Filomat 31:17 (2017), 5475–5488
https://doi.org/10.2298/FIL1717475A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fixed Point Theorems in Complete Modular Metric Spaces and an Application to Anti-periodic Boundary Value Problems

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Abstract. In this paper existence and uniqueness of fixed points for a general class of contractive and non-expansive mappings on modular metric spaces is discussed. As an application of the theoretical results, the existence of a solution of anti-periodic boundary value problems for nonlinear first order differential equations of Carathéodory's type is considered in the framework of modular metric spaces.

1. Introduction

The notions of metric modular and modular metric spaces (or metric modular spaces) have been introduced recently by Chistyakov [7, 9]. Metric modulars generate metric spaces by providing a weaker convergence called the modular convergence having a non-metrizable topology. Modular metric spaces are extensions of metric spaces, metric linear spaces, and classical modular linear spaces founded by Nakano as extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions [13, 14]. For a detailed overview of metric modulars and modular metric spaces, see [9].

In [8], Chistyakov establishes a fixed point theorem for contractive maps in modular metric spaces. In [12], authors also prove the existence of fixed point theorems for contraction mappings and Kannan type contraction mappings in modular metric spaces.

In this paper, our main aim is to prove two fixed point theorems on modular metric spaces for non-expansive mappings discussed by Bogin [3] on complete metric spaces. Bogin [3] proved the following fixed point theorem.

Theorem 1.1. *Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \tag{1}$$

where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Inspired by the study of Bogin, we prove a fixed point theorem for self mappings on modular metric spaces satisfying the condition (1). We also study contractive mappings of a similar type and give some particular cases.

2010 *Mathematics Subject Classification*. 47H10 46A80, 54H25, 55M20.
Keywords. Modular metric; Fixed point; Boundary value problem.
Received: 17 July 2017; Accepted: 27 October 2017
Communicated by K. Taş
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In [8, 9], Chistyakov applied his theoretical findings to an initial value problem for first order differential equations of Carathéodory's type. In this study, we present an application of fixed point results to modular metric space on an anti-periodic boundary value problem for Carathéodory's type ordinary differential equations using a variant of a modular metric space given in the problem studied by Chistyakov in [8]. Because of their appearance on many applications, anti-periodic problems are studied extensively in the last twenty years, see [1, 2, 4, 8–11, 15] for example. The existence and uniqueness of solutions for such problems have received a great attention, see [5, 16, 17] and references therein. In [11], authors proved existence results for a nonlinear Carathéodory's type anti-periodic first order problem using a Leray-Schauder alternative. Motivated by the studies in [8, 11], as an application of our theorems, we consider anti-periodic first order boundary value problems of Carathéodory's type.

The paper is organized as follows: In Section 2 basic concepts on metric modular and modular metric space are given. Existence and uniqueness theorems for self mappings on modular metric spaces are proved in Section 3. As an example of metric modular spaces, the metric modular space of anti-periodic mappings of bounded generalized φ-variations is introduced in Section 4. Finally, the existence of solutions of anti-periodic boundary value problem for first order differential equations of Carathéodory's type is investigated in Section 5.

2. Modular metric spaces essentials

In this section, we give some relevant definitions and results on modular metric spaces which will be used in our main results. For further information, see [7, 9].

Let X be a nonempty set, $\lambda > 0$, $w : (0, \infty) \times X \times X \longrightarrow [0, \infty]$. We write $w_\lambda(x, y) = w(\lambda, x, y)$ for all $\lambda > 0$, $x, y \in X$ so that $w = \{w_\lambda\}_{\lambda>0}$ for which $w_\lambda : X \times X \longrightarrow [0, \infty]$.

Definition 2.1. *A function $w : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ is said to be a* **(metric) modular** *on X if it satisfies the following three conditions:*

a) $x = y$ iff $w_\lambda(x, y) = 0$ for all $\lambda > 0$;

b) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$;

c) $w_{\lambda+\mu}(x, z) \leq w_\lambda(x, y) + w_\mu(y, z)$ for all $\lambda, \mu > 0$.

for all $x, y, z \in X$.

If, instead of a), the function w satisfies only

$$w_\lambda(x, x) = 0 \text{ for all } \lambda > 0 \tag{2}$$

then w is said to be a **pseudomodular** on X.

If w satisfies (2) and given $x, y \in X$, if there exists a number $\lambda > 0$, possibly depending on x and y, such that $w_\lambda(x, y) = 0$, then $x = y$, the function w is called a **strict modular** on X.

A modular (pseudomodular, strict modular) w on X is said to be **convex** if, instead of c), for all $\lambda, \mu > 0$, it satisfies the inequality

$$w_{\lambda+\mu}(x, z) \leq \frac{\lambda}{\lambda + \mu} w_\lambda(x, y) + \frac{\mu}{\lambda + \mu} w_\mu(y, z). \tag{3}$$

It is shown in [7] that a convex modular satisfies

$$w_\lambda(x, y) \leq \frac{\mu}{\lambda} w_\mu(x, y) \leq w_\mu(x, y). \tag{4}$$

for all $x, y \in X$ and $0 < \mu \leq \lambda$. Indeed, using the condition (c) of Definition 2.1 it can be seen that, a modular (pseudomodular) w satisfies

$$w_{\lambda_2}(x, y) \leq w_{\lambda_1}(x, y) \tag{5}$$

for $\lambda_1 < \lambda_2$ and all $x, y \in X$.

Definition 2.2. *[7] Let w be a pseudomodular on X and $x_0 \in X$. Then the sets*

$$X_w = X_w(x_0) = \{x \in X : w_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

$$X_w^* = X_w^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0, \text{ such that } w_\lambda(x, x_0) < \infty\}$$

are said to be modular metric spaces (around x_0).

It can be observed that, $X_w \subset X_w^*$ holds. If w is a metric modular on X, then the modular space X_w can be equipped with a (nontrivial) metric generated by w given by

$$d_w(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq \lambda\}$$

for any $x, y \in X_w$. If w is a convex modular on X, then $X_w = X_w^*$ holds and they are endowed with the metric

$$d_w^*(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}.$$

These distances are called Luxemburg distances.

Definition 2.3. *[7, 8] Let X_w and X_w^* be modular metric spaces.*

- *The sequence $\{x_n\}$ in X_w (or X_w^*) is said to be w-convergent to $x \in X$ if and only if $w_\lambda(x_n, x) \to 0$ as $n \to \infty$ for some $\lambda > 0$. Then x is called the modular limit of $\{x_n\}$.*
- *The sequence $\{x_n\}$ in X_w is said to be w-Cauchy if $w_\lambda(x_n, x_m) \to 0$ as $m, n \to \infty$ for some $\lambda > 0$.*
- *A subset M of X_w or X_w^* is said to be w-complete if any w-Cauchy sequence in M is an w-convergent sequence and its w-limit is in M.*

In [8], it is shown that, if w is a pseudomodular on X, the modular metric spaces X_w and X_w^* are closed with respect to w-convergence. In addition, if w is strict, then the modular limit is unique if it exists.

Remark 2.4. *According to the definition of convergence and properties of metric modular, it is easy to see that if $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$ for some $\lambda > 0$, then $\lim_{n\to\infty} w_\mu(x_n, x) = 0$ for all $\mu > \lambda > 0$.*

3. Fixed point theorems in complete modular metric spaces

In this section, we state and prove fixed point theorems for two types of mappings on modular metric spaces. The first type is an analog of the non-expansive mapping discussed by Bogin on metric spaces [3] and the second type is a contraction mapping with a similar structure. These theorems have various consequences given as corollaries.

First, we state the definition of modular contractive mappings given in [8].

Definition 3.1. *Let X be a nonempty set and w be a metric modular on X.*

(i) A map $T : X_w^* \longrightarrow X_w^*$ is said to be w-contractive provided that there exist $0 < k < 1$ and $\lambda_0 > 0$ depending on k such that

$$w_{k\lambda}(Tx, Ty) \leq w_\lambda(x, y)$$

for all $0 < \lambda < \lambda_0$ and $x, y \in X_w^*$.

(ii) A map $T : X_w^* \longrightarrow X_w^*$ is said to be strong w-contractive provided that there exist $0 < k < 1$ and $\lambda_0 > 0$ depending on k such that

$$w_{k\lambda}(Tx, Ty) \leq k w_\lambda(x, y)$$

for all $0 < \lambda < \lambda_0$ and $x, y \in X_w^*$.

The following fixed point theorems have been proved in [8].

Theorem 3.2. *Let X be a nonempty set and w be a strict convex metric modular on X. Let X_w^* be a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ be a w-contractive self mapping.*

If for every $\lambda > 0$ there exists an $x = x(\lambda) \in X_w^*$ such that $w_\lambda(x, Tx) < \infty$ then T has a fixed point in X_w^*.

If in addition $w_\lambda(x, y) < \infty$ for all $x, y \in X_w^*$ and every $\lambda > 0$, then the fixed point of T is unique.

Theorem 3.3. *Let X be a nonempty set and w be a strict metric modular on X. Let X_w^* be a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ be a strong w-contractive self mapping.*

If for every $\lambda > 0$ there exists an $x = x(\lambda) \in X_w^*$ such that $w_\lambda(x, Tx) < \infty$ then T has a fixed point in X_w^*.

If in addition $w_\lambda(x, y) < \infty$ for all $x, y \in X_w^*$ and every $\lambda > 0$, then the fixed point of T is unique.

We start with a variant of the definitions of w-contraction and strong w-contraction.

Definition 3.4. *Let w be a metric modular on X.*

(i) A map $T : X_w^* \to X_w^*$ is said to be a **Bogin-type w-contraction**, *if there exist $0 < k < 1$ and λ_0 depending on k, such that*

$$w_{k\lambda}(Tx, Ty) \le a w_\lambda(x, y) + b[w_{2\lambda}(x, Tx) + w_{2\lambda}(y, Ty)] + c[w_{2\lambda}(x, Ty) + w_{2\lambda}(y, Tx)] \tag{6}$$

holds for all $0 < \lambda < \lambda_0$, $x, y \in X_w^*$ and $a, b, c \ge 0$ with $a + 2b + 2c = 1$.

(ii) A map $T : X_w^* \to X_w^*$ is said to be a **strong Bogin-type w-contraction**, *if there exist $0 < k < 1$, λ_0 depending on k and $a, b, c \ge 0$ satisfying $a + 4b + 4c = k < 1$, such that the inequality (6) holds for all $0 < \lambda < \lambda_0$, $x, y \in X_w^*$.*

In the following, we prove an auxiliary result needed in the proofs of the fixed point theorems.

Lemma 3.5. *Let X be a nonempty set, w be a metric modular on X and $T : X_w^* \to X_w^*$ be a Bogin-type or a strong Bogin-type w-contractive map. Suppose that for every $\lambda > 0$, there exists an $x_0 \in X_w^*$ such that $w_\lambda(x_0, Tx_0) < \infty$. Then, the sequence $\{x_n\} := \{T^n x_0\}$ satisfies*

$$w_{k\lambda}(x_{n+1}, x_{n+2}) \le w_\lambda(x_n, x_{n+1}), \tag{7}$$

for all $\lambda < \lambda_0$.

Proof. Starting with $x_0 \in X_w^*$ with $w_\lambda(x_0, Tx_0) < \infty$, we construct the sequence $\{x_n\} \in X_w^*$ as $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Note that for all $x, y \in X_w^*$ the relation (5) holds. Then the inequality (6) with $x = x_n$ and $y = x_{n+1}$ yields

$$\begin{aligned}
w_{k\lambda}(Tx_n, Tx_{n+1}) &= w_{k\lambda}(x_{n+1}, x_{n+2}) \\
&\le a w_\lambda(x_n, x_{n+1}) + b[w_{2\lambda}(x_n, Tx_n) + w_{2\lambda}(x_{n+1}, Tx_{n+1})] \\
&+ c[w_{2\lambda}(x_n, Tx_{n+1}) + w_{2\lambda}(x_{n+1}, Tx_n)] \\
&= a w_\lambda(x_n, x_{n+1}) + b[w_{2\lambda}(x_n, x_{n+1}) + w_{2\lambda}(x_{n+1}, x_{n+2})] \\
&+ c[w_{2\lambda}(x_n, x_{n+2}) + w_{2\lambda}(x_{n+1}, x_{n+1})] \\
&\le a w_\lambda(x_n, x_{n+1}) + b[w_\lambda(x_n, x_{n+1}) + w_\lambda(x_{n+1}, x_{n+2})] \\
&+ c[w_\lambda(x_n, x_{n+1}) + w_\lambda(x_{n+1}, x_{n+2})],
\end{aligned} \tag{8}$$

for all $\lambda < \lambda_0$. Taking into account that $k\lambda < \lambda$, we obtain

$$\begin{aligned}
w_{k\lambda}(x_{n+1}, x_{n+2}) &\le (a + b + c) w_\lambda(x_n, x_{n+1}) + (b + c) w_\lambda(x_{n+1}, x_{n+2}) \\
&\le (a + b + c) w_\lambda(x_n, x_{n+1}) + (b + c) w_{k\lambda}(x_{n+1}, x_{n+2}),
\end{aligned}$$

and conclude

$$w_{k\lambda}(x_{n+1}, x_{n+2}) \le \frac{a + b + c}{1 - b - c} w_\lambda(x_n, x_{n+1}). \tag{9}$$

If T is Bogin or strong Bogin type w-contraction, then $a + 2b + 2c \leq 1$, so that $\dfrac{a+b+c}{1-b-c} \leq 1$. Hence, the inequality (9) becomes

$$w_{k\lambda}(x_{n+1}, x_{n+2}) \leq w_\lambda(x_n, x_{n+1}), \tag{10}$$

which completes the proof. □

Our main theorem is an analog of the fixed point theorem by Bogin [3] in the framework of modular metric spaces.

Theorem 3.6. *Let w be a strict convex metric modular on X and X_w^* be a w-complete modular metric space induced by w. Assume that $T : X_w^* \to X_w^*$ is a Bogin-type w-contractive self mapping.*

If for every $\lambda > 0$ there exists an $x \in X_w^*$ satisfying $w_\lambda(x, Tx) < \infty$ then the mapping T has a fixed point in X_w^*.

If in addition, $w_\lambda(x, y) < \infty$ for all $x, y \in X_w^*$, $\lambda > 0$ then the fixed point of T is unique.

Proof. Let x_0 be any element in X_w^* satisfying $w_\lambda(x_0, Tx_0) < \infty$. Define the sequence $\{x_n\} \in X_w^*$ as $x_n = T^n x_0$ for all $n \in \mathbb{N}$ and assume that $w_\lambda(x_n, x_{n+1}) > 0$ for all $\lambda > 0$ and $n \in \mathbb{N}$. Indeed, if $w_{\lambda_f}(x_{n_0}, x_{n_0+1}) = 0$ for some $\lambda_f > 0$ and $n_0 \in \mathbb{N}$, then x_{n_0} would be a fixed point of T.

Regarding Lemma 3.5, the sequence $\{x_n\}$ satisfies

$$w_{k\lambda}(x_{n+1}, x_{n+2}) \leq w_\lambda(x_n, x_{n+1}) \tag{11}$$

for all $0 < \lambda < \lambda_0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will show that the sequence $\{x_n\}$ is w-Cauchy.

First, we observe that since w is convex, then for any positive integers m, n with $n > m$ we have

$$\begin{aligned} w_{\lambda_c}(x_m, x_n) &\leq \frac{\lambda_m}{\lambda_c} w_{\lambda_m}(x_m, x_{m+1}) + \frac{\lambda_{m+1}}{\lambda_c} w_{\lambda_{m+1}}(x_{m+1}, x_{m+2}) \\ &+ \cdots + \frac{\lambda_{n-1}}{\lambda_c} w_{\lambda_{n-1}}(x_{n-1}, x_n), \end{aligned} \tag{12}$$

where $\lambda_c = \lambda_m + \lambda_{m+1} + \cdots + \lambda_{n-1}$. On the other hand, due to the fact that $k^n\lambda < \lambda < \lambda_0$, the inequality (11) gives

$$w_{k^{n+1}\lambda}(x_{n+1}, x_{n+2}) = w_{k(k^n\lambda)}(x_{n+1}, x_{n+2}) \leq w_{k^n\lambda}(x_n, x_{n+1}),$$

or, inductively,

$$w_{k^{n+1}\lambda}(x_{n+1}, x_{n+2}) \leq w_\lambda(x_0, x_1) < \infty \tag{13}$$

for all $n \in \mathbb{N}_0$ and $0 < \lambda < \lambda_0$. Let $\lambda_1 = (1-k)\lambda_0 < \lambda_0$. We have then

$$w_{k^n\lambda_1}(x_n, x_{n+1}) \leq w_{\lambda_1}(x_0, x_1) < \infty \tag{14}$$

for all $n \in \mathbb{N}_0$. Choose $\lambda_l = k^l\lambda_1$ for $l = m, m+1, \ldots, n-1$ in (12) and take into account (14). This results in

$$\begin{aligned} w_{\lambda_c}(x_m, x_n) &\leq \frac{k^m\lambda_1}{\lambda_c} w_{k^m\lambda_1}(x_m, x_{m+1}) + \frac{k^{m+1}\lambda_1}{\lambda_c} w_{k^{m+1}\lambda_1}(x_{m+1}, x_{m+2}) \\ &+ \cdots + \frac{k^{n-1}\lambda_1}{\lambda_c} w_{k^{n-1}\lambda_1}(x_{n-1}, x_n) \\ &= \frac{\lambda_1}{\lambda_c} \sum_{l=m}^{n-1} k^l w_{k^l\lambda_1}(x_l, x_{l+1}) \\ &\leq \frac{\lambda_1}{\lambda_c} w_{\lambda_1}(x_0, x_1) \sum_{l=m}^{n-1} k^l \\ &= \frac{\lambda_1}{\lambda_c} w_{\lambda_1}(x_0, x_1) k^m \frac{1-k^{n-m}}{1-k}, \end{aligned} \tag{15}$$

where $\lambda_c = \sum_{l=m}^{n-1} k^l \lambda_1 = k^m \lambda_1 \frac{1-k^{n-m}}{1-k}$. Since $\lambda_1 = (1-k)\lambda_0$, then $\lambda_0 = \frac{\lambda_1}{1-k}$ and we have $\lambda_c = k^m(1-k^{n-m})\lambda_0 < \lambda_0$. Hence, the inequalities (4) and (15) imply

$$0 \le w_{\lambda_0}(x_m, x_n) \le \frac{\lambda_c}{\lambda_0} w_{\lambda_c}(x_m, x_n) \le \frac{\lambda_1}{\lambda_0} w_{\lambda_1}(x_0, x_1) k^m \frac{1-k^{n-m}}{1-k} \le k^m w_{\lambda_1}(x_0, x_1).$$

As a result we have

$$\lim_{m\to\infty} w_{\lambda_0}(x_m, x_n) = 0.$$

Therefore $\{x_n\}$ is a w-Cauchy sequence in X_w^*. Since X_w^* is w-complete by the assumption, then the sequence $\{x_n\}$ is w-convergent to some $x \in X_w^*$, that is,

$$\lim_{n\to\infty} w_{\lambda_0}(x_n, x) = 0, \tag{16}$$

and since w is a strict modular, the limit is unique.

Now, we will show that $Tx = x$ holds. Using the third condition of metric modular and noting that $Tx_n = x_{n+1}$, we have

$$w_{(k+1)\lambda_0}(Tx, x) \le w_{k\lambda_0}(Tx, Tx_n) + w_{\lambda_0}(x_{n+1}, x). \tag{17}$$

Since w is of Bogin-type w-contractive, we have

$$\begin{aligned} w_{k\lambda_0}(Tx, Tx_n) &\le a w_{\lambda_0}(x, x_n) + b[w_{2\lambda_0}(x, Tx) + w_{2\lambda_0}(x_n, x_{n+1})] \\ &\quad + c[w_{2\lambda_0}(x, x_{n+1}) + w_{2\lambda_0}(x_n, Tx)] \\ &\le a w_{\lambda_0}(x, x_n) \\ &\quad + (b+c)[w_{\lambda_0}(x, x_{n+1}) + w_{\lambda_0}(x_n, x_{n+1}) + w_{\lambda_0}(Tx, Tx_{n+1})] \\ &\le a w_{\lambda_0}(x, x_n) \\ &\quad + (b+c)[w_{\lambda_0}(x, x_{n+1}) + w_{\lambda_0}(x_n, x_{n+1}) + w_{k\lambda_0}(Tx, Tx_{n+1})] \end{aligned}$$

which gives

$$\begin{aligned} (1-b-c) w_{k\lambda_0}(Tx, Tx_n) &\le a w_{\lambda_0}(x, x_n) \\ &\quad + (b+c)[w_{\lambda_0}(x, x_{n+1}) + w_{\lambda_0}(x_n, x_{n+1})]. \end{aligned}$$

Taking limit of both sides as $n \to \infty$, we obtain

$$\lim_{n\to\infty} w_{k\lambda_0}(Tx_n, Tx) = 0. \tag{18}$$

Thus we get $w_{(k+1)\lambda_0}(Tx, x) = 0$. Since w is strict, $Tx = x$ holds, i.e., x is a fixed point of T.

To show the uniqueness, we assume that x and y are two fixed points of T, i.e., $x, y \in X_w^*$ for which $Tx = x$ and $Ty = y$ hold. Since w is convex, we have

$$\begin{aligned} w_{\lambda_0}(x, y) \le \tfrac{k\lambda_0}{\lambda_0} w_{k\lambda_0}(x, y) &= k w_{k\lambda_0}(Tx, Ty) \\ &\le ka w_{2\lambda_0}(x, y) + kb[w_{2\lambda_0}(x, Tx) + w_{2\lambda_0}(y, Ty)] \\ &\quad + kc[w_{2\lambda_0}(x, Ty) + w_{2\lambda}(y, Tx)] \\ &\le ka w_{\lambda_0}(x, y) + kb[w_{\lambda_0}(x, x) + w_{\lambda_0}(y, y)] \\ &\quad + kc[w_{\lambda_0}(x, y) + w_{\lambda_0}(y, x)] \\ &= k(a+2c) w_{\lambda_0}(x, y). \end{aligned} \tag{19}$$

By the assumption $w_\lambda(x, y) < \infty$, we get $(1 - k(a+2c)) w_{\lambda_0}(x, y) \le 0$. Due to the fact that $a + 2c \le a + 2b + 2c = 1$ and $0 < k < 1$, $w_{\lambda_0}(x, y) = 0$ holds. Since w is strict, we have $x = y$. This completes the proof of the theorem. □

We next prove another fixed point theorem in which the condition on the constants a, b, c is changed so that the mapping T becomes strong Bogin-type w-contractive.

Theorem 3.7. *Let w be a strict metric modular on X and X_w^* be a w-complete modular metric space induced by w. Assume that $T : X_w^* \to X_w^*$ is a strong Bogin-type w-contractive self mapping.*

If for every $\lambda > 0$ there exists an $x \in X_w^*$ satisfying $w_\lambda(x, Tx) < \infty$ then the mapping T has a fixed point in X_w^*.

If in addition, $w_\lambda(x, y) < \infty$ for all $x, y \in X_w^*$, $\lambda > 0$ then the fixed point of T is unique.

Proof. Let $v_\lambda(x, y) = \dfrac{w_\lambda(x, y)}{\lambda}$ for all $\lambda > 0$ and $x, y \in X$. Then v is a strict convex metric modular and the modular metric space $X_v^* = X_w^*$ is v-complete. Since T is a strong Bogin-type w-contractive self mapping, (6) implies

$$\begin{aligned} k\lambda v_{k\lambda}(Tx, Ty) &\leq a\lambda v_\lambda(x, y) \\ &\quad +2b\lambda[v_{2\lambda}(x, Tx) + v_{2\lambda}(y, Ty)] + 2\lambda c[v_{2\lambda}(x, Ty) + v_{2\lambda}(y, Tx)] \\ &= \lambda[av_\lambda(x, y) \\ &\quad +2b(v_{2\lambda}(x, Tx) + v_{2\lambda}(y, Ty)) + 2c(v_{2\lambda}(x, Ty) + v_{2\lambda}(y, Tx))] \end{aligned}$$

which gives

$$v_{k\lambda}(Tx, Ty) \leq \frac{a}{k}v_\lambda(x, y) + \frac{2b}{k}[v_{2\lambda}(x, Tx) + v_{2\lambda}(y, Ty)] + \frac{2c}{k}[v_{2\lambda}(x, Ty) + v_{2\lambda}(y, Tx)].$$

Since $k = a + 4b + 4c$, the condition (6) holds for the strict convex metric modular v and T becomes a Bogin-type v-contractive self mapping on X_v^*. Applying Theorem 3.6 for v, we get the result. □

Theorem 3.7 has various consequences. Observe that for special choices of the constants a, b, c in Theorem 3.7 we obtain w-contraction mappings of Kannan, Chatterjee, Reich and Banach types. Below we discuss these contractions.

For $a = 0$ and $c = 0$ in (6) the mapping T becomes the so-called Kannan type contraction mapping. A different version of this mapping is studied in [12].

Corollary 3.8. *Let X be a nonempty set and w be a strict metric modular on X. Let X_w^* be a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ be a self mapping of* **Kannan type**, *that is, for some $0 < k < 1$, there exists $\lambda_0 > 0$ such that the inequality*

$$w_{k\lambda}(Tx, Ty) \leq \frac{k}{4}[w_{2\lambda}(x, Tx) + w_{2\lambda}(y, Ty)] \tag{20}$$

holds for all $x, y \in X_w^*$ and all $0 < \lambda < \lambda_0$. If for every $\lambda > 0$ there exists an $x \in X_w^*$ satisfying $w_\lambda(x, Tx) < \infty$ then the mapping T has a fixed point in X_w^*. If in addition $w_\lambda(x, y) < \infty$ for all $x, y \in X_w^*$ and every $\lambda > 0$, then the fixed point of T is unique.

Also, by taking $a = 0$ and $b = 0$ in (6) we obtain the so-called Chatterjee type contractive mapping.

Corollary 3.9. *Let X be a nonempty set and w be a strict metric modular on X. Let X_w^* be a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ be a self mapping of* **Chatterjee type**, *that is, for some $0 < k < 1$, there exists $\lambda_0 > 0$ such that the inequality*

$$w_{k\lambda}(Tx, Ty) \leq \frac{k}{4}[w_{2\lambda}(x, Ty) + w_{2\lambda}(y, Tx)] \tag{21}$$

holds for all $x, y \in X_w^*$ and all $0 < \lambda < \lambda_0$. If for every $\lambda > 0$ there exists an $x \in X_w^*$ satisfying $w_\lambda(x, Tx) < \infty$ then the mapping T has a fixed point in X_w^*. If in addition $w_\lambda(x, y) < \infty$ for all $x, y \in X_w^*$ and every $\lambda > 0$, then the fixed point of T is unique.

Another type of contraction known as Reich type contraction follows from (6) with $c = 0$.

Corollary 3.10. *Let X be a nonempty set and w be a strict metric modular on X. Let X_w^* be a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ be a self mapping of* **Reich type**, *that is, for some $0 < k < 1$, there exists $\lambda_0 > 0$ such that the inequality*

$$w_{k\lambda}(Tx, Ty) \leq \frac{k}{5}[w_\lambda(x, y) + w_{2\lambda}(x, Tx) + w_{2\lambda}(y, Ty)] \tag{22}$$

holds for all $x, y \in X_w^*$ and all $0 < \lambda < \lambda_0$. If for every $\lambda > 0$ there exists an $x \in X_w^*$ satisfying $w_\lambda(x, Tx) < \infty$ then the mapping T has a fixed point in X_w^*. If for every $\lambda > 0$ and all fixed points x, y of T we have $w_\lambda(x, y) < \infty$ then the fixed point of T is unique.

Remark 3.11. *The special choices of a, b and c for $a = 1$, $b = c = 0$ and $a < 1$, $b = c = 0$ correspond to parts (i) and (ii) of Definition 3.1, on which the self mapping T may be called as* **nonexpansive** *and* **Banach type** *mappings, respectively. In the corresponding cases, Theorem 3.2 and Theorem 3.3 become consequences of our main results given in Theorem 3.6 and Theorem 3.7, respectively.*

4. A metric modular space of anti-periodic mappings of bounded generalized φ-variations

In this section, inspired by the examples in [8] (Section 4), we introduce a metric modular space of real valued mappings defined on a finite interval and satisfying anti-periodic boundary conditions. This space is going to be used in the application of the main results to anti-periodic boundary value problems for nonlinear first order differential equations of Carathéodory's type discussed in the next section.

We define a metric modular space of anti-periodic mappings of bounded generalized φ-variations in few steps as follows.

Step 1.

Let $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a given continuous, convex, nondecreasing and unbounded function satisfying $\varphi(x) = 0$ iff $x = 0$. Let $\tilde{X}$ be the set of real valued functions on $[0, L]$ for some $L > 0$, that is,

$$\tilde{X} := \{u | u : [0, L] \to \mathbb{R}\}.$$

Define $w : (0, \infty) \times \tilde{X} \times \tilde{X} \to [0, \infty]$ for all $\lambda > 0$ and $u, v \in X$ as

$$w_\lambda(u, v) = \sup_\pi \sum_{i=1}^{n} \varphi\left(\frac{|[u(t_i) + v(t_{i-1})] - [u(t_{i-1}) + v(t_i)]|}{\lambda(t_i - t_{i-1})}\right)(t_i - t_{i-1}), \tag{23}$$

where the supremum is taken over all partitions $\pi = \{t_i\}_{i=0}^n$ of the interval $[0, L]$, that is, $0 = t_0 < t_1 < t_2 < \cdots < t_n = L$. Then, it is known that $w_\lambda(u, v)$ is a convex pseudomodular on $\tilde{X}$ (see [7, 8]).

Step 2.

Let $u_0(t) = u_0 \in \tilde{X}$ be a constant function. Define the convex pseudomodular metric space $\tilde{X}_w^*$ as

$$\tilde{X}_w^* = \tilde{X}_w^*(u_0) = \{u \in \tilde{X} : \exists\lambda = \lambda(u) > 0, \text{ such that } w_\lambda(u, u_0) < \infty\}.$$

The space $\tilde{X}_w^*$ is denoted by $GV_\varphi([0, L])$ and is called the **space of mappings of bounded generalized φ-variations**, see [6]. Then $u \in \tilde{X}_w^* = GV_\varphi([0, L])$ if and only if $u : [0, L] \to \mathbb{R}$ and there exists a constant $\lambda = \lambda(u) > 0$ such that

$$w_\lambda(u, u_0) = \sup_\pi \sum_{i=1}^{n} \varphi\left(\frac{|u(t_i) - u(t_{i-1})|}{\lambda(t_i - t_{i-1})}\right)(t_i - t_{i-1}) < \infty.$$

Clearly, $w_\lambda(u, u_0)$ is independent of u_0.

Step 3.

Let X be the set of anti-periodic real valued functions on $[0, L]$ for some $L > 0$, that is,

$$X := \{u | u : [0, L] \to \mathbb{R}, u(0) = -u(L)\} \subset \tilde{X}. \tag{24}$$

Then w_λ defined in (23) is a convex pseudomodular on X.

Lemma 4.1. *The mapping w_λ defined in* (23) *is a strict convex metric modular on X.*

Proof. We need to show that

$$w_\lambda(u,v) = 0 \Longrightarrow u(t) = v(t) \text{ for all } u,v \in X \text{ and } t \in [0,L].$$

It is obvious that for any $s,t \in [0,L]$, with $s \neq t$,

$$\varphi\left(\frac{|[u(t)+v(s)]-[u(s)+v(t)]|}{\lambda|t-s|}\right)|t-s| \leq w_\lambda(u,v).$$

Since φ has an inverse, then

$$|u(t)-v(t)+v(s)-u(s)| \leq \lambda|t-s|\varphi^{-1}\left(\frac{w_\lambda(u,v)}{|t-s|}\right).$$

Let $w_\lambda(u,v)=0$. Then, regarding the properties of φ we have

$$|u(t)-v(t)+v(s)-u(s)| \leq \lambda|t-s|\varphi^{-1}(0) = 0,$$

which implies that

$$u(t)-v(t) = u(s)-v(s). \tag{25}$$

Solving the system resulting by taking $s=0$ and $s=L$ in (25) and using the anti-periodicity property of the functions u and v we obtain $u(t)-v(t)=0$ for any $t \in [0,L]$. □

Step 4.
Now we define

$$X_w^* = \tilde{X}_w^* \cap X = GV_\varphi([0,L]) \cap X = \{u \in GV_\varphi([0,L]) | u(0) = -u(L)\}. \tag{26}$$

Then, X_w^* is a metric modular space. We will show that X_w^* is w-complete.

Lemma 4.2. *The metric modular space*

$$X_w^* = \{u \in X : \exists\lambda = \lambda(u) > 0, \text{ such that } w_\lambda(u,u_0) < \infty\}$$

is w-complete.

Proof. Let $\{u_n\} \subset X_w^*$ be a w-Cauchy sequence. Then

$$w_\lambda(u_n,u_m) \to 0 \text{ as } m,n \to \infty.$$

for some $\lambda = \lambda(\{u_n\}) > 0$. Therefore, for $n,m \in \mathbb{N}$, $t \in (0,L)$ we have

$$\begin{aligned} 2|u_n(t)-u_m(t)| &= |2[u_n(t)-u_m(t)] - u_n(0) + u_m(0) - u_n(L) + u_m(L)| \\ &\leq |[u_n(t)+u_m(0)] - [u_n(0)+u_m(t)]| \\ &\quad + |[u_n(t)+u_m(L)] - [u_n(L)+u_m(t)]| \\ &\leq \lambda|t-0|\varphi^{-1}\left(\frac{w_\lambda(u_n,u_m)}{|t-0|}\right) + \lambda|L-t|\varphi^{-1}\left(\frac{w_\lambda(u_n,u_m)}{|L-t|}\right) \end{aligned}$$

which implies that $\lim\limits_{n,m\to\infty} |u_n(t)-u_m(t)| = 0$. Since $\mathbb{R}$ is complete then, the sequence $\{u_n\}$ converges to some $u:[0,L] \to \mathbb{R}$ with $u(0) = -u(L)$, that is, for all $t \in [0,L]$, $\lim_{n\to\infty} |u_n(t)-u(t)| = 0$ holds for $u \in X$.

It remains to show that $\lim_{n\to\infty} w_\lambda(u_n(t), u(t)) = 0$. From the lower semicontinuity of w_λ (see [7], p.27) we have

$$w_\lambda(u_n, u) \le \liminf_{m\to\infty} w_\lambda(u_n, u_m)$$

for every $n \in \mathbb{N}$. Since $\{u_n\}$ is w-Cauchy,

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} \text{ such that } \forall n, m \ge N(\varepsilon) \implies w_\lambda(u_n, u_m) < \varepsilon.$$

Hence, for all $n \ge N(\varepsilon)$

$$\limsup_{m\to\infty} w_\lambda(u_n, u_m) \le \sup_{m \ge N(\varepsilon)} w_\lambda(u_n, u_m) < \varepsilon.$$

Then we conclude that for every $\varepsilon > 0$, $\exists\, N(\varepsilon) \in \mathbb{N}$ such that

$$w_\lambda(u_n, u) \le \liminf_{m\to\infty} w_\lambda(u_n, u_m) \le \limsup_{m\to\infty} w_\lambda(u_n, u_m) < \varepsilon.$$

Then, $\{u_n\}$ is w-convergent to u. Since X_w^* is closed under the modular convergence, then $u \in X_w^*$, that is X_w^* is w-complete. $\square$

Step 5.

If in addition, the function φ satisfies the *Orlicz condition at infinity*, that is, $\dfrac{\varphi(y)}{y} \to \infty$ as $y \to \infty$, then $w_1(u, 0)$ is called the φ-variation of the function $u : [0, L] \longrightarrow \mathbb{R}$; the function u with $w_1(u, 0) < \infty$ is said to be of bounded φ-variation on $[0, L]$ and

$$w_\lambda(u, v) = w_\lambda(u - v, 0) = w_1\left(\frac{u - v}{\lambda}, 0\right), \ \ \lambda > 0.$$

For the functions $u : [0, L] \longrightarrow \mathbb{R}$ in the space $\tilde{X}_w^* = GV_\varphi([0, L])$ it is known that (see [8, 9]),

$$u \in GV_\varphi([0, L]) \Leftrightarrow w_\lambda(u, 0) = w_1(u/\lambda, 0) < \infty \; for\ some\ \lambda = \lambda(u) > 0$$

$$\Leftrightarrow u \in AC[0, L], u' \in L^1[0, L] \; with\ w_\lambda(u, 0) = \int_0^L \varphi\left(\frac{|u'(t)|}{\lambda}\right) dt < \infty,$$

where $AC[0, L]$ is the space of all absolutely continuous real valued functions on $[0, L]$ and $L^1[0, L]$ is the space of Lebesgue integrable functions on $[0, L]$.

5. Anti-periodic boundary value problem for nonlinear first order Carathéodory's type ordinary differential equations

In this section, following the notations of Section 4, we will apply the fixed point result given in Theorem 3.6 to the following anti-periodic boundary value problem for Carathéodory's type nonlinear first order ordinary differential equations:

$$\begin{cases} u'(t) & = & f(t, u(t)), \quad a.e. \ t \in [0, L] \\ u(0) & = & -u(L). \end{cases} \tag{27}$$

Here $f : [0, L] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory's type function which satisfies the following conditions:

C1. For every $u \in \mathbb{R}$ $f(., u)$ is Lebesgue measurable on $[0, L]$ and there exists a point $v_0 \in \mathbb{R}$ such that $\int_0^L \varphi\left(\frac{|f(t,v_0)|}{\lambda}\right) dt < \infty$ for some $\lambda = \lambda(f(., v_0)) > 0$,

C2. For a.e. $t \in [0, L]$, and all $u, v \in \mathbb{R}$, there exists a constant $K > 0$ such that

$$|f(t, u) - f(t, v)| \le K|u - v|.$$

where φ is a function satisfying the Orlicz condition at infinity and $\lambda > 0$.

For $\mu \in \mathbb{R}$, problem (27) can be written as

$$\begin{cases} u'(t) + \mu u(t) &= f(t, u(t)) + \mu u(t), \quad t \in [0, L] \\ u(0) &= -u(L) \end{cases} \tag{28}$$

which is equivalent to the integral equation

$$u(t) = \int_0^L G(t, s)[f(s, u(s)) + \mu u(s)]ds, \tag{29}$$

see [11], where $G(t, s)$ is the Green function defined by

$$G(t, s) = \begin{cases} \dfrac{e^{\mu(L+s-t)}}{e^{\mu L} + 1}, & 0 \le s \le t \le L, \\ \dfrac{-e^{\mu(s-t)}}{e^{\mu L} + 1}, & 0 \le t < s \le L. \end{cases} \tag{30}$$

A function $u : [0, L] \to \mathbb{R}$ is a solution of (27) if $u \in GV_\phi[0, L]$ and satisfies (27) or, equivalently (28).

Let X^*_w be the modular metric space (26) generated by the metric modular (23). We will discuss the existence of solution $u(t)$ of (27) in the space X^*_w.

For the case of $\mu = 1$, consider the integral operator

$$Fu(t) = \int_0^L G(t, s)[f(s, u(s)) + u(s)]ds, \tag{31}$$

where $u \in X^*_w$, $t \in [0, L]$ and $G(t, s)$ is the corresponding Green function given in (30). Observe that, if $u \in X^*_w$ is a fixed point of F, then it becomes a solution of (27). We next prove that the operator F has a fixed point by following the proof technique used by Chistyakov in [8, 9] for an initial value problem associated with a differential equation of a similar type.

Theorem 5.1. *If the function f satisfies the conditions (C1) and (C2), then the operator F given by (31) maps X^*_w into itself and*

$$w_{M\lambda}(Fu, Fv) \le w_\lambda(u, v) \tag{32}$$

where $M = (K + 1)(L + 1)L$, for all $u, v \in X^*_w$, $\lambda > 0$.

Proof. Let $u \in X^*_w$, that is, $u \in GV_\varphi([0, L])$ and $u(0) = -u(L)$. Now, $Fu(0) = -Fu(L)$ and thus $Fu \in X = \{v | v : [0, L] \to \mathbb{R}, v(0) = -v(L)\}$. Now, we will show that $Fu \in GV_\varphi([0, L])$, i.e.,

$$w_\lambda(Fu, 0) = \int_0^L \varphi\left(\frac{|(Fu)'(t)|}{\lambda}\right)dt < \infty$$

holds, which will imply $Fu \in X^*_w$.

Observe that, the condition (C2) implies

$$\begin{aligned} |f(t, u(t))| = |f(t, u(t)) - f(t, v_0) + f(t, v_0)| &\le |f(t, u(t)) - f(t, v_0)| + |f(t, v_0)| \\ &\le K|u(t) - v_0| + |f(t, v_0)|. \end{aligned}$$

Since $u(0) = -u(L)$, we have $u(t) = -u(L) + \int_0^t u'(s)ds$, for a.e. $t \in [0, L]$ and thus we get

$$|f(t, u(t))| \le K \int_0^L |u'(s)|ds + K|u(L) + v_0| + |f(t, v_0)|.$$

Notice that, since $u \in X_w^*$, i.e., $u \in GV_\varphi([0,L])$, there exists a constant $\lambda_1 = \lambda_1(u) > 0$ such that

$$w_{\lambda_1}(u,0) = \int_0^L \varphi\left(\frac{|u'(t)|}{\lambda_1}\right) dt < \infty \tag{33}$$

and by (C1) there exists another constant $\lambda_2 = \lambda_2(f(.,v_0)) > 0$ such that

$$\int_0^L \varphi\left(\frac{|f(t,v_0)|}{\lambda_2}\right) dt < \infty. \tag{34}$$

Since φ is a convex function, for $\lambda_0 = KL\lambda_1 + \lambda_2 + 1$ so that $\frac{KL\lambda_1}{\lambda_0} + \frac{\lambda_2}{\lambda_0} + \frac{1}{\lambda_0} = 1$, we obtain

$$\begin{aligned} \varphi\left(\frac{1}{\lambda_0}\left[K\int_0^L |u'(t)|dt + K|u(L)+v_0| + |f(t,v_0)|\right]\right) &= \varphi\left(\frac{KL\lambda_1}{\lambda_0}\left(\frac{1}{L}\int_0^L \frac{|u'(t)|}{\lambda_1}dt\right) + \frac{1}{\lambda_0}K|u(L)+v_0| + \frac{\lambda_2}{\lambda_0}\frac{|f(t,v_0)|}{\lambda_2}\right) \\ &\leq \frac{KL\lambda_1}{\lambda_0}\varphi\left(\frac{1}{L}\int_0^L \frac{|u'(t)|}{\lambda_1}dt\right) \\ &\quad + \frac{1}{\lambda_0}\varphi(K|u(L)+v_0|) + \frac{\lambda_2}{\lambda_0}\varphi\left(\frac{|f(t,v_0)|}{\lambda_2}\right). \end{aligned}$$

Monotonicity of φ, Jensen's integral inequality and estimates (33) and (34) give,

$$\begin{aligned} \int_0^L \varphi\left(\frac{|f(t,u(t))|}{\lambda_0}\right)dt &\leq \int_0^L \varphi\left(\frac{1}{\lambda_0}\left[K\int_0^L |u'(t)|dt + K|u(L)+v_0| + |f(t,v_0)|\right]\right)dt \\ &\leq \frac{KL\lambda_1}{\lambda_0}\int_0^L \varphi\left(\frac{|u'(t)|}{\lambda_1}\right)dt + \frac{L}{\lambda_0}\varphi(K|u(L)+v_0|) \\ &\quad + \frac{\lambda_2}{\lambda_0}\int_0^L \varphi\left(\frac{|f(t,v_0)|}{\lambda_2}\right)dt \\ &:= C_1 < \infty. \end{aligned}$$

Jensen's inequality implies,

$$\varphi\left(\frac{1}{L\lambda_0}\int_0^L |f(t,u(t))|dt\right) \leq \frac{1}{L}\int_0^L \varphi\left(\frac{|f(t,u(t))|}{\lambda_0}\right)dt \leq \frac{C_1}{L}$$

which yields

$$\int_0^L |f(t,u(t))|dt \leq \lambda_0 L\varphi^{-1}\left(\frac{C_1}{L}\right) < \infty.$$

Therefore $f \in L^1[0,L]$. Moreover,

$$|Fu(t)| \leq \int_0^L |G(t,s)||f(s,u(s)) + u(s)|ds$$

and using the above arguments and the facts that, $|G(t,s)| \leq \dfrac{e^L}{e^L+1} \leq 1$ for $t,s \in [0,L]$ and $\displaystyle\int_0^L |G(t,s)|ds = \frac{e^L-1}{e^L+1} \leq 1$ gives

$$\begin{aligned} \int_0^L \varphi\left(\frac{|Fu(t)|}{\lambda_3}\right)dt &\leq \int_0^L \varphi\left(\frac{1}{\lambda_3}\left[\int_0^L |G(t,s)||f(s,u(s)) + u(s)|ds\right]\right) \\ &\leq \frac{L\lambda_0}{\lambda_3}\int_0^L \varphi\left(\frac{|f(t,u(t))|}{\lambda_0}\right)dt + \frac{L^2\lambda_1}{\lambda_3}\int_0^L \varphi\left(\frac{|u'(t)|}{\lambda_1}\right)dt + \frac{L^2}{\lambda_3}\varphi(|u(L)|) \\ &:= C_2 < \infty \end{aligned}$$

for $\lambda_3 = L(\lambda_0 + L\lambda_1 + 1)$.

The operator Fu defined in (31) can be written as

$$Fu(t) = e^{-t}\int_0^t e^s[f(s,u(s)) + u(s)]ds - \frac{e^{-L}}{1+e^{-L}}e^{-t}\int_0^L e^s[f(s,u(s)) + u(s)]ds$$

from which it can be seen that the operator $Fu \in AC[0,L]$ for all $u \in X_w^*$, and $(Fu)'(t) = f(t,u(t)) + u(t) - Fu(t)$. Now we have

$$\begin{aligned}
w_\lambda(Fu,0) &= \int_0^L \varphi\left(\frac{|(Fu)'(t)|}{\lambda}\right)dt \\
&= \int_0^L \varphi\left(\frac{|f(t,u(t)) + u(t) - Fu(t)|}{\lambda}\right)dt && (35)\\
&\leq \frac{\lambda_0}{\lambda}\int_0^L \varphi\left(\frac{|f(t,u(t))|}{\lambda_0}\right)dt + \frac{L\lambda_1}{\lambda}\int_0^L \varphi\left(\frac{|u'(t)|}{\lambda_1}\right)dt + \frac{L}{\lambda}\varphi(|u(L)|) && (36)\\
&\quad + \frac{\lambda_3}{\lambda}\int_0^L \varphi\left(\frac{|Fu(t)|}{\lambda_3}\right) && (37)\\
&:= C < \infty && (38)
\end{aligned}$$

for $\lambda = L\lambda_1 + \lambda_0 + \lambda_3 + 1$. Thus, F maps X_w^* into itself.

Let $u, v \in X_w^*$ and $\lambda > 0$, then

$$\begin{aligned}
w_{(K+1)(L+1)L\lambda}(Fu,Fv) &= w_{(K+1)(L+1)L\lambda}(Fu - Fv, 0) = \int_0^L \varphi\left(\frac{|(Fu-Fv)'(t)|}{(K+1)(L+1)L\lambda}\right)dt \\
&= \int_0^L \varphi\left(\frac{|f(t,u(t)) - f(t,v(t)) + (u-v) + (Fu-Fv)|}{(K+1)(L+1)L\lambda}\right)dt. && (39)
\end{aligned}$$

Observe that

$$\begin{aligned}
|f(t,u(t)) - f(t,v(t)) + (u-v) + (Fu-Fv)| &\leq (K+1)(L+1)|u(t) - v(t)| \\
&\leq (K+1)(L+1)\int_0^L |(u-v)'(s)|ds
\end{aligned}$$

holds for a.e. $t \in [0,L]$ by the condition (C2). Monotonicity of φ gives

$$\varphi\left(\frac{|f(t,u(t)) - f(t,v(t)) + (u-v) + (Fu-Fv)|}{(K+1)(L+1)L\lambda}\right) \leq \varphi\left(\frac{1}{L}\int_0^L \frac{|(u-v)'(s)|}{\lambda}ds\right)$$

and Jensen's inequality implies

$$\varphi\left(\frac{1}{L}\int_0^L \frac{|(u-v)'(s)|}{\lambda}ds\right) \leq \frac{1}{L}\int_0^L \varphi\left(\frac{|(u-v)'(s)|}{\lambda}\right)ds = \frac{1}{L}w_\lambda(u-v,0) = \frac{1}{L}w_\lambda(u,v).$$

Thus we get

$$\varphi\left(\frac{|f(t,u(t)) - f(t,v(t)) + (u-v) + (Fu-Fv)|}{(K+1)(L+1)L\lambda}\right)dt \leq \frac{1}{L}w_\lambda(u,v).$$

Therefore,

$$w_{(K+1)(L+1)L\lambda}(Fu,Fv) \leq w_\lambda(u,v).$$

□

Theorem 5.2. *If $(K+1)(L+1)L < 1$ holds, then the anti-periodic boundary value problem (27) has at least one solution in $GV_\varphi([0,L])$.*

Proof. The metric modular defined by (23) is a strict convex modular on $X = \{u | u : [0,L] \longrightarrow \mathbb{R}, u(0) = -u(L)\}$ by Lemma 4.1 and the modular space (26) is w-complete by Lemma 4.2. Theorem 5.1 implies that $F : X_w^* \to X_w^*$ is a self mapping satisfying the inequality $w_{(K+1)(L+1)L\lambda}(Fu, Fv) \leq w_\lambda(u,v)$. If $(K+1)(L+1)L < 1$ holds, then condition (6) is satisfied for $a = 1$, $b = c = 0$ in Theorem 3.6. Now, it remains to show that for every $\lambda > 0$ there exists $u \in X_w^*$ such that $w_\lambda(u, Fu) < \infty$. Clearly, for constant function $u_0 \in X_w^*$ we have

$$
\begin{aligned}
w_\lambda(u_0, Fu_0) = w_\lambda(Fu_0 - u_0, 0) &= \int_0^L \varphi\left(\frac{|(Fu_0)' - (u_0)'|}{\lambda}\right) dt = \int_0^L \varphi\left(\frac{|f(t,u_0) + u_0 - Fu_0|}{\lambda}\right) dt \\
&\leq \frac{\lambda_0}{\lambda}\int_0^L \varphi\left(\frac{|f(t,u_0)|}{\lambda_0}\right) dt + \frac{L\lambda_1}{\lambda}\int_0^L \varphi\left(\frac{|u_0'|}{\lambda_1}\right) dt + \frac{L}{\lambda}\varphi(|u_0(L)|) \qquad (40) \\
&\quad + \frac{\lambda_3}{\lambda}\int_0^L \varphi\left(\frac{|Fu_0|}{\lambda_3}\right) \qquad (41) \\
&= \frac{\lambda_0}{\lambda}\int_0^L \varphi\left(\frac{|f(t,u_0)|}{\lambda_0}\right) dt + \frac{L}{\lambda}\varphi(|u_0|) + \frac{\lambda_3}{\lambda}\int_0^L \varphi\left(\frac{|Fu_0|}{\lambda_3}\right) \qquad (42) \\
&< \infty. \qquad (43)
\end{aligned}
$$

Here, λ_0, λ_1 and λ_2 can be chosen so that $\lambda = L\lambda_1 + \lambda_0 + \lambda_3 + 1$ with $\lambda_0 = KL\lambda_1 + \lambda_2 + 1$ and $\lambda_3 = L(\lambda_0 + L\lambda_1 + 1)$ holds. Actually, the only constant function satisfying the anti-periodicity condition $u(0) = -u(L)$ is the zero function $u \equiv 0$. Thus, Theorem 3.6 implies that the operator F has a fixed point in X_w^*, which means that, the anti-periodic boundary value problem (27) has at least one solution in $GV_\varphi([0,L])$. □

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