag{11}$$

$$J_m j_n - J_n j_m = (-1)^n\, 2^{n+1}\, J_{m-n}, \tag{12}$$

$$j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3.2^n, \tag{13}$$

$$j_n J_{m+1} + 2j_{n-1}\, J_m = j_{m+n}, \tag{14}$$

$$j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \tag{15}$$

$$j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}\,(2^{2r} - 1), \tag{16}$$

$$j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} = 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r}, \tag{17}$$

and summation formulas

$$\sum_{i=2}^{n} J_i = \frac{J_{n+2} - 3}{2}, \tag{18}$$

$$\sum_{i=1}^{n} j_i = \frac{j_{n+2} - 5}{2}. \tag{19}$$

Several authors worked on Jacobsthal numbers and polynomials in [5]-[13].

Sum formulas for odd and even Jacobsthal and Jacobsthal-Lucas numbers were given in [8] respectively as follows,

$$\sum_{i=0}^{n} J_{2i+1} = \frac{2J_{2n+2} + n + 1}{3}, \tag{20}$$

$$\sum_{i=0}^{n} J_{2i} = \frac{2J_{2n+1} - n - 2}{3}\,. \tag{21}$$

and

$$\sum_{i=0}^{n} j_{2i+1} = 2J_{2n+2} - n - 1, \tag{22}$$

$$\sum_{i=0}^{n} j_{2i} = J_{2n+2} + n + 1\,. \tag{23}$$

Identities for Jacobsthal numbers were given in [5] as follows,

$$\begin{cases} J_n J_{n+1} + 2J_{n-1}J_n = J_{2n} = J_n\, j_n, \\ J_n J_{m+1} + 2J_{n-1}J_m = J_{n+m}, \\ J_{2n+1} = J_{n+1}^2 + 2\,J_n^2, \\ J_m J_{n-1} - J_{m-1}J_n = (-1)^n.2^{n-1}J_{m-n}. \end{cases} \tag{24}$$

Now, we be talked about the history of the quaternions:
The quaternions were first described by Irish mathematician Sir William Rowan Hamilton in 1843, [14]. The description is a kind of extension of complex numbers to higher spatial dimensions. The set of real quaternions, denoted by H, is defined by

$$H = \{\, q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \ \mid\ q_0, q_1, q_2, q_3 \in \mathbb{R}\,\} \tag{25}$$

where

$$i^2 = j^2 = k^2 = -1\,,\ \ i\,j = -j\,i = k\,,\quad j\,k = -k\,j = i\,,\quad k\,i = -i\,k = j\,.$$

After the work of Hamilton, in 1849, Cockle introduced the set of split quaternions [15] which can be represented as

$$H_S = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\} \tag{26}$$

where

$$i^2 = -1,\ \ j^2 = k^2 = 1,\ \ i\,j\,k = 1.$$

Several authors worked on different quaternions and their generalizations. ([16]-[20],[27]-[31]). In 2013, Akyiğit and et al. [17] defined split Fibonacci quaternions and split Lucas quaternions and obtained some identities for them. Complex split quaternions defined by Kula and Yaylı in 2007, [28]. In 1963, Horadam [21] firstly introduced the n-th Fibonacci quaternion and generalized Fibonacci quaternions, which can be represented as

$$H_F = \{Q_n = F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3}\ |F_n,\ n-th\ \text{ Fibonacci number}\} \tag{27}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\ \ i\,j = -j\,i = k\,,\quad j\,k = -k\,j = i\,,\quad k\,i = -i\,k = j$$

and $n \geq 1$.

In 1969, Iyer ([26],[27]) derived many relations for the Fibonacci quaternions. Also, in 1973, Swamy [30] considered generalized Fibonacci quaternions as a new quaternion as follows:

$$P_n = H_n + i\,H_{n+1} + j\,H_{n+2} + k\,H_{n+3} \tag{28}$$

where

$$\begin{cases} H_n = H_{n-1} + H_{n-2}, \\ H_1 = p, \\ H_2 = p + q, \\ H_n = (p-q)F_n + qF_{n+1},\ \ n \geq 1. \end{cases}$$

(See [30] for generalized Fibonacci quaternions). In 1977, Iakin ([24],[25]) introduced higher order quaternions and gave some identities for these quaternions. In 1993, Horadam ([22],[23]) extended into quaternions to the complex Fibonacci numbers defined by Harman [20]. In 2012, Halıcı [18] gave generating functions and Binet's formulas for Fibonacci and Lucas quaternions. In 2013, Halıcı [19] defined complex Fibonacci quaternions as follows

$$H_{FC} = \{R_n = C_n + e_1\,C_{n+1} + e_2\,C_{n+2} + e_3\,C_{n+3}\ |\,C_n = F_n + i\,F_{n+1},\ i^2 = -1\} \tag{29}$$

where

$$\begin{gathered} {e_1}^2 = {e_2}^2 = {e_3}^2 = e_1\,e_2\,e_3 = -1\,, \\ e_1\,e_2 = -e_2\,e_1 = e_3\,,\ e_2\,e_3 = -e_3\,e_2 = e_1\,,\ e_3\,e_1 = -e_1\,e_3 = e_2,\ n \geq 1. \end{gathered}$$

In 2009, Ata and Yaylı [16] defined dual quaternions with dual numbers [1)] coefficient as follows:

$$H(\mathbb{D}) = \{Q = A + Bi + Cj + Dk\ |\ A,\,B,\,C,\,D \in \mathbb{D}\,, i^2 = j^2 = k^2 = -1 = i\,j\,k\} \tag{30}$$

In 2014, Nurkan and Güven [29] defined dual Fibonacci quaternions as follows:

$$H(\mathbb{D}) = \{\tilde{Q}_n = \tilde{F}_n + i\tilde{F}_{n+1} + j\tilde{F}_{n+2} + k\tilde{F}_{n+3}\,|\,\tilde{F}_n = F_n + \epsilon F_{n+1},\ \epsilon^2 = 0, \epsilon \neq 0\}, \tag{31}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\ \ i\,j = -j\,i = k\,,\quad j\,k = -k\,j = i\,,\quad k\,i = -i\,k = j$$

$n \geq 1$ and $\tilde{Q}_n = Q_n + \varepsilon\,Q_{n+1}$. Essentially, these quaternions in equations (26) and (27) must be called dual coefficient quaternion and dual coefficient Fibonacci quaternions, respectively. Majernik [32] defined dual quaternions as follows:

$$H_{\mathbb{D}} = \left\{ \begin{matrix} Q = a + b\,i + c\,j + d\,k\ |\ a\,,\,b\,,\,c\,,\,d \in \mathbb{R},\ i^2 = j^2 = k^2 = i\,j\,k = 0, \\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0 \end{matrix} \right\}. \tag{32}$$

For more details on dual quaternions, see [33]. It is clear that $H(\mathbb{D})$ and $H_{\mathbb{D}}$ are different sets. In 2015, Yüce and Torunbalcı Aydın [34] defined dual Fibonacci quaternions as follows:

$$H_{\mathbb{D}} = \{Q_n = F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3}\ |\ F_n,\ n-th\ \text{ Fibonacci number}\}, \tag{33}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

The Lucas sequence (L_n) and D_n^L which is the $n-th$ term of the dual Lucas quaternion sequence (D_n^L) are defined by the following recurrence relations:

$$\begin{cases} L_{n+2} = L_{n+1} + L_n,\ \forall\,n \geq 0 \\ L_0 = 2,\ L_1 = 1 \end{cases} \tag{34}$$

and

$$D_n^L = L_n + i\,L_{n+1} + j\,L_{n+2} + k\,L_{n+3}, \tag{35}$$

[1)] Dual number: $A = a + \varepsilon\,b\,,\ a,b \in \mathbb{R}\,,\ \varepsilon^2 = 0\,,\ \varepsilon \neq 0.$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,.$$

In 2015, Szynal-Liana and Wloch defined the Jacobsthal quaternions and the Jacobsthal- Lucas quaternions respectively as follows [35]

$$JQ_n = J_n + i\,J_{n+1} + j\;J_{n+2} + k\,J_{n+3}\,, \tag{36}$$

$$JLQ_n = j_n + i\,j_{n+1} + j\;j_{n+2} + k\,j_{n+3}\,. \tag{37}$$

In [35], using (7)-(17) relations between Jacobsthal and Jacobsthal-Lucas numbers are given as follows

$$JQ_{n+1} + JQ_n = 2^n(1 + 2i + 4j + 8k), \tag{38}$$

$$JQ_{n+1} - JQ_n = \frac{1}{3}[2^n(1 + 2i + 4j + 8k) + 2(-1)^n(1 - i + j - k)], \tag{39}$$

$$JQ_{n+r} + JQ_{n-r} = \frac{1}{3}[2^{n-r}(2^{2r} + 1)(1 + 2i + 4j + 8k) - 2(-1)^{n-r}(1 - i + j - k)], \tag{40}$$

$$JQ_{n+r} - JQ_{n-r} = \frac{1}{3}[2^{n-r}(2^{2r} - 1)(1 + 2i + 4j + 8k)], \tag{41}$$

$$N(JQ_n) = JQ_n.\overline{JQ}_n = \frac{1}{9}[85.2^{2n} + 10.2^n(-1)^n + 4], \tag{42}$$

$$JLQ_{n+1} + JLQ_n = 3.2^n(1 + 2i + 4j + 8k), \tag{43}$$

$$JLQ_{n+1} - JLQ_n = 2^n(1 + 2i + 4j + 8k) - 2(-1)^n(1 - i + j - k), \tag{44}$$

$$JLQ_{n+r} + JLQ_{n-r} = 2^{n-r}(2^{2r} + 1)(1 + 2i + 4j + 8k) + 2(-1)^{n-r}(1 - i + j - k), \tag{45}$$

$$JLQ_{n+r} - JLQ_{n-r} = [2^{n-r}(2^{2r} - 1)(1 + 2i + 4j + 8k)], \tag{46}$$

$$N(JLQ_n) = 85.2^{2n} + 10.2^n(-1)^n + 4, \tag{47}$$

$$JQ_n + JLQ_n = 2.JQ_{n+1}, \tag{48}$$

In this paper, we will give the Jacobsthal quaternions as follows

$$Q_J = \{JQ_n = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3} \mid J_n,\ nth\ \text{Jacobsthal number}\} \tag{49}$$

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\;\; i\,j = -j\,i = k\,,\quad j\,k = -k\,j = i\,,\quad k\,i = -i\,k = j \tag{50}$$

and $n \geq 1$. The scaler and the vector part of the Jacobsthal quaternion JQ_n are denoted by

$$S_{Q_n} = J_n \;\text{ and }\; V_{Q_n} = iJ_{n+1} + jJ_{n+2} + kJ_{n+3}. \tag{51}$$

Let JQ_n and JR_n be two Jacobsthal quaternions such that

$$JQ_n = J_n + i\,J_{n+1} + j\;J_{n+2} + k\,J_{n+3} \tag{52}$$

and

$$JR_n = K_n + i\,K_{n+1} + j\;K_{n+2} + k\,K_{n+3} \tag{53}$$

where K_n is $n-th$ Jacobsthal number.

Then, the addition, subtraction and multiplication of the Jacobsthal quaternions are the same as for real quaternions.

The conjugate of the Jacobsthal quaternion JQ_n is denoted by $\overline{JQ_n}$ and it is

$$\overline{JQ_n} = J_n - i\,J_{n+1} - j\,J_{n+2} - k\,J_{n+3}. \tag{54}$$

The norm of JQ_n is defined as

$$N_{JQ_n} = \left\|JQ_n\right\| = JQ_n\ \overline{JQ_n} = J_n^2 + J_{n+1}^2 + J_{n+2}^2 + J_{n+3}^2. \tag{55}$$

2. The Properties of the Jacobsthal Quaternions

Theorem 2.1. *Let J_n and JQ_n be the $n-th$ terms of the Jacobsthal sequence (J_n) and the Jacobsthal quaternion sequence (JQ_n), respectively. In this case, for $n \geq 1$ we can give the following relations:*

$$JQ_n + \overline{JQ_n} = 2J_n \tag{56}$$

$$JQ_n^2 = 2J_n.JQ_n - JQ_n.\overline{JQ_n} \tag{57}$$

$$JQ_{n+1} + 2JQ_n = JQ_{n+2} \tag{58}$$

$$JQ_n - iJQ_{n+1} - j\,JQ_{n+2} - kJQ_{n+3} = J_n + J_{n+2} + J_{n+4} + J_{n+6} \tag{59}$$

$$JQ_nJQ_m + 2JQ_{n-1}JQ_{m-1} = 2JQ_{n+m-1} - J_{n+m-1} - J_{n+m+1} - J_{n+m+3} - J_{n+m+5}. \tag{60}$$

Proof. (56): From (1.52) and (1.54) proof can easily be done.

(57): By (1.52) and (1.55)

$$\begin{aligned} JQ_n^2 =& J_n^2 - J_{n+1}^2 - J_{n+2}^2 - J_{n+3}^2 + 2J_n(i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ =& 2J_n(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) - (J_n^2 + J_{n+1}^2 + J_{n+2}^2 + J_{n+3}^2) \\ =& 2J_n.JQ_n - JQ_n.\overline{JQ_n} \end{aligned}$$

(58): By the equations (1.52) and

$$JQ_{n+1} = J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4} \tag{61}$$

we get,

$$\begin{aligned} JQ_{n+1} + 2JQ_n =& (J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4}) + 2(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}\,) \\ =& (J_{n+1} + 2J_n) + i\,(J_{n+2} + 2J_{n+1}) + j\,(J_{n+3} + 2J_{n+2}) + k\,(J_{n+4} + 2J_{n+3}) \\ =& J_{n+2} + i\,J_{n+3} + j\,J_{n+4} + k\,J_{n+5} \\ =& JQ_{n+2}\,. \end{aligned}$$

(59): By using (1.52) and conditions (1.50) we get

$$\begin{aligned} JQ_n - i\,JQ_{n+1} - j\,JQ_{n+2} - k\,JQ_{n+3} =& (J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ & - i\,(J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4}\,) \\ & - j\,(J_{n+2} + i\,J_{n+3} + j\,J_{n+4} + k\,J_{n+5}) \\ & - k(J_{n+3} + i\,J_{n+4} + j\,J_{n+5} + k\,J_{n+6}) \\ =& J_n + J_{n+2} + J_{n+4} + J_{n+6}\,. \end{aligned}$$

(60): By using (1.52), we get

$$
\begin{aligned}
JQ_n\,JQ_m =& J_n\,J_m - J_{n+1}\,J_{m+1} - J_{n+2}\,J_{m+2} - J_{n+3}\,J_{m+3}\\
&+ i\,(J_n\,J_{m+1} + J_{n+1}\,J_m + J_{n+2}\,J_{m+3} - J_{n+3}\,J_{m+2})\\
&+ j\,(J_n\,J_{m+2} - J_{n+1}\,J_{m+3} + J_{n+2}\,J_m + J_{n+3}\,J_{m+1})\\
&+ k\,(J_n\,J_{m+3} + J_{n+1}\,J_{m+2} - J_{n+2}\,J_{m+1} + J_{n+3}\,J_m)
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
2JQ_{n-1}\,JQ_{m-1} =& 2(J_{n-1}\,J_{m-1} - J_n\,J_m - J_{n+1}\,J_{m+1} - J_{n+2}\,J_{m+2})\\
&+ 2i\,(J_{n-1}\,J_m + J_n\,J_{m-1} + J_{n+1}\,J_{m+2} - J_{n+2}\,J_{m+1})\\
&+ 2j\,(J_{n-1}\,J_{m+1} - J_n\,J_{m+2} + J_{n+1}\,J_{m-1} + J_{n+2}\,J_m)\\
&+ 2k\,(J_{n-1}\,J_{m+2} + J_n\,J_{m+1} - J_{n+1}\,J_m + J_{n+2}\,J_{m-1})
\end{aligned}
\tag{63}
$$

Finally, adding equations (62) and (63) side by side and using (24), we obtain

$$
\begin{aligned}
JQ_n\,JQ_m + 2JQ_{n-1}\,JQ_{m-1} =& (J_{n+m-1} - J_{n+m+1} - J_{n+m+3} - J_{n+m+5})\\
&\quad + i\,(2J_{n+m}) + j\,(2J_{n+m+1}) + k\,(J_{n+m+2})\\
=& 2(J_{n+m-1} + i\,J_{n+m} + j\,J_{n+m+1} + k\,J_{n+m+2})\\
&\quad - (J_{n+m-1} + J_{n+m+1} + J_{n+m+3} + J_{n+m+5}\\
=& 2JQ_{n+m-1} - J_{n+m-1} - J_{n+m+1} - J_{n+m+3} - J_{n+m+5}.
\end{aligned}
$$

□

Theorem 2.2. *Let JQ_n be the Jacobsthal quaternion and JLQ_n be Jacobsthal-Lucas quaternion. The following relations are satisfied*

$$
\begin{aligned}
&JQ_{n+1} + 2JQ_{n-1} = JLQ_n\,,\\
&2JQ_{n+1} - JQ_n = JLQ_n\,.
\end{aligned}
\tag{64}
$$

Proof. From equations (52) and (8), it follows that

$$
\begin{aligned}
JQ_{n+1} + 2JQ_{n-1} =& \quad (J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4}) + 2(J_{n-1} + i\,J_n + j\,J_{n+1} + k\,J_{n+2})\\
=& \quad (J_{n+1} + 2J_{n-1}) + i\,(J_{n+2} + 2J_n) + j\,(J_{n+3} + 2J_{n+1}) + k\,(J_{n+4} + 2J_{n+2})\\
=& \quad j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3}\\
=& \quad JLQ_n\,.
\end{aligned}
$$

and

$$
\begin{aligned}
2JQ_{n+1} - JQ_n =& \quad 2(J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4}) - (J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3})\\
=& \quad (2J_{n+1} - J_n) + i\,(J_{n+2} - J_{n+1}) + j\,(2J_{n+3} - J_{n+2}) + k\,(2J_{n+4} - J_{n+3})\\
=& \quad j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3}\\
=& \quad JLQ_n\,.
\end{aligned}
$$

where we used (8) and $2J_{n+1} - J_n = j_n$ [3]. □

Theorem 2.3. *Let JQ_n be the Jacobsthal quaternion and $\overline{JQ}_n$ be conjugate of JQ_n. Then, we can give the following relations between these quaternions:*

$$JQ_n^2 = JQ_n\,(2J_n - \overline{JQ}_n), \tag{65}$$

$$JQ_n\overline{JQ}_n + 2JQ_{n-1}\overline{JQ}_{n-1} = J_{2n-1} + J_{2n+1} + J_{2n+3} + J_{2n+5}, \tag{66}$$

$$JQ_n^2 + 2JQ_{n-1}^2 = 2JQ_{2n-1} - (J_{2n-1} + J_{2n+1} + J_{2n+3} + J_{2n+5}) = 2\,JQ_{2n-1} - JQ_n.\overline{JQ}_n - 2JQ_{n-1}.\overline{JQ}_{n-1}. \tag{67}$$

Proof. (65): By using (52) and (55) we get

$$\begin{aligned} JQ_n^2 &= (J_n^2 - J_{n+1}^2 - J_{n+2}^2 - J_{n+3}^2) + 2i\,(J_nJ_{n+1}) + 2j\,(J_nJ_{n+2}) + 2k\,(J_nJ_{n+3}) \\ &= 2J_n(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) - (J_n^2 + J_{n+1}^2 + J_{n+2}^2 + J_{n+3}^2) \\ &= 2J_n.JQ_n - JQ_n.\overline{JQ}_n = JQ_n.(2J_n - \overline{JQ}_n). \end{aligned}$$

(66): By using (55) we get

$$\begin{aligned} JQ_n.\overline{JQ}_n + 2JQ_{n-1}.\overline{JQ}_{n-1} &= (J_n^2 + 2J_{n-1}^2) + (J_{n+1}^2 + 2J_n^2) + (J_{n+2}^2 + 2J_{n+1}^2) + (J_{n+3}^2 + 2J_{n+2}^2) \\ &= J_{2n-1} + J_{2n+1} + J_{2n+3} + J_{2n+5} \end{aligned} \tag{68}$$

(67): By using (55) and (68) we get

$$\begin{aligned} JQ_n^2 + 2JQ_{n-1}^2 =& (J_n^2 + 2J_{n-1}^2) - (J_{n+1}^2 + 2J_n^2) - (J_{n+2}^2 + 2J_{n+1}^2) - (J_{n+3}^2 + 2J_{n+2}^2) \\ &+ 2\,[\,i\,(J_nJ_{n+1} + 2J_{n-1}J_n) + j\,(J_nJ_{n+2} + 2J_{n-1}J_{n+1}) + k\,(J_nJ_{n+3} + 2J_{n-1}J_{n+2})] \\ =& [J_{2n-1} - J_{2n+1} - J_{2n+3} - J_{2n+5}] + 2[i\,J_{2n} + j\,J_{2n+1} + k\,J_{2n+2}] \\ =& 2\,[J_{2n-1} + i\,J_{2n} + j\,J_{2n+1} + k\,J_{2n+2}] - [J_{2n-1} + J_{2n+1} + J_{2n+3} + J_{2n+5}] \\ =& 2\,JQ_{2n-1} - (J_{2n-1} + J_{2n+1} + J_{2n+3} + J_{2n+5}) \\ =& 2\,JQ_{2n-1} - JQ_n.\overline{JQ}_n - 2JQ_{n-1}.\overline{JQ}_{n-1} \end{aligned}$$

where we used relations (24). □

Theorem 2.4. *Let JQ_n be the $n-th$ term of the Jacobsthal quaternion sequence. Then, we have the following identities*

$$\sum_{s=1}^{n} JQ_s = \frac{1}{2}[JQ_{n+2} - JQ_2], \tag{69}$$

$$\sum_{s=0}^{p} JQ_{n+s} = \frac{1}{2}[JQ_{n+p+2} - JQ_{n+1}], \tag{70}$$

$$\sum_{s=1}^{n} JQ_{2s-1} = \frac{2JQ_{2n}}{3} + \frac{1}{3}[n(2\,JQ_2 - JQ_3) - 2JQ_0], \tag{71}$$

$$\sum_{s=1}^{n} JQ_{2s} = \frac{2JQ_{2n+1}}{3} - \frac{1}{3}[n(2\,JQ_2 - JQ_3) - 2JQ_1]. \tag{72}$$

Proof. (69): we get

$$\begin{aligned} \sum_{s=1}^{n} JQ_s &= \sum_{s=1}^{n} J_s + i\sum_{s=1}^{n} J_{s+1} + j\sum_{s=1}^{n} J_{s+2} + k\sum_{s=1}^{n} J_{s+3} \\ &= \frac{1}{2}[(J_{n+2} - 1) + i\,(J_{n+3} - 3) + j\,(J_{n+4} - 5) + k\,(J_{n+5} - 11)] \\ &= \frac{1}{2}[(J_{n+2} - J_2) + i\,(J_{n+3} - J_3) + j\,(J_{n+4} - J_4) + k\,(J_{n+5} - J_5)] \\ &= \frac{1}{2}[J_{n+2} + i\,J_{n+3} + j\,J_{n+4} + k\,J_{n+5} - (J_2 + i\,J_3 + j\,J_4 + k\,J_5)] \\ &= \frac{1}{2}[JQ_{n+2} - JQ_2]\,. \end{aligned}$$

(70): Hence, we can write

$$\begin{aligned}\sum_{s=0}^{p} JQ_{n+s} =&(J_n + \ldots + J_{n+p}) + i(J_{n+1} + \ldots + J_{n+p+1}) + j(J_{n+2} + \ldots + J_{n+p+2}) + k(J_{n+3} + \ldots + J_{n+p+3})\\ =&\frac{1}{2}[(J_{n+p+2} - J_{n+1}) + i\,(J_{n+p+3} - J_{n+2}) + j\,(J_{n+p+4} - J_{n+3}) + k\,(J_{n+p+5} - J_{n+4})]\\ =&\frac{1}{2}[J_{n+p+2} + i\,J_{n+p+3} + j\,J_{n+p+4} + k\,J_{n+p+5} - (J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4})]\\ =&\frac{1}{2}[JQ_{n+p+2} - JQ_{n+1}]\,.\end{aligned}$$

(71): Using (20) and (21), we get

$$\begin{aligned}\sum_{s=1}^{n} JQ_{2s-1} =&(J_1 + J_3 + \ldots + J_{2n-1}) + i(J_2 + J_4 + \ldots + J_{2n}) + j(J_3 + J_5 + \ldots + J_{2n+1}) + k(J_4 + J_6 + \ldots + J_{2n+2})\\ =&[\frac{(2\,J_{2n} + n)}{3} + i\,\frac{(2\,J_{2n+1} - n - 2)}{3} + j\,\frac{(2\,J_{2n+2} + n - 2)}{3} + k\,\frac{(2\,J_{2n+3} - n - 6)}{3}]\\ =&\frac{2}{3}[J_{2n} + i\,J_{2n+1} + j\,J_{2n+2} + k\,J_{2n+3}] + \frac{1}{3}[n\,(1 - i + j - k) - 2(i + j + 3k)]\\ =&\frac{2JQ_{2n}}{3} + \frac{1}{3}[n\,(2\,JQ_2 - JQ_3) - 2\,JQ_0)]\,.\end{aligned}$$

(72): Using (20) and (21), we obtain

$$\begin{aligned}\sum_{s=1}^{n} JQ_{2s} =&(J_2 + J_4 + \ldots + J_{2n}) + i(J_3 + J_5 + \ldots + J_{2n+1}) + j(J_4 + J_6 + \ldots + J_{2n+2}) + k(J_5 + J_7 + \ldots + J_{2n+3})\\ =&[\frac{(2\,J_{2n+1} - n - 2}{3} + i\,\frac{(2\,J_{2n+2} + n - 2)}{3} + j\,\frac{(2\,J_{2n+3} - n - 6)}{3} + k\,\frac{(2\,J_{2n+4} + n - 10)}{3}]\\ =&\frac{2}{3}[J_{2n+1} + i\,J_{2n+2} + j\,J_{2n+3} + k\,J_{2n+4}] + \frac{1}{3}[-n\,(1 - i + j - k) - 2(1 + i + 3j + 5k)]\\ =&\frac{2JQ_{2n+1}}{3} - \frac{1}{3}[-n\,(2\,JQ_2 - JQ_3) - 2\,JQ_1)]\,.\end{aligned}$$

□

Theorem 2.5. *Let* $\overline{JQ}_n$ *be the conjugate of the Jacobsthal quaternions* $JQ_n = J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}$ *and* $\overline{JLQ}_n$ *be the conjugate of the Jacobsthal-Lucas quaternions* $JLQ_n = j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3}$. *Then*

$$JLQ_n\,\overline{JQ}_n - \overline{JLQ}_n\,JQ_n = (-1)^{n-1}.2^n(4i + 4j + 12k), \tag{73}$$

$$JLQ_n\,\overline{JQ}_n + \overline{JLQ}_n\,JQ_n = 2\,\left[(J_{2n} + J_{2n+2} + J_{2n+4} + J_{2n+6}) + (-1)^n.2^n(-8i - 4j + 4k)\right], \tag{74}$$

$$JLQ_n\,JQ_n - \overline{JLQ}_n\,\overline{JQ}_n = 2\,\left[(J_{2n} - J_{2n+2} - J_{2n+4} - J_{2n+6}) + (-1)^n.2^n(8i - 4j - 4k)\right]. \tag{75}$$

Proof.
(73): Using the relations (12), (36) and (37), we get

$$\begin{aligned} JLQ_n\,\overline{JQ_n}\; - \overline{JLQ_n}\,JQ_n =&(j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3})\,(J_n - i\,J_{n+1} - j\,J_{n+2} - k\,J_{n+3}) \\ &-(j_n - i\,j_{n+1} - j\,j_{n+2} - k\,j_{n+3})\,(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ =&2i\,[-(j_nJ_{n+1} - J_nj_{n+1})] + 2j\,[-(j_nJ_{n+2} - J_nj_{n+2})] + 2k\,[-(j_nJ_{n+3} - J_nj_{n+3})] \\ =&(-1)^{n-1}.2^{n+2}(i\,J_1 + j\,J_2 + k\,J_3) \\ =&(-1)^{n-1}.2^{n}(4i + 4j + 12k). \end{aligned}$$

(74): Using the relations (12), (36) and (37), follows

$$\begin{aligned} JLQ_n\,\overline{JQ_n}\; + \overline{JLQ_n}\,JQ_n =&(j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3})\,(J_n - i\,J_{n+1} - j\,J_{n+2} - k\,J_{n+3}) \\ &+(j_n - i\,j_{n+1} - j\,j_{n+2} - k\,j_{n+3})\,(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ =&2\,[j_nJ_n + j_{n+1}J_{n+1} + j_{n+2}J_{n+2} + j_{n+3}J_{n+3}] + 2i\,[-(j_{n+2}J_{n+3} - J_{n+2}j_{n+3})] \\ &+ 2j\,[j_{n+1}J_{n+3} - J_{n+1}j_{n+3}] + 2k\,[-(j_{n+1}J_{n+2} - J_{n+1}j_{n+2})] \\ =&2[(J_{2n} + J_{2n+2} + J_{2n+4} + J_{2n+6}) + (-1)^n.2^n(-8i + 4j + 4k)]. \end{aligned}$$

(75): Using the relations (12), (36) and (37), we find

$$\begin{aligned} JLQ_n\,JQ_n\; + \overline{JLQ_n}\,\overline{JQ_n} =&(j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3})\,(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3}) \\ &+(j_n - i\,j_{n+1} - j\,j_{n+2} - k\,j_{n+3})\,(J_n - i\,J_{n+1} - j\,J_{n+2} - k\,J_{n+3}) \\ =&2\,[j_nJ_n - j_{n+1}J_{n+1} - j_{n+2}J_{n+2} - j_{n+3}J_{n+3}] + 2i\,[(j_{n+2}J_{n+3} - J_{n+2}j_{n+3})] \\ &+ 2j\,[-j_{n+1}J_{n+3} + J_{n+1}j_{n+3}] + 2k\,[(j_{n+1}J_{n+2} - J_{n+1}j_{n+2})] \\ =&2[(J_{2n} - J_{2n+2} - J_{2n+4} - J_{2n+6}) + (-1)^n.2^n(8i - 4j - 4k)]. \end{aligned}$$

□

Theorem 2.6. ***(Binet's Formulas)****. Let JQ_n and JLQ_n be $n - th$ terms of the Jacobsthal quaternion (JQ_n) and the Jacobsthal-Lucas quaternion (JLQ_n), respectively. For $n \geq 1$, the Binet's formulas for these quaternions are as follows:*

$$JQ_n = \frac{1}{\alpha - \beta}\left[\underline{\alpha}\;\alpha^n - \underline{\beta}\,\beta^n\right] \tag{76}$$

and

$$JLQ_n = (\underline{\underline{\alpha}}\;\alpha^n + \underline{\underline{\beta}}\,\beta^n\,) \tag{77}$$

respectively, where

$$\alpha - \beta = 3, \quad \underline{\alpha} = 1 + 2\,i + 4\,j + 8\,k\,, \quad \underline{\beta} = 1 - i + j - k$$

and

$$\underline{\underline{\alpha}} = 3 + 6\,i + 12\,j + 24\,k\,, \quad \underline{\underline{\beta}} = 3 - 3\,i + 3\,j - 3\,k\,.$$

Proof. The characteristic equation of recurrence relations

$JQ_{n+2} = JQ_{n+1} + 2JQ_n$ and $JLQ_{n+2} = JLQ_{n+1} + 2JLQ_n$ is $t^2 - t - 2 = 0$.

The roots of this equation are $\alpha = 2$ and $\beta = -1$ where $\alpha + \beta = 1$, $\alpha - \beta = 3$, $\alpha\beta = -2$.

Using recurrence relation and initial values $JQ_0 = (0, 1, 1, 3)$, $JQ_1 = (1, 1, 3, 5)$ the Binet's formula for JQ_n, we get

$$JQ_n = A\,\alpha^n + B\,\beta^n = \frac{1}{3}\left[\,(1 + 2i + 4j + 8k)\,2^n - (1 - i + j - k)\,(-1)^n\right],$$

where $A = \dfrac{JQ_1 - JQ_0\,\beta}{\alpha - \beta}$, $B = \dfrac{\alpha\,JQ_0 - JQ_1}{\alpha - \beta}$ and $\underline{\alpha} = 1 + 2i + 4j + 8k,\quad \underline{\beta} = 1 - i + j - k.$

Similarly, the Binet's formula for JLQ_n is obtained as follows:

$$JLQ_n = [\,(3 + 6i + 12j + 24k)\,2^n + (3 - 3i + 3j - 3k)\,(-1)^n]$$

where

$$\underline{\underline{\alpha}} = 3 + 6i + 12j + 24k,\quad \underline{\underline{\beta}} = 3 - 3i + 3j - 3k$$

respectively. □

Theorem 2.7. ***(Cassini Identity)****. Let JQ_n and JLQ_n be the $n-th$ terms of the Jacobsthal quaternion sequence (JQ_n) and the Jacobsthal-Lucas quaternion sequence (JLQ_n), respectively. For $n \geq 1$, the Cassini identities for JQ_n and JLQ_n are as follows:*

$$JQ_{n-1}JQ_{n+1} - JQ_n^2 = (-1)^n\,2^{n-1}(7 + 5i + 7j + 5k). \tag{78}$$

$$JLQ_{n-1}JLQ_{n+1} - JLQ_n^2 = (-2)^{n-1}\,3^2(7 + 5i + 7j + 5k). \tag{79}$$

Proof. For the proof of (78) and (79), we will use relations of Jacobsthal number and Jacobsthal-Lucas number [5, 6] as follows:

$$J_mJ_{n-1} - J_{m-1}J_n = (-1)^n\,2^{n-1}\,J_{m-n} \tag{80}$$

$$j_mj_{n-1} - j_{m-1}j_n = (-2)^{n-1}\,3^2\,j_{m-n} \tag{81}$$

(78): Using the relations (3) and (80), we get

$$\begin{aligned}JQ_{n-1}\,JQ_{n+1} - JQ_n^2 =& (J_{n-1} + i\,J_n + j\,J_{n+1} + k\,J_{n+2})\,(J_{n+1} + i\,J_{n+2} + j\,J_{n+3} + k\,J_{n+4})\\ &- (J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3})\,(J_n + i\,J_{n+1} + j\,J_{n+2} + k\,J_{n+3})\\ =& [(J_{n-1}J_{n+1} - J_n^2) - (J_nJ_{n+2} - J_{n+1}^2) - (J_{n+1}J_{n+3} - J_{n+2}^2) - (J_{n+2}J_{n+4} - J_{n+3}^2)]\\ &+ i\,[-(J_nJ_{n+1} - J_{n-1}J_{n+2}) - (J_{n+2}J_{n+3} - J_{n+1}J_{n+4})]\\ &+ j\,[-(J_nJ_{n+2} - J_{n-1}J_{n+3}) - J_nJ_{n+4} - (J_nJ_{n+2} - J_{n+1}^2) + J_{n+2}^2]\\ &+ k\,[-(J_nJ_{n+3} - J_{n-1}J_{n+4}) - (J_{n+2}J_{n+1} - J_{n+1}J_{n+2})]\\ =& (-1)^n\,2^{n-1}\,(7 + 5i + 7j + 5k).\end{aligned}$$

(79): Using the relations (4) and (81), we obtain

$$\begin{aligned}JLQ_{n-1}\,JLQ_{n+1} - JLQ_n^2 =& (j_{n-1} + i\,j_n + j\,j_{n+1} + k\,j_{n+2})\,(j_{n+1} + i\,j_{n+2} + j\,j_{n+3} + k\,j_{n+4})\\ &- (j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3})\,(j_n + i\,j_{n+1} + j\,j_{n+2} + k\,j_{n+3})\\ =& [(j_{n-1}j_{n+1} - j_n^2) - (j_nj_{n+2} - j_{n+1}^2) - (j_{n+1}j_{n+3} - j_{n+2}^2) - (j_{n+2}j_{n+4} - j_{n+3}^2)]\\ &+ i\,[-(j_nj_{n+1} - j_{n-1}j_{n+2}) - (j_{n+2}j_{n+3} - j_{n+1}j_{n+4})]\\ &+ j\,[-(j_nj_{n+2} - j_{n-1}j_{n+3}) - j_nj_{n+4} - (j_nj_{n+2} - j_{n+1}^2) + j_{n+2}^2]\\ &+ k\,[-(j_nj_{n+3} - j_{n-1}j_{n+4}) - (j_{n+2}j_{n+1} - j_{n+1}j_{n+2})]\\ =& (-2)^{n-1}\,3^2\,(7 + 5i + 7j + 5k).\end{aligned}$$

□

We will give an example in which we check in a particular case the Cassini identity for the Jacobsthal quaternions.

Example 1. Let JQ_1, JQ_2, JQ_3 and JQ_4 be the Jacobsthal quaternions such that

$$\begin{cases} JQ_1 = 1 + i + 3j + 5k, \\ JQ_2 = 1 + 3i + 5j + 11k, \\ JQ_3 = 3 + 5i + 11j + 21k, \\ JQ_4 = 5 + 11i + 21j + 43k. \end{cases}$$

In this case,

$$\begin{aligned} JQ_1\,JQ_3 - JQ_2^2 &= (1 + i + 3j + 5k)(3 + 5i + 11j + 21k) - (1 + 3i + 5j + 11k)^2 \\ &= (-140 + 16i + 24j + 32k) - (-154 + 6i + 10j + 22k) \\ &= (14 + 10i + 14j + 10k) \\ &= (-1)^2\, 2\,(7 + 5i + 7j + 5k) \end{aligned} \tag{82}$$

and

$$\begin{aligned} JQ_2\,JQ_4 - JQ_3^2 &= (1 + 3i + 5j + 11k)(5 + 11i + 21j + 43k) - (3 + 5i + 11j + 21k)^2 \\ &= (-606 + 10i + 38j + 106k) - (-578 + 30i + 66j + 126k) \\ &= (-28 - 20i - 28j - 20k) \\ &= (-1)^3\, 2^2\,(7 + 5i + 7j + 5k). \end{aligned} \tag{83}$$

Example 2. Let JLQ_1, JLQ_2, JLQ_3 and JLQ_4 be the Jacobsthal-Lucas quaternions such that

$$\begin{cases} JLQ_1 = 1 + 5i + 7j + 17k, \\ JLQ_2 = 5 + 7i + 17j + 31k, \\ JLQ_3 = 7 + 17i + 31j + 65k, \\ JLQ_4 = 17 + 31i + 65j + 127k. \end{cases}$$

In this case,

$$\begin{aligned} JLQ_1\,JLQ_3 - JLQ_2^2 &= (1 + 5i + 7j + 17k)(7 + 17i + 31j + 65k) - (5 + 7i + 17j + 31k)^2 \\ &= (-1400 - 20i + 44j + 220k) - (-1274 + 70i + 170j + 310k) \\ &= (-126 - 90i + 126j + 90k) \\ &= (-2)\, 3^2\,(7 + 5i + 7j + 5k) \end{aligned} \tag{84}$$

and

$$\begin{aligned} JLQ_2 JLQ_4 - JLQ_3^2 &= (5 + 7i + 17j + 31k)(17 + 31i + 65j + 127k) - (7 + 17i + 31j + 65k)^2 \\ &= (-5174 + 418i + 686j + 1090k) - (-5426 + 238i + 434j + 910k) \\ &= (252 - 1120i + 252j - 910k) \\ &= (-2)^2\, 3^2 (7 + 5i + 7j + 5k). \end{aligned} \tag{85}$$

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