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Common Fixed Point Results for Generalized (ψ, β)-Geraghty Contraction Type Mapping in Partially Ordered Metric-like Spaces with Application

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Abstract. Harandi [A. A. Harandi, *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl., 2012 (2012), 10 pages] introduced the notion of metric-like spaces as a generalization of partial metric spaces and studied some fixed point theorems in the context of the metric-like spaces. In this paper, we utilize the notion of the metric-like spaces to introduce and prove some common fixed points theorems for mappings satisfying nonlinear contractive conditions in partially ordered metric-like spaces. Also, we introduce an example and an application to support our work. Our results extend and modify some recent results in the literature.

1. Introduction and preliminaries

The notion of partial metric space was introduced by Matthews [24] in 1994 as a part of the study of denotational semantics of dataflow networks. He showed that the Banach contraction principle can be generalized to the partial metric context for applications in program verification.

Definition 1.1. *[24] Let Y be a nonempty set. A function $p : Y \times Y \to [0, \infty)$ is called a partial metric if for all $y, w, z \in Y$, the following conditions are satisfied:*

1. $y = z \Leftrightarrow p(y, y) = p(y, z) = p(z, z)$,
2. $p(y, y) \leq p(y, z)$,
3. $p(y, z) = p(z, y)$,
4. $p(y, z) \leq p(y, w) + p(w, z) - p(w, w)$.

The pair (Y, p) is called a partial metric space.

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A sequence $\{y_n\}$ in a partial metric space (Y, p) converges to a point $y \in Y$ if $\lim_{n\to\infty} p(y_n, y) = p(y, y)$. A sequence $\{y_n\}$ of elements of Y is called $p-$Cauchy if the $\lim_{n,m\to\infty} p(y_n, y_m)$ exists as a finite number. The partial metric space (Y, p) is called complete if for each p-Cauchy sequence $\{y_n\}$, there is some $y \in Y$ such that

$$p(y, y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n,m\to\infty} p(y_n, y_m).$$

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(y, z) = \max\{y, z\}$ for all $y, z \in \mathbb{R}^+$. For more examples of partial metric spaces, see [5], [22], and [29].

Moreover, an important development is reported in fixed point theory via ordered metric spaces. The existence of a fixed point in partially ordered sets has been considered in [16], [17], [30], and [31]. Tarski's theorem is used in [25] to show the existence of solutions for fuzzy equations and in [26] to prove existence theorems for fuzzy differential equations. In [13], [25],[26], and [28] some applications to ordinary differential equations and matrix equations are investigated. In [12] and [15], some fixed point theorems were proved for a mixed monotone mapping in a metric space endowed with partial order and the authors applied their results to problems of existence and uniqueness of solutions to some boundary value problems. Harandi [1] introduced a new extension of the concept of partial metric space, called a metric-like space. He established the existence and uniqueness of fixed points in a metric-like space as well as in a partially ordered metric-like space. The purpose of this paper is to present some common fixed point theorems involving Geraghty contraction type mappings in the context of ordered metric-like spaces. Also, we introduce an example and an application on the existence of a unique solution of an integral equation. Our results extend and modify some recent results in the literature.
Now, we recall the definition of the metric like space:

Definition 1.2. *[1] Let Y is a nonempty set. A function $\sigma : Y \times Y \to [0, \infty)$ is said to be a metric like space (or dislocated metric) on Y if for any $y, w, z \in Y$, the following conditions hold:*

(σ_1) $\sigma(y, z) = 0 \Rightarrow y = z$,

(σ_2) $\sigma(y, z) = \sigma(z, y)$,

(σ_3) $\sigma(y, w) \leq \sigma(y, z) + \sigma(z, w)$.

The pair (Y, σ) is called a metric-like space.

It is clear that every metric space and partial metric space is a metric-like space but the converse is not true.

Example 1.3. *[1] Let $Y = \{0, 1\}$ and*

$$\sigma(y, z) = \begin{cases} 2, & \text{if } y = z = 0; \\ 1, & \text{otherwise}. \end{cases}$$

Then (Y, σ) is a metric-like space but it is not a partial metric space. Note that $\sigma(0, 0) \nleq \sigma(0, 1)$.

Moreover, each metric-like σ on Y generates a topology τ_σ on Y whose base is the family of open σ-balls

$$B_\sigma(y, \epsilon) = \{z \in Y : | \sigma(y, z) - \sigma(y, y) | < \epsilon\}, \ \textit{for all } y \in Y \textit{ and } \epsilon > 0.$$

Let (Y, σ) and (Z, σ) be metric-like spaces, and let $f : Y \to Z$ be a continuous mapping. Then

$$\lim_{n\to\infty} y_n = y \ \Rightarrow \ \lim_{n\to\infty} f y_n = f y.$$

A sequence $\{y_n\}$ of elements of Y is called σ-Cauchy if the limit $\lim_{n,m\to\infty} \sigma(y_n, y_m)$ exists as a finite number. The metric-like space (Y, σ) is called complete if for each σ-Cauchy sequence $\{Y_n\}$, there is some $y \in Y$ such that

$$\lim_{n\to\infty} \sigma(y_n, y) = \sigma(y, y) = \lim_{n,m\to\infty} \sigma(y_n, y_m).$$

Lemma 1.4. *Let (Y,σ) be a metric-like space. Let $\{y_n\}$ be a sequence in Y such that $y_n \to y$ where $y \in Y$ and $\sigma(y,y) = 0$. Then, for all $z \in Y$, we have $\lim_{n\to\infty} \sigma(y_n, z) = \sigma(y,z)$.*

Remark 1.5. *Let $Y = \{0,1\}$, and $\sigma(y,z) = 1$ for each $y,z \in Y$ and $y_n = 1$ for each $n \in \mathbb{N}$. Then it is easy to see that $y_n \to 0$ and $y_n \to 1$ and so in metric-like spaces the limit of a convergent sequence is not necessarily unique.*

Karapınar et al. studied many interesting fixed and common fixed point theorems in metric-like space, see [2]-[4], [11] and [18]-[21]. Also, Aydi et al. studied some nice works in metric like spaces, see [6]-[10].

Definition 1.6. *Let $(Y,\preceq)$ be a partially ordered set and let $f,g : Y \to Y$ be two mappings. Then:*

1. *The elements $y,z \in Y$ are called comparable if $y \preceq z$ or $z \preceq y$ holds.*
2. *f is called nondecreasing w.r.t. $\preceq$ if $y \preceq z$ implies $fy \preceq fz$.*
3. *The pair (f,g) is said to be weakly increasing if $fy \preceq gfy$ and $gy \preceq fgy$ for all $y \in Y$.*
4. *The mapping f is said to be weakly increasing if the pair (f,I) is weakly increasing, where I is denoted to the identity mapping on Y.*

Definition 1.7. *Let $(Y,\preceq)$ be a partially ordered set. Then we say that Y is regular, if whenever $\{w_n\}$ is a nondecreasing sequence in Y with respect to $\preceq$ such that $w_n \to w$, then $w_n \preceq w$ for all $n \in \mathbb{N}$.*

In 1973, Geraghty [14] defined a class of functions Π to be the set of functions $\alpha : [0,\infty) \to [0,1)$ such that if $\{t_n\}$ is a sequence in $[0,+\infty)$ with $\alpha(t_n) \to 1$, then $t_n \to 0$.

The notion of an altering distance function was presented by Khan et al. [23] as follows:

Definition 1.8. *[23] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following conditions hold:*

1. *ψ is continuous and nondecreasing,*
2. *$\psi(t) = 0 \Leftrightarrow t = 0$.*

2. Main Result

At the beginning of this section, we introduce the following lemma which will be used efficiently in the proof of our main result.

Lemma 2.1. *Let (Y,σ) be a metric like space and let $\{w_n\}$ be a sequence in Y such that $\lim_{n\to\infty} \sigma(w_n, w_{n+1}) = 0$. If $\lim_{n,m\to\infty} \sigma(w_n, w_m) \neq 0$, then there exist $\epsilon > 0$ and two sequences $\{n_l\}$ and $\{m_l\}$ of positive integers with $n_l > m_l > l$ such that following three sequences $\{\sigma(y_{2n_l}, y_{2m_l})\}$, $\{\sigma(y_{2n_l-1}, y_{2m_l})\}$, and $\{\sigma(y_{2n_l}, y_{2m_l+1})\}$ converge to ϵ^+ when $l \to \infty$.*

Proof. *Let $\{w_n\}$ be a sequence in (Y,σ) such that $\lim_{n\to\infty} \sigma(w_n, w_{n+1}) = 0$ and $\lim_{n,m\to\infty} \sigma(w_n, w_m) \neq 0$. Then there exist $\epsilon > 0$ and two sequences $\{n_l\}$ and $\{m_l\}$ of positive integers such that n_l is the smallest positive integer for which*

$$n_l > m_l > l,\ \ \sigma(w_{2n_l}, w_{2m_l}) \geq \epsilon.$$

This means that

$$\sigma(w_{2n_l-2}, w_{2m_l}) < \epsilon.$$

The triangular inequality implies that

$$\begin{aligned}\epsilon &\leq \sigma(w_{2n_l}, w_{2m_l}) \leq \sigma(w_{2n_l}, w_{2n_l-1}) + \sigma(w_{2n_l-1}, w_{2n_l-2}) + \sigma(w_{2n_l-2}, w_{2m_l}) \\ &\leq \sigma(w_{2n_l}, w_{2n_l-1}) + \sigma(w_{2n_l-1}, w_{2n_l-2}) + \epsilon.\end{aligned}$$

Letting $l \to \infty$ in the above inequalities, implies that

$$\lim_{l\to\infty} \sigma(y_{2n_l}, y_{2m_l}) = \epsilon^+. \tag{1}$$

Again, from the triangular inequality, we can deduce that

$$\mid \sigma(y_{2n_l}, y_{2m_l+1}) - \sigma(y_{2n_l}, y_{2m_l}) \mid \leq \sigma(y_{2m_l}, y_{2m_l+1}).$$

Letting $l \to \infty$ in the above inequality. Then we have

$$\lim_{l\to\infty} \sigma(y_{2n_l}, y_{2m_l+1}) = \epsilon^+.$$

Similarly, one can easily show that

$$\lim_{l\to\infty} \sigma(y_{2n_l-1}, y_{2m_l}) = \epsilon^+.$$

□

Definition 2.2. *Let $(Y, \preceq, \sigma)$ be a partially ordered metric like space and $f, g : Y \to Y$ be two mappings. Then we say that the pair (f, g) is of generalized (ψ, β)-Geraghty contraction type if there exist $\alpha \in \Pi, \psi \in \Psi$, and a continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*

$$\psi(\sigma(fy, gz)) \leq \alpha(M_{y,z})\beta(M_{y,z}), \tag{2}$$

holds for all comparable elements $y, z \in Y$, where

$$M_{y,z} = \max\{\sigma(y, z), \sigma(y, fy), \sigma(z, gz)\}.$$

Theorem 2.3. *Let $(Y, \preceq, \sigma)$ be a complete partially ordered metric like space and $f, g : Y \to Y$ be two mappings satisfying the following conditions:*

1. *The pair (f, g) is weakly increasing.*
2. *The pair (f, g) is of generalized (ψ, β)-Geraghty contraction type.*
3. *Either f or g is continuous.*

Then f and g have a common fixed point $u \in Y$ with $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the common fixed point of f and g is unique.

Proof. Choose $y_0 \in Y$. Let $y_1 = fy_0$ and $y_2 = gy_1$. Continuing in this way, we construct a sequence $\{y_n\}$ in Y defined by:

$$y_{2n+1} = fy_{2n} \text{ and } y_{2n+2} = gy_{2n+1}.$$

Since the pair (f, g) is weakly increasing, we have

$$y_1 = fy_0 \preceq gfy_0 = y_2 = fy_1 =\preceq ...y_{2n} \preceq gfy_{2n} = y_{2n+2} \preceq ....$$

Thus $y_n \preceq y_{n+1}$, for all $n \in \mathbb{N}$. If there exists some $k \in \mathbb{N}$ such that $\sigma(y_{2k}, y_{2k+1}) = 0$. Then $y_{2k} = y_{2k+1}$ and hence y_{2k} is a fixed point of f. To show that y_{2k} is also a fixed point of g it is enough to show that $y_{2k} = y_{2k+1} = y_{2k+2}$.

Assume $\sigma(y_{2k+1}, y_{2k+2}) \neq 0$. Since $y_{2k} \preceq y_{2k+1}$, then by (2), we have

$$\begin{aligned}
& \psi(\sigma(y_{2k+1}, y_{2k+2})) \\
= \; & \psi(\sigma(fy_{2k}, gy_{2k+1})) \\
\leq \; & \alpha(M_{y_{2k},y_{2k+1}})\beta(M_{y_{2k},y_{2k+1}}) \\
\leq \; & \alpha(\max\{\sigma(y_{2k}, y_{2k+1}), \sigma(y_{2k}, fy_{2k}), \sigma(y_{2k+1}, gy_{2k+1})\})\beta(\max\{\sigma(y_{2k}, y_{2k+1}), \\
& \sigma(y_{2k}, fy_{2k}), \sigma(y_{2k+1}, gy_{2k+1})\}) \\
= \; & \alpha(\max\{\sigma(y_{2k}, y_{2k+1}), \sigma(y_{2k}, y_{2k+1}), \sigma(y_{2k+1}, y_{2k+2})\})\beta(\max\{\sigma(y_{2k}, y_{2k+1}), \\
& \sigma(y_{2k}, y_{2k+1}), \sigma(y_{2k+1}, y_{2k+2})\}) \\
= \; & \alpha(\sigma(y_{2k+1}, y_{2k+2}))\beta(\sigma(y_{2k+1}, y_{2k+2})) \\
< \; & \beta(\sigma(y_{2k+1}, y_{2k+2})) \\
\leq \; & \psi(\sigma(y_{2k+1}, y_{2k+2})),
\end{aligned} \tag{3}$$

which is a contradiction. So $\sigma(y_{2k+1}, y_{2k+2}) = 0$, that is, $y_{2k} = y_{2k+1} = y_{2k+2}$. Thus y_{2k} is a common fixed point for f and g.

Now, we assume that $\sigma(y_n, y_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. If n is even, then $n = 2t$ for some $t \in \mathbb{N}$.

$$\begin{aligned}
& \psi(\sigma(y_n, y_{n+1})) \\
= \; & \psi(\sigma(y_{2r}, y_{2r+1})) \\
= \; & \psi(\sigma(fy_{2r}, gy_{2r-1})) \\
\leq \; & \alpha(\max\{\sigma(y_{2r-1}, y_{2r}), \sigma(y_{2r-1}, gy_{2r-1}), \sigma(y_{2r}, fy_{2r})\})\beta(\max\{\sigma(y_{2r-1}, y_{2r}), \\
& \sigma(y_{2r-1}, gy_{2r-1}), \sigma(y_{2r}, fy_{2r})\}) \\
= \; & \alpha(\max\{\sigma(y_{2r-1}, y_{2r}), \sigma(y_{2r}, y_{2r+1})\})\beta(\max\{\sigma(y_{2r-1}, y_{2r}), \sigma(y_{2r}, y_{2r+1})\}) \\
< \; & \beta(\max\{\sigma(y_{2r-1}, y_{2r}), \sigma(y_{2r}, y_{2r+1})\}).
\end{aligned} \tag{4}$$

Assume

$$\max\{\sigma(y_{2r-1}, y_{2r}), \sigma(y_{2r}, y_{2r+1})\} = \sigma(y_{2r}, y_{2r+1}).$$

By (4), we get

$$\psi(\sigma(y_{2r}, y_{2r+1})) < \psi(\sigma(y_{2r}, y_{2r+1}),$$

which is a contradiction. Thus

$$\max\{\sigma(y_{2r-1}, y_{2r}), \sigma(y_{2r}, y_{2r+1})\} = \sigma(y_{2r}, y_{2r-1}).$$

Therefore

$$\psi(\sigma(y_{2n}, y_{2n+1})) < \psi(\sigma(y_{2n-1}, y_{2n})).$$

Since ψ is an altering distance function, we conclude that

$$\sigma(y_{2n}, y_{2n+1}) \leq \sigma(y_{2n-1}, y_{2n}) \tag{5}$$

holds for all $n \in \mathbb{N}$. If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N}$. By (2), we have

$$\begin{aligned}
& \psi(\sigma(y_n, y_{n+1})) \\
= \; & \psi(\sigma(y_{2r+1}, y_{2r+2})) \\
= \; & \psi(\sigma(fy_{2r}, gy_{2r+1})) \\
\leq \; & \alpha(\max\{\sigma(y_{2r}, y_{2r+1}), \sigma(y_{2r}, fy_{2r}), \sigma(y_{2r+1}, gy_{2r+1})\}) \\
& \beta(\max\{\sigma(y_{2r}, y_{2r+1}), \sigma(y_{2r}, fy_{2r}), \sigma(y_{2r+1}, gy_{2r+1})\}) \\
= \; & \alpha(\sigma(y_{2r}, y_{2r+1}), \sigma(y_{2r+1}, y_{2r+2})\})\beta(\sigma(y_{2r}, y_{2r+1}), \sigma(y_{2r+1}, y_{2r+2})\}) \\
< \; & \beta(\sigma(y_{2r}, y_{2r+1}), \sigma(y_{2r+1}, y_{2r+2})).
\end{aligned} \tag{6}$$

Assume that $\max\{\sigma(y_{2r}, y_{2r+1}), \sigma(y_{2r+1}, y_{2r+2})\} = \sigma(y_{2r+1}, y_{2r+2})$.
By (6), we get $\psi(\sigma(y_{2r+1}, y_{2r+2})) < \psi(\sigma(y_{2r+1}, y_{2r+2}))$, which is a contradiction. Thus

$$\max\{\sigma(y_{2r}, y_{2r+1}), \sigma(y_{2r+1}, y_{2r+2})\} = \sigma(y_{2r}, y_{2r+1}).$$

Therefore

$$\psi(\sigma(y_n, y_{n+1})) < \psi(\sigma(y_{n-1}, y_n)).$$

Since ψ is an altering distance function, we deduce that

$$\sigma(y_{2n+1}, y_{2n+2}) \leq \sigma(y_{2n}, y_{2n+1}) \tag{7}$$

holds for all $n \in \mathbb{N}$. Combining (5) and (7) together, we have

$$\sigma(y_n, y_{n+1}) \leq \sigma(y_{n-1}, y_n) \tag{8}$$

holds for all $n \in \mathbb{N}$. Therefore, the sequence $\{\sigma(y_n, y_{n+1})\}$ is a decreasing sequence. Thus, there exists $u \geq 0$ such that

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} \sigma(y_n, y_{n+1}) = u.$$

Now we prove that $u = 0$. Suppose the contrary, that is $u > 0$. From (4) and (6), we have

$$\psi(\sigma(y_n, y_{n+1})) \leq \alpha(\sigma(y_{n-1}, y_n))\beta(\sigma(y_{n-1}, y_n)).$$

Taking the limit lim sup in the above inequality, implies that $\psi(u) < \beta(u) \leq \psi(u)$, which is a contradiction. Therefore $u = 0$. This implies that

$$w_n = \sigma(y_n, y_{n+1}) \to 0 \ \ as \ \ n \to \infty. \tag{9}$$

Now, we prove that

$$\lim_{n,m\to\infty} \sigma(y_n, y_m) = 0.$$

Suppose that

$$\lim_{n,m\to\infty} \sigma(y_n, y_m) \neq 0.$$

By Lemma 2.1, there exist $\epsilon > 0$ and two sequences $\{y_{n_l}\}$ and $\{y_{m_l}\}$ of $\{y_n\}$ with $2n_l > 2m_l \geq l$ such that the following three sequences

$$\{\sigma(y_{2n_l}, y_{2m_l})\}, \{\sigma(y_{2n_l-1}, y_{2m_l})\}, \{\sigma(y_{2n_l}, y_{2m_l+1})\} \tag{10}$$

converge to ϵ^+ when $l \to \infty$. From (2), we have

$$\begin{aligned} \psi(\sigma(y_{2n_l}, y_{2m_l+1})) &= \psi(\sigma(fy_{2m_l}, gy_{2n_l-1})) \\ &\leq \alpha(M_{y_{2m_l}, y_{2n_l-1}})\beta(M_{y_{2m_l}, y_{2n_l-1}}), \end{aligned} \tag{11}$$

where

$$\begin{aligned} M_{y_{2m_l}, y_{2n_l-1}} &= \max\{\sigma(y_{2n_l-1}, y_{2m_l}), \sigma(y_{2n_l-1}, gy_{2n_{l-1}}), \sigma(y_{2m_l}, fy_{2m_l})\} \\ &= \max\{\sigma(y_{2n_l-1}, y_{2m_l}), \sigma(y_{2n_l-1}, y_{2n_l}), \sigma(y_{2m_l}, y_{2m_l+1})\}. \end{aligned}$$

Letting $l \to \infty$ in (11) and using the properties of ψ, α and β, we conclude that

$$\begin{aligned} \psi(\epsilon) &\leq \alpha(\epsilon)\beta(\epsilon) \\ &< \beta(\epsilon) \\ &\leq \psi(\epsilon), \end{aligned}$$

a contradiction. Therefore

$$\lim_{n,m\to\infty} \sigma(y_n, y_m) = 0,$$

that is, $\{y_n\}$ is a $\sigma-$Cauchy sequence in Y. From the completeness of (Y, σ), there exists $w \in Y$ such that

$$\lim_{n\to\infty} \sigma(y_n, w) = \sigma(w, w) = \lim_{n,m\to\infty} \sigma(y_n, y_m) = 0. \tag{12}$$

Since f and g are continuous, we get

$$\lim_{n\to\infty} \sigma(y_{n+1}, gw) = \sigma(fy_{n+1}, gw) = \sigma(fw, gw), \tag{13}$$

$$\lim_{n\to\infty} \sigma(fw, y_{n+1}) = \sigma(fw, gy_n) = \sigma(fw, gw). \tag{14}$$

By Lemma 1.4 and (12), we obtain

$$\lim_{n\to\infty} \sigma(y_{n+1}, gw) = \sigma(w, gw), \tag{15}$$

and

$$\lim_{n\to\infty} \sigma(fw, y_{n+1}) = \sigma(fw, w). \tag{16}$$

Combining (13) and (15), we deduce that $\sigma(w, gw) = \sigma(fw, gw)$. Also, by (14) and (16), we deduce that $\sigma(fw, w) = \sigma(fw, gw)$. So

$$\sigma(w, gw) = \sigma(fw, w) = \sigma(fw, gw). \tag{17}$$

Now we show that $\sigma(w, gw) = 0$. Suppose to the contrary, that is, $\sigma(w, gw) > 0$. Since $w \preceq w$, we obtain

$$\begin{aligned} \psi(\sigma(w, gw)) &= \psi(\sigma(fw, gw)) \\ &\leq \alpha(M_{w,w})\beta(M_{w,w}), \end{aligned} \tag{18}$$

where

$$\begin{aligned} M_{w,w} &= \max\{\sigma(w, w), \sigma(w, fw), \sigma(w, gw)\} \\ &= \max\{\sigma(w, gw), \sigma(w, gw)\} = \sigma(w, gw). \end{aligned}$$

Therefore, from (18), we get

$$\psi(\sigma(w, gw)) \leq \alpha(\sigma(w, gw))\beta(\sigma(w, gw)) < \beta(\sigma(w, gw) \leq \psi(\sigma(w, gw), \tag{19}$$

which is a contradiction. Therefore we have $\sigma(w, gw) = 0$. Hence $gw = w$. From (17), we conclude that $\sigma(w, fw) = 0$. Thus $fw = w$. So w is a common fixed point of f and g. To prove the uniqueness of the common fixed point, we assume that v is another fixed point of f and g. Now we show that $\sigma(v, v) = 0$. Suppose to the contrary, that is, $\sigma(v, v) > 0$. Since $v \preceq v$, we have

$$\begin{aligned} \psi(\sigma(v, v)) &= \psi(\sigma(fv, gv)) \\ &\leq \alpha(\sigma(v, v))\beta(\sigma(v, v)) \\ &< \beta(\sigma(v, v)) \\ &\leq \psi(\sigma(v, v)) \end{aligned}$$

which is a contradiction. Thus $\sigma(v, v) = 0$. So by the additional conditions on Y, we conclude that w and v are comparable. Now assume that $\sigma(w, v) \neq 0$. Then

$$\begin{aligned} \psi(\sigma(w, v)) &= \psi(\sigma(fw, gv)) \\ &\leq \alpha(\sigma(w, v))\beta(\sigma(w, v)) \\ &< \beta(\sigma(w, v)) \\ &\leq \psi(\sigma(w, v)) \end{aligned}$$

which is a contradiction. Thus $\sigma(w, v) = 0$. Hence $w = v$. Thus f and g have a unique common fixed point. □

The continuity of f or g in Theorem 2.3 can be dropped if we assume that Y is regular. Thus, we have the following theorem:

Theorem 2.4. *Let $(Y, \preceq, \sigma)$ be a complete partially ordered metric like space and $f, g : Y \to Y$ be two mappings satisfying the following conditions:*

1. *The pair (f, g) is weakly increasing.*
2. *The pair (f, g) is generalized (ψ, β)-Geraghty contraction type.*
3. *Y is regular.*

Then f and g have a common fixed point $u \in Y$ with $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the common fixed point of f and g is unique.

Proof. *Following the proof of Theorem 2.3, we construct a sequences $\{y_n\}$ in Y such that $y_n \to u \in Y$ with $\sigma(u, u) = 0$. Now, we prove that u is a common fixed point of f and g.*
By regularity of Y, we have

$$y_n \preceq u \text{ for all } n \in \mathbb{N}.$$

So for any $n \in \mathbb{N}$, the elements y_n and u are comparable.
Now, we prove that $\sigma(u, gu) = 0$. Suppose to the contrary, that is $\sigma(u, gu) > 0$.
By (2), we have

$$\begin{aligned}
&\psi(\sigma(y_{2n+1}, gu)) \\
= \ &\psi(\sigma(fy_{2n}, gu)) \\
\le \ &\alpha(\max\{\sigma(y_{2n}, u), \sigma(y_{2n}, fy_{2n}), \sigma(u, gu)\})\beta(\max\{\sigma(y_{2n}, u), \sigma(y_{2n}, fy_{2n}), \sigma(u, gu)\}) \\
= \ &\alpha(\max\{\sigma(y_{2n}, u), \sigma(y_{2n}, y_{2n+1}), \sigma(u, gu)\})\beta(\max\{\sigma(y_{2n}, u), \sigma(y_{2n}, y_{2n+1}), \sigma(u, gu)\}).
\end{aligned}$$

Letting $n \to +\infty$ in above inequalities, we conclude that

$$\psi(\sigma(u, gu)) \le \alpha(\sigma(u, gu))\beta(\sigma(u, gu)).$$

Using the properties of ψ, α and β, we conclude that $\psi(\sigma(u, gu)) < \psi(\sigma(u, gu))$, a contradiction. Thus, $\sigma(u, gu) = 0$. Hence u is a fixed point of g. Using similar arguments as above, we can show that u is a fixed point of f. The uniqueness of the common fixed point of f and g is obtained by similar arguments as those given in the proof of Theorem 2.3. □

Corollary 2.5. *Let $(Y, \preceq, \sigma)$ be a partially ordered complete metric like space and $f : Y \to Y$ be a mapping satisfying the following conditions:*

1. *There exist $\psi \in \Psi$, $\alpha \in \Pi$ and a continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(t) \le \psi(t)$ for all $t > 0$ such that*
$$\psi(\sigma(fx, fy)) \le \alpha(\max\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy)\})\beta(\max\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy)\})$$
holds for all comparable $x, y \in Y$.
2. *$fy \preceq f(fy)$ for all $y \in Y$.*
3. *f is continuous.*

Then f has a fixed point $u \in Y$ such that $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the fixed point of f is unique.

Proof. *It follows from Theorem 2.3 by putting $g = f$.* □

The continuity of f in Corollary 2.5 can be dropped:

Corollary 2.6. *Let $(Y, \preceq, \sigma)$ be a partially ordered complete metric like space and $f : Y \to Y$ be a mapping satisfying the following conditions:*

1. *There exist $\psi \in \Psi$, $\alpha \in \Pi$ and a continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*
$$\psi(\sigma(fx, fy)) \leq \alpha(\max\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy)\})\beta(\max\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy)\})$$
holds for all comparable $x, y \in Y$.
2. *$fy \preceq f(fy)$ for all $y \in Y$.*
3. *Y is regular.*

Then f has a fixed point $u \in Y$ such that $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the fixed point of f is unique.

Proof. It follows from Theorem 2.4 by putting $g = f$. □

The following two corollaries are direct results of Theorem 2.3 and Theorem 2.4 respectively.

Corollary 2.7. *Let $(Y, \preceq, \sigma)$ be a partially ordered complete metric like space and $f, g : Y \to Y$ be two mappings satisfying the following conditions:*

1. *There exist $\psi \in \Psi$, $\alpha \in \Pi$ and a continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*
$$\psi(\sigma(fx, gy)) \leq \alpha(\sigma(x, y))\beta(\sigma(x, y))$$
holds for all comparable $x, y \in Y$.
2. *The pair (f, g) is weakly increasing.*
3. *Either f or g is continuous.*

Then f and g have a common fixed point $u \in Y$ such that $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the common fixed point of f and g is unique.

The continuity of f or g in Corollary 2.9 can be dropped:

Corollary 2.8. *Let $(Y, \preceq, \sigma)$ be a partially ordered complete metric like space and $f, g : Y \to Y$ be two mappings satisfying the following conditions:*

1. *There exist $\psi \in \Psi$, $\alpha \in \Pi$ and a continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*
$$\psi(\sigma(fx, gy)) \leq \alpha(\sigma(x, y))\beta(\sigma(x, y))$$
holds for all comparable $x, y \in Y$.
2. *The pair (f, g) is weakly increasing.*
3. *Y is regular.*

Then f and g have a common fixed point $u \in Y$ such that $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the common fixed point of f and g is unique.

The following two corollaries are consequence results of Corollary 2.9 and Corollary 2.10, respectively.

Corollary 2.9. *Let $(Y, \preceq, \sigma)$ be a partially ordered complete metric like space and $f : Y \to Y$ be a mapping satisfying the following conditions:*

1. *There exist $\psi \in \Psi$, $\alpha \in \Pi$ and a continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*
$$\psi(\sigma(fx, fy)) \leq \alpha(\sigma(x, y))\beta(\sigma(x, y))$$
holds for all comparable $x, y \in Y$.
2. $fy \preceq f(fy)$ *for all $y \in Y$.*
3. f *is continuous.*

Then f has fixed point $u \in Y$ such that $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the fixed point of f is unique.

The continuity of f or g in Corollary 2.9 can be dropped:

Corollary 2.10. *Let $(Y, \preceq, \sigma)$ be a partially ordered complete metric like space and $f : Y \to Y$ be a mapping satisfying the following conditions:*

1. *There exist $\psi \in \Psi$, $\alpha \in \Pi$ and a continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(t) < \psi(t)$ for all $t > 0$ such that*
$$\psi(\sigma(fx, fy)) \leq \alpha(\sigma(x, y))\beta(\sigma(x, y))$$
holds for all comparable $x, y \in Y$.
2. $fy \preceq f(fy)$ *for all $y \in Y$.*
3. Y *is regular.*

Then f has a fixed point $u \in Y$ such that $\sigma(u, u) = 0$. Furthermore, assume that if $z_1, z_2 \in Y$ such $\sigma(z_1, z_1) = \sigma(z_2, z_2) = 0$ implies that z_1 and z_2 are comparable. Then the fixed point of f is unique.

Example 2.11. *Let $Y = \{0, 1, 4\}$ be equipped with the following partial order $\preceq$:*

$$\preceq := \{(0, 0), (1, 1), (4, 4), (1, 0)\}$$

Define a metric like function $\sigma : Y \times Y \to \mathbb{R}$ by $\sigma(0, 0) = 0$, $\sigma(1, 1) = \sigma(4, 4) = 8$, $\sigma(1, 4) = \sigma(4, 1) = 4$, $\sigma(4, 0) = \sigma(0, 4) = 4$, and $\sigma(0, 1) = \sigma(1, 0) = 4$. It is easy to see that (Y, σ) is a complete metric like space. Also, define $f, g : Y \to \mathbb{R}^+$ by

$$f = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, g = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is an easy matter to see that the pair (f, g) is weakly increasing mapping with respect to $\preceq$ and that f and g are continuous. Define $\psi : [0; \infty) \to [0; \infty)$ by $\psi(t) = \frac{1}{e}t$, and $\beta(t) = \frac{1}{1.1e}t$. Also, define $\alpha : [0, +\infty) \to [0, 1)$ by $\alpha(t) = e^{\frac{-t}{16}}$ if $t > 0$ and $\alpha(0) = 0$. We next verify that the functions (f, g) satisfies the inequality (2). For that, given $y, z \in Y$ with $y \preceq z$. Then we have the following cases:
Case 1: *$y = 0$ and $z = 0$. Then*

$$\psi(\sigma(f0, g0)) = \psi(\sigma(0, 0)) = 0 \leq \alpha(M_{1,0})\beta(M_{1,0}).$$

Case 2: *$y = 1$ and $z = 1$. Then*

$$\psi(\sigma(f1, g1)) = \psi(\sigma(0, 1)) = \psi(4) = \frac{4}{e},$$

and

$$M_{1,1} = \max\{8, 4, 8\} = 8.$$

So

$$\alpha(M_{1,1})\beta(M_{1,1}) = \alpha(8)\beta(8) = e^{-\frac{1}{2}}\frac{8}{1.1e}.$$

Hence

$$\psi(\sigma(f1, g1)) \leq \alpha(M_{1,1})\beta(M_{1,1}).$$

Case 3: $y = 4$ *and* $z = 4$. *Then*

$$\psi(\sigma(f4, g4)) = \psi(\sigma(1, 0)) = \psi(4) = \frac{4}{e},$$

and

$$M_{4,4} = \max\{8, 4, 4\} = 8.$$

So

$$\alpha(M_{4,4})\beta(M_{4,4}) = \alpha(8)\beta(8) = e^{-\frac{1}{8}}\frac{8}{1.1e}.$$

Hence

$$\psi(\sigma(f4, g4)) \leq \alpha(M_{4,4})\beta(M_{4,4}).$$

Case 4: $1 \leq 0$. *Then, we have two subcases:*
Subcase I: $y = 1$ *and* $z = 0$.. *Then*

$$\psi(\sigma(f1, g0)) = \psi(\sigma(0, 0)) = 0 \leq \alpha(M_{1,0})\beta(M_{1,0}).$$

Subcase II: $y = 0$ *and* $z = 1$. *Then*

$$\psi(\sigma(f0, g1)) = \psi(\sigma(0, 1)) = \psi(4) = \frac{4}{e},$$

and

$$M_{0,1} = \max\{0, 0, 8\} = 8.$$

So

$$\alpha(M_{0,1})\beta(M_{0,1}) = \alpha(8)\beta(8) = e^{-\frac{1}{2}}\frac{8}{1.1e}.$$

Hence

$$\psi(\sigma(f0, g1)) \leq \alpha(M_{0,1})\beta(M_{0,1}).$$

Thus, all the conditions of Theorem 2.3 are satisfied and hence f and g have a common fixed point. Indeed, 0 *is a common fixed point of f and g.*

3. Application

Let $Y = C([0, 1], \mathbb{R})$ be the set of real continuous functions defined on $[0, 1]$. Take the metric-like $\sigma : Y \times Y \to [0, \infty)$ given by

$$\sigma(y, z) = \| y - z \|_\infty = \sup_{t \in [0,1]} | y(t) - z(t) |,$$

for all $y, z \in Y$. Then (Y, σ) is a complete metric-like space. Consider the integral equation

$$g(t) + \int_0^1 S(t, r) f(r, y(r)) dr; \quad t \in [0, 1] \tag{20}$$

The purpose of this section is to give an existence solution to (3.1) that belongs to $Y = C(I; R)$ (the set of continuous real functions defined on $I = [0, 1]$), by using the obtained result in Corollary 2.5. We endow Y with the partial order $\preceq$ given by:

$$y \preceq z \Leftrightarrow y(t) \leq z(t) \ \ for\ all\ t \in [0, 1].$$

We suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are two continuous. Now,we de
ne $S : [0,1] \times [0,1] \to [0,\infty)$

$$Fy(t) = g(t) + \int_0^1 S(t,r) f(r, y(r))dr; \ \ t \in [0,1] \tag{21}$$

for all $y \in Y$. Then, a solution to (20) is a fixed point of f. We will prove the following result.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

1. *there exists $\zeta : Y \times Y \to [0,1)$ such that for all $r \in [0,1]$ and for all $y, z \in Y$*

$$0 \leq \mid f(r, y(r)) - f(r, z(r)) \mid \leq \zeta(y,z) \mid y(r) - z(r) \mid,$$

2. *there exists $\alpha : [0,\infty) \to [0,1)$ such that*

$$\lim_{n\to\infty} \alpha(t_n) = 1 \ \Rightarrow \ \lim_{n\to\infty} t_n = 0,$$

and

$$\| \int_0^1 S(t,r)\zeta(y,z)dr \|_\infty \leq (\| y - z \|_\infty).$$

Then the integral equation (20) has a unique solution in Y.

Proof. *Clearly, any fixed point of (21) is a solution to (20). By conditions (i) and (ii), we obtain*

$$\begin{aligned}
\mid f(y)(t) - f(z)(t) \mid &= \left| \int_0^1 S(t,r)[f(r,y(r)) - f(r,z(r))]dr \right| \\
&\leq \int_0^1 S(t,r) \mid f(r,y(r)) - f(r,z(r)) \mid dr \\
&\leq \int_0^1 S(t,r)\zeta(y,z) \mid f(r,y(r)) - f(r,z(r)) \mid dr \\
&\leq \int_0^1 S(t,r)\zeta(y,z) \| y - z \|_\infty \, dr \\
&\leq \sigma(y,z) \int_0^1 S(t,r)\zeta(y,z)dr \\
&\leq \alpha(\sigma(y,z))\sigma(y,z).
\end{aligned}$$

Then we have

$$\| f(y)(t) - f(z)(t) \|_\infty \leq \alpha(\sigma(y,z))\sigma(y,z).$$

Thus for all$y, z \in Y$, we obtain

$$\sigma(fy, fz) \leq \alpha(\sigma(y,z))\sigma(y,z)$$

This implies that the hypotheses of Corollary 2.5 hold. Thus the operator f has a unique fixed point, that is, the integral equation (21) has a unique solution in Y. □

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References

[1] A. A. Harandi, *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl., 2012 (2012), 10 pages.

[2] S. Al-Mezel, C.-M. Chen, E. Karapınar and V. Rakocevc, *Fixed point results for various α-admissible contractive mappings on metric-like spaces*, Abstract and Applied Analysis Volume 2014 (2014), Article ID 379358.

[3] H. Alsulami, S.Gülyaz, E. Karapınar, I. Erhan, *An Ulam stability result on quasi-b-metric-like spaces*, Open Mathematics, Volume 14, Issue 1 (Jan 2016).

[4] H. H. Alsulami, E.Karapınar and H.Piri, *Fixed Points of Modified F-Contractive Mappings in Complete Metric-Like Spaces*, Journal of Function Spaces, Volume 2015 (2015), Article ID 270971, 9 pages

[5] H. Aydi, *Fixed point results for weakly contractive mappings in ordered partial metric spaces*. J. Adv. Math. Stud. **4**(2),1-12(2011)

[6] H. Aydi, A. Felhi, *Best proximity points for cyclic Kannan-Chatterjea- Ciric type contractions on metric-like spaces*, Journal of Nonlinear Sciences and Application, 9, 2458–2466 (2016)

[7] H. Aydi, A. Felhi, S. Sahmim, *On common fixed points for (alpha,psi)-contractions and generalized cyclic contractions in b-metric-like spaces and consequences*, Journal of Nonlinear Sciences and Application, 9, 2492–2510 (2016)

[8] H. Aydi, A. Felhi, *On best proximity points for various alpha-proximal contractions on metric-like spaces*, Journal of Nonlinear Sciences and Application, 9, 5202–5218 (2016)

[9] H. Aydi, A. Felhi, H. Afshari, *New Geraghty type contractions on metric-like spaces*, Journal of Nonlinear Sciences and Application, 10, 780–788 (2017).

[10] H. Aydi, A. Felhi, S. Sahmim, *Common fixed points via implicit contractions on b-metric-like spaces*, Journal of Nonlinear Sciences and Application, 10, 1524–1537 (2017)

[11] H.Aydi and E. Karapınar, *Fixed point results for generalized $\alpha - \psi$-contractions in metric-like spaces and applications*, Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 133, pp. 1-15.

[12] L. Ciric, N. Cakid, M. Rajovic, JS. Uma, *Monotone generalized nonlinear contractions in partially ordered metric spaces*. Fixed Point Theory Appl. **2008**, Article ID 131294 (2008)

[13] D. ÓRegan, A. Petrusel, *Fixed point theorems for generalized contractions in ordered metric spaces*. J. Math. Anal. Appl. **341**, 1241-1252 (2008)

[14] M. Geraghty, *On contractive mappings*. Proc. Am. Math. Soc. 40, 604608 (1973).

[15] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*.Nonlinear Anal. **65**, 1379-1393 (2006)

[16] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*. Nonlinear Anal. **71**, 3403-3410 (2009)

[17] J. Harjani, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*. Nonlinear Anal. **72**, 1188-1197 (2010)

[18] B. Hazarika, E. Karapınar, R. Arab, M. Rabbani, *Metric-like spaces to prove existence of solution for nonlinear quadratic integral equation and numerical method to solve it*, Journal of Computational and Applied Mathematicshttps://doi.org/10.1016/j.cam.2017.07.012

[19] E. Karapınar, it "A Short Survey on Dislocated Metric Spaces via Fixed-Point Theory" Edited by J. Banaś et al. (eds.), Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness, DOI 10.1007/978-981-10-3722-$1_1$3

[20] E.Karapınar, M.Kutbi, H. Piri And D. O'Regan, *Fixed points of conditionally F-contractions in complete metric-like spaces*, Fixed Point Theory and Applications 2015, 2015:126

[21] E. Karapınar, P.Salimi, *Dislocated Metric Space to Metric Spaces with some fixed point theorems*, Fixed Point Theory and Applications, (2013), 2013:222.

[22] E. Karapınar, W. Shatanawi,*On weakly (C, ψ, φ)-contractive mappings in ordered partial metric spaces*. Abstr. Appl. Anal. **2012**, Article ID 495892 (2012)

[23] MS, Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*. Bull Aust Math Soc. 30,19 (1994). doi:10.1017/S0004972700001659

[24] S. G. Matthews, *Partial metric topology*. Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., **728**(1994), 183197.

[25] JJ. Nieto, R. Odríguez-López, *Existence of extremal solutions for quadratic fuzzy equations*. Fixed Point Theory Appl.**2005**, 321-342 (2005)

[26] JJ. Nieto, R. Odríguez-López, *Applications of contractive like mapping principles to fuzzy equations*. Rev. Mat.Complut. **19**, 361-383 (2006)

[27] JJ. Nieto, R. Odríguez-López, *Applications of contractive like mapping principles to fuzzy equations*. Rev. Mat.

[28] ACM. Ran, MCB. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*. Proc. Am.Math. Soc. **132**, 1435-1443

[29] S. Romaguera, O. Valero, *A quantitative computational model for complete partial metric spaces via formal balls*. Math. Struct. Comput. Sci. **19**(3), 541-563 (2009)

[30] Shatanawi W, Alrawashdeh A, *Common fixed points of almost generalized (ψ, φ)-contractive mappings in ordered metric spaces*. Fixed Point Theory Appl. 2012: 80 (2012)

[31] Y. Wu, *New fixed point theorems and applications of mixed monotone operator*. J. Math. Anal. Appl. **341**, 883-893(2008)