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On Generalized Fractional Operators and a Gronwall Type Inequality with Applications

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Abstract. In this paper, we obtain the Gronwall type inequality for generalized fractional operators unifying Riemann-Liouville and Hadamard fractional operators. We apply this inequality to the dependence of the solution of differential equations, involving generalized fractional derivatives, on both the order and the initial conditions. More properties for the generalized fractional operators are formulated and the solutions of initial value problems in certain new weighted spaces of functions are established as well.

1. Introduction and Preliminaries

Fractional Calculus generalizes ordinary differentiation and integration to arbitrary order. This calculus has been attracting the interest of a big number of scientists because it was shown that it gives good results when this calclus is applied to model real world phenomena [1–5]. Furthermore, there have been attempts to find new fractional operators with different kernels in order to better model these phenomena [6–9]. Many authors discussed theoretical and application aspects of differential equations within fractional integrals and derivatives [10–14].

Integral inequalities play a significant role in the development of the theory of differential and integral equations [15]. One of the most popular inequalities is the Gronwall inequality [16] which has always been attracting many scientists because of its applications in many areas of mathematics. In [17], a generalized Gronwall inequality with application on a fractional differential equation involving Riemann-Liouville derivatives was considered. While in [18], the Gronwall inequality was proved for Hadamarad fractional derivative and in [19] it was obtained for $q-$fractional operators. Other types of inequalities were considered in [20, 21]

In this paper, we generalize the Gronwall inequality to differential equations involving a two-parameter generalized fractional derivative [22, 23] which provides the Riemann-Liouville and Hadamard fractional derivatives when one of the parameters is fixed at different values. We use this inequality to investigate the dependence of the solution on both the initial conditions and the order of the differential equation.

Before we pass to our main results, we review and introduce some notations, definitions, theorems and lemmas that will be necessary to proceed.

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The left-sided Riemann-Liouville fractional derivative of order $n-1 \leq \alpha < n$, [1–3] of a function $g : [t_0, \infty) \longrightarrow \mathbb{R}$ is given by

$$D^{\alpha}_{t_0+} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} g(\tau)\, d\tau. \tag{1}$$

The corresponding left-sided Riemann–Liouville integral operator of order $\alpha > 0$ [1–3] of a continuous function $g : [t_0, \infty) \longrightarrow \mathbb{R}$ is given by

$$J^{\alpha}_{t_0+} g(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} g(\tau)\, d\tau. \tag{2}$$

J. Hadamard [24, 25], introduced a new type of fractional derivatives and integrals of the form:

$$\left(\mathcal{D}^{\alpha}_{t_0+} g\right)(t) = \frac{1}{\Gamma(n-\alpha)} \delta^n \int_{t_0}^{t} \left(\ln \frac{t}{\tau}\right)^{n-\alpha-1} g(\tau) \frac{d\tau}{\tau},\ \alpha \in [n-1, n), \tag{3}$$

and

$$\left(\mathcal{J}^{\alpha}_{t_0+} g\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\ln \frac{t}{\tau}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau},\ (0 \leq t_0),\ \alpha > 0, \tag{4}$$

where $\delta = \left(t \frac{d}{dt}\right)$ is the so-called δ-derivative. The Caputo modification of Hadamard fractional derivatives were discussed in [26–28]

In this paper, we use the generalized fractional integral operator of order $\alpha \in [n-1, n)$, $\rho > 0$, $t_0 \geq 0$ and $t \in (t_0, \infty)$ given by

$$\left(\mathcal{J}^{\alpha,\rho}_{t_0+} g\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_0}^{t} (t^{\rho} - \tau^{\rho})^{\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{5}$$

and the generalized fractional derivative operator

$$\left(\mathcal{D}^{\alpha,\rho}_{t_0+} g\right)(t) = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_{t_0}^{t} \left(\frac{t^{\rho} - \tau^{\rho}}{\rho}\right)^{n-\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}},\ \alpha \in [n-1, n), \tag{6}$$

where $\gamma = \left(t^{1-\rho} \frac{d}{dt}\right)$.

The relation between the above latter two fractional operators is as follows:

$$\left(\mathcal{D}^{\alpha,\rho}_{t_0+} g\right)(t) = \gamma^n \left(\mathcal{J}^{n-\alpha,\rho}_{t_0+} g\right)(t),\ \alpha \in [n-1, n). \tag{7}$$

Note that the generalized operators (5)-(6) are reduced to Riemann–Liouville fractional operators as $\rho \to 1$ and Hadamard fractional operators as $\rho \to 0^+$.

The generalized Caputo fractional derivatives were discussed in [29].

Definition 1.1. *Let $G = [t_0, b]$, $(0 < t_0 < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$ and the parameters $\rho > 0$ and $0 \leq \mu < 1$.*

(i) We denote by $C[t_0, b]$ the space of continuous functions g on G with the norm

$$\|g\|_C = \max\left\{|g(t)| : t \in [t_0, b]\right\}.$$

(ii) The weighted space $C_{\mu,\rho}[t_0, b]$ of functions g on $(t_0, b]$ is defined by

$$C_{\mu,\rho}[t_0, b] = \left\{g : [t_0, b] \to \mathbb{R},\ \left(\frac{t^{\rho} - t_0^{\rho}}{\rho}\right)^{\mu} g(t) \in C[t_0, b]\right\},$$

$C_{0,\rho}\left[t_0,b\right]=C\left[t_0,b\right]$, with the norm

$$\left\|g\right\|_{C_{\mu,\rho}}=\max\left\{\left|\Big(\frac{t^{\rho}-t_0^{\rho}}{\rho}\Big)^{\mu}g\left(t\right)\right|:t\in\left[t_0,b\right]\right\},\ \left\|g\right\|_{C_{0,\rho}}=\left\|g\right\|_{C}.$$

(iii) The weighted space $C^{\mu}_{1-\mu,\rho}\left[t_0,b\right]$ of functions g on $(t_0,b]$ is defined by

$$C^{\mu}_{1-\mu,\rho}\left[t_0,b\right]=\left\{g:\left[t_0,b\right]\to\mathbb{R},\ g\left(t\right)\in C_{1-\mu,\rho}\left[t_0,b\right],\ \mathcal{D}^{\mu,\rho}_{t_{0+}}g\left(t\right)\in C_{1-\mu,\rho}\left[t_0,b\right]\right\}.$$

(iv)The weighted space $C^{n}_{\gamma,\mu,\rho}\left[t_0,b\right]$ of functions g on $(t_0,b]$ is definied by

$$C^{n}_{\gamma,\mu,\rho}\left[t_0,b\right]=\left\{g:\left[t_0,b\right]\to\mathbb{R},\ \gamma^{k}g\left(t\right)\in C\left[t_0,b\right],\ k=0,1,...(n-1),\ \gamma^{n}g\left(t\right)\in C_{\mu,\rho}\left[t_0,b\right]\right\},$$

with the norm

$$\left\|g\right\|_{C^{n}_{\gamma,\mu,\rho}}=\sum_{k=0}^{n-1}\left\|\gamma^{k}g\right\|_{C}+\left\|\gamma^{n}g\right\|_{C_{\mu,\rho}},\ C^{0}_{\gamma,\mu,\rho}\left[t_0,b\right]=C_{\mu,\rho}\left[t_0,b\right].$$

(v) The weighted space $AC^{n}_{\gamma,\mu,\rho}\left[t_0,b\right]$ of functions g on $(t_0,b]$ is definied by

$$AC^{n}_{\gamma,\mu,\rho}\left[t_0,b\right]=\left\{g:\left[t_0,b\right]\to\mathbb{R},\ \gamma^{k}g\left(t\right)\in AC\left[t_0,b\right],\ k=0,1,...(n-1),\ \gamma^{n}g\left(t\right)\in C_{\mu,\rho}\left[t_0,b\right]\right\},$$

where $AC\left[t_0,b\right]$ be the space of absolutely continuous functions on $[t_0,b]$.

Remark 1.2. *Let $\alpha>0,\beta>0$, $\rho>0$ and $\tau>0$ then*

$$\begin{aligned}\int_{\tau}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\left(\frac{s^{\rho}-\tau^{\rho}}{\rho}\right)^{\beta}\frac{ds}{s^{1-\rho}} &= \left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha+\beta+1}\int_{0}^{1}\left(1-z\right)^{\alpha}z^{\beta}dz\\ &= \frac{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{\Gamma\left(\alpha+\beta+2\right)}\Big(\frac{t^{\rho}-\tau^{\rho}}{\rho}\Big)^{\alpha+\beta+1}.\end{aligned}$$

The inner integral is evaluated with the help of the substitution $s^{\rho}=\tau^{\rho}+z(t^{\rho}-\tau^{\rho})$ and the definition of the beta function.

Lemma 1.3. *Let $0<t_0<b<+\infty$, $\alpha>0,\mu>0$ then*

(i) If $\mu>\alpha>0$, then the fractional integration operator $\left(\mathcal{J}^{\alpha,\rho}_{t_{0+}}\right)$ is bounded from $C_{\mu,\rho}\left[t_0,b\right]$ into $C_{\mu-\alpha,\rho}\left[t_0,b\right]$:

$$\left\|\mathcal{J}^{\alpha,\rho}_{t_{0+}}g\right\|_{C_{\mu-\alpha,\rho}}\leq K_1\left\|g\right\|_{C_{\mu,\rho}},\ K_1=\frac{\Gamma\left(1-\mu\right)}{\Gamma\left(1+\alpha-\mu\right)}.$$

In particular $\left(\mathcal{J}^{\alpha,\rho}_{t_{0+}}\right)$ is bounded in the space $C_{\mu,\rho}\left[t_0,b\right]$.

(ii) If $\mu\leq\alpha$, then the fractional operator $\left(\mathcal{J}^{\alpha,\rho}_{t_{0+}}\right)$ is bounded from $C_{\mu,\rho}\left[t_0,b\right]$ into $C\left[t_0,b\right]$:

$$\left\|\mathcal{J}^{\alpha,\rho}_{t_{0+}}g\right\|_{C}\leq K_2\left\|g\right\|_{C_{\mu,\rho}},\ K_2=\left(\frac{b^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-\mu}\frac{\Gamma\left(1-\mu\right)}{\Gamma\left(1+\alpha-\mu\right)}.$$

(iii) For any $g\in C\left[t_0,b\right]$. The fractional operator $\left(\mathcal{J}^{\alpha,\rho}_{t_{0+}}\right)$ is a mapping from $C\left[t_0,b\right]$ to $C\left[t_0,b\right]$ and

$$\left\|\mathcal{J}^{\alpha,\rho}_{t_{0+}}g\right\|_{C}\leq\frac{1}{\Gamma\left(\alpha+1\right)}\left(\frac{b^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha}\left\|g\right\|_{C}.$$

Proof. (i) when $\mu > \alpha$, by Remark 1.2 we have

$$\begin{aligned}
\left\|\mathcal{J}_{t_{0+}}^{\alpha,\rho} g\right\|_{C_{\mu-\alpha,\rho}} &= \max\left\{\left|\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\mu-\alpha} \mathcal{J}_{t_{0+}}^{\alpha,\rho} g(t)\right| : t \in (t_0, b]\right\} \\
&\leq \max_{t\in(t_0,b]}\left\{\left|\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\mu-\alpha} \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}}\right|\right\} \\
&\leq \max_{t\in(t_0,b]}\left\{\left|\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\mu-\alpha} \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}}{\left(\tau^\rho - t_0^\rho\right)^{\mu}} \left(\tau^\rho - t_0^\rho\right)^{\mu} g(\tau) \frac{d\tau}{\tau^{1-\rho}}\right|\right\} \\
&\leq \|g\|_{C_{\mu,\rho}} \left|\rho^{-\mu+\alpha} \left(t^\rho - t_0^\rho\right)^{\mu-\alpha} \frac{\rho^{1-\alpha+\mu}}{\Gamma(\alpha)} \int_{t_0}^{t} \left(t^\rho - \tau^\rho\right)^{\alpha-1} \left(\tau^\rho - t_0^\rho\right)^{-\mu} \frac{d\tau}{\tau^{1-\rho}}\right| \\
&\leq \|g\|_{C_{\mu,\rho}} \left|\rho^{-\mu+\alpha} \left(t^\rho - t_0^\rho\right)^{\mu-\alpha} \frac{\rho^{1-\alpha+\mu}}{\Gamma(\alpha)} \int_{t_0}^{t} \left(t^\rho - \tau^\rho\right)^{\alpha-1} \left(\tau^\rho - t_0^\rho\right)^{-\mu} \frac{d\tau}{\tau^{1-\rho}}\right| \\
&\leq \|g\|_{C_{\mu,\rho}} \left|\rho^{-\mu+\alpha} \left(t^\rho - t_0^\rho\right)^{\mu-\alpha} \frac{\rho^{\mu-\alpha}}{\Gamma(\alpha)} \left(t^\rho - t_0^\rho\right)^{\alpha-\mu} \frac{\Gamma(\alpha)\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}\right| \\
&\leq \|g\|_{C_{\mu,\rho}} \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}.
\end{aligned}$$

and hence

$$\left\|\mathcal{J}_{t_{0+}}^{\alpha,\rho} g\right\|_{C_{\mu-\alpha,\rho}} \leq \|g\|_{C_{\mu,\rho}} \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}$$

where

$$K_1 = \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}.$$

(ii) When $\mu \leq \alpha$, by Remark 1.2 we have

$$\begin{aligned}
\left\|\mathcal{J}_{t_{0+}}^{\alpha,\rho} g\right\|_{C} &= \max\left\{\left|\mathcal{J}_{t_{0+}}^{\alpha,\rho} g(t)\right| : t \in (t_0, b]\right\} \\
&\leq \max_{t\in(t_0,b]}\left\{\left|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}}\right|\right\} \\
&\leq \max_{t\in(t_0,b]}\left\{\left|\frac{\rho^{1-\alpha+\mu}}{\Gamma(\alpha)} \int_{t_0}^{t} \left(t^\rho - \tau^\rho\right)^{\alpha-1} \left(\tau^\rho - t_0^\rho\right)^{-\mu} \rho^{-\mu} \left(\tau^\rho - t_0^\rho\right)^{\mu} g(\tau) \frac{d\tau}{\tau^{1-\rho}}\right|\right\} \\
&\leq \|g\|_{C_{\mu,\rho}} \left|\frac{\rho^{1-\alpha+\mu}}{\Gamma(\alpha)} \int_{t_0}^{t} \left(t^\rho - \tau^\rho\right)^{\alpha-1} \left(\tau^\rho - t_0^\rho\right)^{-\mu} \frac{d\tau}{\tau^{1-\rho}}\right| \\
&\leq \|g\|_{C_{\mu,\rho}} \left|\frac{1}{\Gamma(\alpha)} \rho^{\mu-\alpha} \left(t^\rho - t_0^\rho\right)^{\alpha-\mu} \frac{\Gamma(\alpha)\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}\right| \\
&\leq \|g\|_{C_{\mu,\rho}} \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)} \left(\frac{b^\rho - t_0^\rho}{\rho}\right)^{\alpha-\mu}.
\end{aligned}$$

(iii) For any $g \in C[t_0, b]$, one has

$$\begin{aligned}
\left\|\mathcal{J}_{t_{0+}}^{\alpha,\rho} g\right\|_C &= \max_{t\in(t_0,b]} \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}} \right| \le \frac{1}{\Gamma(\alpha)} \left\|g\right\|_C \left| \int_{t_0}^{t} (t^\rho - \tau^\rho)^{\alpha-1} \frac{d\tau}{\tau^{1-\rho}} \right| \\
&= \frac{1}{\Gamma(\alpha)} \left\|g\right\|_C \left| \left(\mathcal{J}_{t_{0+}}^{\alpha,\rho} 1\right)(t) \right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} \left(t^\rho - t_0^\rho\right)^\alpha \right| \left\|g\right\|_C \\
&\le \frac{1}{\Gamma(\alpha+1)} \left(\frac{b^\rho - t_0^\rho}{\rho}\right)^\alpha \left\|g\right\|_C .
\end{aligned}$$

□

Lemma 1.4. *[29]*

(i) Let $\alpha \in (0,1], \beta \ge 0,\ 0 \le t_0,\ \rho > 0$. Then we have

$$\left(\mathcal{J}_{t_{0+}}^{\alpha,\rho}\right)\left(\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta\right)(t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha+\beta}. \tag{8}$$

In particular

$$\left(\mathcal{J}_{t_{0+}}^{\alpha,\rho} 1\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \frac{d\tau}{\tau^{1-\rho}} = \frac{1}{\Gamma(\alpha+1)} \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha.$$

(ii) If $\beta \ge 0$ and $0 < \alpha \le 1$, then

$$\mathcal{D}_{t_{0+}}^{\alpha,\rho}\left(\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta\right)(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\beta-\alpha}. \tag{9}$$

In particular

$$\left(\mathcal{D}_{t_{0+}}^{\alpha,\rho} 1\right)(t) = \frac{\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{-\alpha}}{\Gamma(1-\alpha)},$$

and for $k = 0, 1, .., [\alpha] + 1$, we have

$$\left(\mathcal{D}_{t_{0+}}^{\alpha,\rho} \left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^{\alpha-k}\right)(t) = 0. \tag{10}$$

Lemma 1.5. *Let $\mu, \alpha \in (0,1)$, $\rho > 0$ and $g \in C_{\mu,\rho}[t_0, b]$.*

If $\mu < \alpha$ then $\mathcal{J}_{t_{0+}}^{\alpha,\rho} g \in C[t_0, b]$ and

$$\mathcal{J}_{t_{0+}}^{\alpha,\rho} g(t_0) = \lim_{t\to t_0} \mathcal{J}_{t_{0+}}^{\alpha,\rho} g(t) = 0. \tag{11}$$

Proof. Let $g \in C_{\mu,\rho}[t_0, b]$ then $\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\mu g(t) \in C[t_0, b]$ and

$$\left| \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\mu g(t) \right| \le M,\ t \in [t_0, b],\ M > 0.$$

Therefore, by Remark 1.2 we have

$$\begin{aligned}\mathcal{J}_{t_{0+}}^{\alpha,\rho} g(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau^{1-\rho}} \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \rho^{\mu} \left(\tau^\rho - t_0^\rho\right)^{-\mu} \rho^{-\mu} \left(\tau^\rho - t_0^\rho\right)^{\mu} g(\tau) \frac{d\tau}{\tau^{1-\rho}},\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{J}_{t_{0+}}^{\alpha,\rho} g\right\| &\leq M \frac{\rho^{1-\alpha+\mu}}{\Gamma(\alpha)} \int_{t_0}^{t} \left(t^\rho - \tau^\rho\right)^{\alpha-1} \left(\tau^\rho - t_0^\rho\right)^{-\mu} \frac{d\tau}{\tau^{1-\rho}} \\ &\leq M \mathcal{J}_{t_{0+}}^{\alpha,\rho} \left(\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^{-\mu}\right)(t).\end{aligned}$$

By using Lemma 1.4-i we have

$$\left\|\mathcal{J}_{t_{0+}}^{\alpha,\rho} g\right\| \leq M \frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)} \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-\mu}.$$

Since

$$\lim_{t \to t_0} \left(t^\rho - t_0^\rho\right)^{\alpha-\mu} = 0, \ \mu < \alpha,$$

we obtain the result in (11). □

Lemma 1.6. *[29]*

(i) The fractional integral operators $\left(\mathcal{J}_{t_{0+}}^{\alpha,\rho}\right)$ *satisfy the semigroup property*

$$\left(\mathcal{J}_{t_{0+}}^{\alpha,\rho} \mathcal{J}_{t_{0+}}^{\beta,\rho} g\right)(t) = \left(\mathcal{J}_{t_{0+}}^{\alpha+\beta,\rho} g\right)(t), \ \rho > 0, \alpha > 0, \beta > 0. \tag{12}$$

(ii) The fractional derivative operators $\left(\mathcal{D}_{t_{0+}}^{\alpha,\rho}\right)$ *satisfy the semigroup property*

$$\left(\mathcal{D}_{t_{0+}}^{\alpha,\rho} \mathcal{J}_{t_{0+}}^{\beta,\rho} g\right)(t) = \left(\mathcal{J}_{t_{0+}}^{\beta-\alpha,\rho} g\right)(t), \ \left(\mathcal{D}_{t_{0+}}^{\alpha,\rho} \mathcal{J}_{t_{0+}}^{\alpha,\rho} g\right)(t) = g(t). \tag{13}$$

(iii) Let $\alpha \in (0,1)$, *if* $g(t) \in C_{\mu,\rho}[t_0, b]$ *and* $\mathcal{J}_{t_{0+}}^{1-\alpha,\rho} g \in C_{\mu,\rho}^{1}[t_0, b]$. *Then, we have*

$$\left(\mathcal{J}_{t_{0+}}^{\alpha,\rho}\left(\mathcal{D}_{t_{0+}}^{\alpha,\rho} g\right)\right)(t) = g(t) - \frac{\left(\mathcal{J}_{t_{0+}}^{1-\alpha,\rho} g\right)(t_0)}{\Gamma(\alpha)} \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1}, \text{ for all } t \in (t_0, b]. \tag{14}$$

(iv) If $g(t) \in C[t_0, b]$ *and* $\mathcal{J}_{t_{0+}}^{1-\alpha,\rho} g \in C^1[t_0, b]$, *then the relation* (14) *holds for any* $t \in [t_0, b]$.

Lemma 1.7. *[29] Let* $n \in \mathbb{N}$. *The space* $C_{\gamma,\mu,\rho}^{n}[t_0, b]$ *consists of those and only those functions g which are represented in the form*

$$g(t) = \frac{1}{(n-1)!} \int_{t_0}^{t} \left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^{n-1} \varphi(\tau) \frac{d\tau}{\tau^{1-\rho}} + \sum_{k=0}^{n-1} c_k \left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^{k},$$

where $\varphi \in C[t_0, b]$ *and* c_k $(k = 0, 1, ..., n-1)$ *are arbitrary constants, such that*

$$\varphi(t) = \mathcal{D}_{t_{0+}}^{n,\rho} g(t), \ c_k = \frac{\mathcal{D}_{t_{0+}}^{k,\rho} g(t_0)}{k!}, \ (k = 0, 1, ..., n-1).$$

Let us recall the definition of the Mittag–Leffler function.

Definition 1.8. *[1–3] Let $\alpha > 0, \beta \in \mathbb{R}$ and $z \in \mathbb{C}$. The Mittag–Leffler function is defined by the series*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)},\ E_{\alpha}(z) = E_{\alpha,1}(z),\ E_1(z) = \exp(z). \tag{15}$$

When $\alpha, \beta > 0$ the series is convergent.

Next, we give the relations connecting the function definied by (15) and the generalized integrals.

Lemma 1.9. *(i) Let $\lambda \in \mathbb{R},\ \alpha, \beta > 0$ and $\mu \geq 0$.*

$$\mathcal{J}_{t_0+}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta E_{\mu,\beta+1}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right](t) = \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\beta+\alpha} E_{\mu,\beta+\alpha+1}\left(\lambda\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\mu\right). \tag{16}$$

(ii) For $\beta \geq 1,\ \mu > 0, \lambda \in \mathbb{R}$, we have

$$\mathcal{D}_{t_0+}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta E_{\mu,\beta+1}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right](t) = \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\beta-\alpha} E_{\mu,\beta-\alpha+1}\left(\lambda\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\mu\right). \tag{17}$$

In particular, when $\beta = 0$, we have

$$\mathcal{D}_{t_0+}^{\alpha,\rho}\left[E_{\mu,1}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right](t) = \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha} E_{\mu,1-\alpha}\left(\lambda\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\mu\right). \tag{18}$$

Proof. (i) For $\alpha, \beta, \mu, \nu > 0,\ \lambda \in \mathbb{R}$ by the virtue of (5) and (15), we have

$$\mathcal{J}_{t_0+}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta E_{\mu,\nu}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right](t) = \mathcal{J}_{t_0+}^{\alpha,\rho}\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\nu + k\mu)}\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^{k\mu+\beta}\right)(t).$$

Now, interchanging the order of integration and summation and then evaluating the inner integral by means of Beta- function, one gets

$$\mathcal{J}_{t_0+}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta E_{\mu,\nu}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right] = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\nu + k\mu)} \frac{\Gamma(\beta + k\mu + 1)}{\Gamma(\beta + k\mu + \alpha + 1)}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{k\mu+\beta+\alpha}.$$

In particular, when $\nu = \beta + 1$, we have

$$\begin{aligned}\mathcal{J}_{t_0+}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta E_{\mu,\beta+1}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right](t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\beta + k\mu + \alpha + 1)}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{k\mu+\beta+\alpha}\\ &= \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\beta+\alpha} \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\beta + k\mu + \alpha + 1)}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{k\mu}\\ &= \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\beta+\alpha} E_{\mu,\alpha+\beta+1}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right).\end{aligned}$$

(ii) For $\alpha, \beta, \mu, \nu > 0,\ \lambda \in \mathbb{R}$, using Lemma 1.4-ii, we have the following

$$\begin{aligned}\mathcal{D}_{t_0+}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta E_{\mu,\nu}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\nu+k\mu)} \mathcal{D}_{t_0+}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^{k\mu+\beta}\right]\\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\nu + k\mu)} \frac{\Gamma(\beta + k\mu + 1)}{\Gamma(\beta + k\mu - \alpha + 1)}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{k\mu+\beta-\alpha}.\end{aligned}$$

In particular, when $\nu = \beta + 1$, we have

$$\mathcal{D}_{t_{0+}}^{\alpha,\rho}\left[\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\beta E_{\mu,\beta+1}\left(\lambda\left(\frac{\tau^\rho - t_0^\rho}{\rho}\right)^\mu\right)\right] = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\beta + k\mu - \alpha + 1)}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{k\mu+\beta-\alpha}$$
$$= \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\beta-\alpha}\sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\beta + k\mu - \alpha + 1)}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{k\mu}$$
$$= \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\beta-\alpha} E_{\mu,\beta-\alpha+1}\left(\lambda\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\mu\right).$$

□

Lemma 1.10. *The solution to the Cauchy problem*

$$\mathcal{D}_{t_{0+}}^{\alpha,\rho} x(t) = -\lambda x(t),\ \mathcal{D}_{t_{0+}}^{\alpha-1,\rho} x(t_0) = \eta,\ t_0 > 0,\ \eta \in \mathbb{R}, \tag{19}$$

with $0 < \alpha < 1$ *and* $\lambda \in \mathbb{R}$ *has the form*

$$x(t) = \eta\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right). \tag{20}$$

Proof. The proof is similar to the proof of Theorem 4.1 in [1]. □

The asymptotical expansion of the Mittag–Leffler function with two parameters was given below.

Lemma 1.11. *[2]Let* $0 < \alpha < 2$, β, ν *be arbitrary real numbers such that* $\frac{\pi\alpha}{2} < \nu < \min(\pi, \pi\alpha)$, *then for an arbitrary integer* $m \geq 1$ *the following expansion holds*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O\left(|z|^{-(m+1)}\right), |z| \longrightarrow \infty,\ \left|\arg z\right| \leq \nu,$$

and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O\left(|z|^{-(m+1)}\right), |z| \longrightarrow \infty,\ \nu \leq \left|\arg z\right| \leq \pi.$$

Example 1.12. *The asymptotic expansion of the Mittag-Leffler function*

$$E_{\alpha,\alpha}\left(M\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^\alpha\right) = \frac{M^{\frac{1}{\alpha}-1}}{\alpha}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{1-\alpha} e^{M^{\frac{1}{\alpha}}\left(\frac{t^\rho - t_0^\rho}{\rho}\right)},\ t \longrightarrow +\infty \tag{21}$$

where M *is a positive constant.*

Definition 1.13. *Let* $G \subset \mathbb{R}$, $[t_0, b] \subset \mathbb{R}$, $g : [t_0, b] \times G \longrightarrow \mathbb{R}$ *be a function,* g *is said to satisfy Lipschitzian condition with respect to the second variable, if for all* $t \in (t_0, b]$ *and for any* $x, \tilde{x} \in G$ *one has*

$$\left|g(t, x) - g(t, \tilde{x})\right| \leq L|x - \tilde{x}|,\ L > 0. \tag{22}$$

with some constant $L > 0$ *independent of* $x, \tilde{x}, t$ *and* g *be bounded on* G.

To conclude this section, we give the classical form of the standard Gronwall inequality described as follows [16, 1919].

Theorem 1.14. *Let $x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$,*

(i) For any $t \in [t_0, T]$

$$x(t) \leq a(t) + \int_{t_0}^{t} b(\tau) x(\tau) d\tau, \tag{23}$$

where $b(t) \geq 0$, then

$$x(t) \leq a(t) + \int_{t_0}^{t} b(\tau) a(\tau) \exp\left(\int_{\tau}^{t} b(s) a(s) ds\right) d\tau. \tag{24}$$

In particular, if $a(t)$ is not decreasing, then

$$x(t) \leq a(t) \exp\left(\int_{t_0}^{t} b(\tau) d\tau\right), \ t \in [t_0, T). \tag{25}$$

(ii) The result remains valid if $\leq$ is replaced by $\geq$ in both (23) *and* (24).

(iii) Both (i) and (ii) remain valid if $\int_{t_0}^{t}$ is replaced by $\int_{t}^{T}$ and $\int_{\tau}^{t}$ by $\int_{t}^{\tau}$ throughout.

The generalized Gronwall inequality with Hadamard derivative and the Riemann-Liouville fractional derivative were presented in [17, 18] respectively. Many authors have established several other very useful Gronwall-like integral inequalities.

2. A generalized Gronwall inequality

In this section, we establish a new version of Gronwall type integral inequality, which generalizes some previous ones.

Theorem 2.1. *Let $\alpha > 0$, $x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$, $T > 0$, $b(t) \leq M$, where M is a constant. If*

$$x(t) \leq a(t) + b(t) \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{26}$$

then

$$x(t) \leq a(t) + \int_{t_0}^{t} \sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n\alpha-1} a(\tau) \frac{d\tau}{\tau^{1-\rho}}, \ t \in [t_0, T). \tag{27}$$

Proof. Define

$$Bx(t) = b(t) \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{28}$$

and the sequence B^k $(k \in \mathbb{N})$ as

$$B^1 = B, \ B^k = BB^{k-1} \ (k \in \mathbb{N} - \{1\}).$$

It follows that

$$x(t) \leq a(t) + Bx(t),$$

which implies that

$$x(t) \leq \sum_{k=1}^{n-1} B^k a(t) + B^n x(t).$$

Now we claim that

$$B^n x(t) \leq \int_{t_0}^{t} \frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{29}$$

and $B^n x(t) \longrightarrow 0$ as $n \longrightarrow \infty$ for $t \in [t_0, T)$.

It is easy to see that (29) is valid for $n = 1$. Assume that it is true for $n = k$, that is,

$$B^k x(t) \leq \int_{t_0}^{t} \frac{(b(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{k\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}. \tag{30}$$

If $n = k + 1$, then the induction implies

$$B^{k+1} x(t) = BB^k x(t) \tag{31}$$

$$\leq \frac{(b(t))^{k+1}}{\rho^{(k+1)\alpha-2}} \int_{t_0}^{t} (t^\rho - s^\rho)^{\alpha-1} \left[\int_{t_0}^{s} \frac{\Gamma(\alpha)^k}{\Gamma(k\alpha)} (s^\rho - \tau^\rho)^{k\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}\right] \frac{ds}{s^{1-\rho}}. \tag{32}$$

Now interchanging the order of integration and utilizing the following particular case of the Fubini's theorem

$$\int_{t_0}^{t} ds \int_{t_0}^{s} f(s,\tau)\, d\tau = \int_{t_0}^{t} d\tau \int_{\tau}^{t} f(s,\tau)\, ds,$$

and assuming that one of these integrals is absolutely convergent, by Remark 1.2, we have

$$B^{k+1} x(t) \leq \frac{(b(t))^{k+1}}{\rho^{(k+1)\alpha-2}} \int_{t_0}^{t} \int_{\tau}^{t} \frac{\Gamma(\alpha)^k}{\Gamma(k\alpha)} (t^\rho - s^\rho)^{\alpha-1} (s^\rho - \tau^\rho)^{k\alpha-1} \frac{ds}{s^{1-\rho}} x(\tau) \frac{d\tau}{\tau^{1-\rho}} \tag{33}$$

$$= \int_{t_0}^{t} \frac{(b(t)\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{(k+1)\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}. \tag{34}$$

Therefore, equation (29) is obtained. Furthermore , since the denominator goes to infinity faster than the numerator (29), one can conclude that

$$B^n x(t) \leq \int_{t_0}^{t} \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}, \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ for } t \in [t_0, T). \tag{35}$$

To complete the proof, we let $n \longrightarrow \infty$ in

$$x(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n x(t) = a(t) + \sum_{k=1}^{n-1} B^k a(t) + B^n x(t),$$

to obtain

$$\begin{aligned} x(t) &\leq a(t) + \sum_{k=1}^{\infty} B^k a(t) \\ &\leq a(t) + \sum_{k=1}^{\infty} \int_{t_0}^{t} \frac{(M\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{k\alpha-1} a(\tau) \frac{d\tau}{\tau^{1-\rho}} \\ &\leq a(t) + \int_{t_0}^{t} \sum_{k=1}^{\infty} \frac{(M\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{k\alpha-1} a(\tau) \frac{d\tau}{\tau^{1-\rho}}. \end{aligned}$$

□

Remark 2.2.
If we take $\rho \to 0^+$ *in Theorem* 2.1*, then the Gronwall's inequality for Hadamard integrals in [18] is recovered.*
If we take $\rho = 1$ *in Theorem* 2.1*, then the Gronwall's inequality for Riemann-Liouville integrals in [17] is obtained.*

Corollary 2.3. *Let* $\alpha > 0$, $x(t), a(t)$ *be non-negative functions and* $b(t) = b \geq 0$. *If*

$$x(t) \leq a(t) + b \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}},$$

then

$$x(t) \leq a(t) + \int_{t_0}^{t} \sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n\alpha-1} a(\tau) \frac{d\tau}{\tau^{1-\rho}}, \ t \in [t_0, T).$$

Corollary 2.4. *Under the hypotheses of Theorem* 2.1*, assume further that* $a(t)$ *is a nondecreasing function for* $t \in [t_0, T)$. *Then*

$$x(t) \leq a(t) E_\alpha \left(b(t) \Gamma(\alpha) \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha} \right), \ t \in [t_0, T).$$

Proof. From (26) and the assumption that $a(t)$ is a nondecreasing function for $t \in [t_0, T)$, we may write

$$x(t) \leq a(t) \left[1 + \sum_{n=1}^{\infty} \int_{t_0}^{t} \frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n\alpha-1} \frac{d\tau}{\tau^{1-\rho}} \right], \ t \in [t_0, T),$$

or

$$x(t) \leq a(t) \left[1 + \sum_{n=1}^{\infty} \mathcal{J}_{t_0}^{n\alpha,\rho} \left((b(t)\Gamma(\alpha))^n \right)(t) \right], \ t \in [t_0, T).$$

Then, with the help of Lemma 1.4 it follows that

$$\begin{aligned}
x(t) &\leq a(t) \left[1 + \sum_{n=1}^{\infty} \left((b(t)\Gamma(\alpha))^n \right) \mathcal{J}_{t_0}^{n\alpha,\rho}(1)(t) \right], \ t \in [t_0, T) \\
&= a(t) \left[1 + \sum_{n=1}^{\infty} \left((b(t)\Gamma(\alpha))^n \right) \frac{\left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{n\alpha}}{\Gamma(n\alpha+1)} \right] \\
&= a(t) \left[\sum_{n=0}^{\infty} \left((b(t)\Gamma(\alpha))^n \right) \frac{\left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{n\alpha}}{\Gamma(n\alpha+1)} \right] \\
&= a(t) E_\alpha \left(b(t) \Gamma(\alpha) \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha} \right).
\end{aligned}$$

The proof is complete. □

Example 2.5. *If* $a(t) = C$, $g(t) = \frac{M}{\Gamma(\alpha)}$ *, then the inequality*

$$x(t) \leq C + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}$$

implies

$$x(t) \leq CE_{\alpha}\left(M\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha}\right).$$

Example 2.6. *If* $a(t) = \frac{C}{\Gamma(\alpha)}\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}$, $b(t) = \frac{M}{\Gamma(\alpha)}$ *, then the inequality*

$$x(t) \leq \frac{C\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1} x(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{36}$$

implies

$$x(t) \leq C\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(M\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha}\right). \tag{37}$$

Proof. Using Theorem 2.1 and Remark 1.2 leads to

$$\begin{aligned}
x(t) &\leq \frac{C\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)} + \int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{M^n\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{n\alpha-1}}{\Gamma(n\alpha)}\frac{C\left(\frac{\tau^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)}\frac{d\tau}{\tau^{1-\rho}} \\
&\leq \frac{C\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{n=1}^{\infty}\frac{CM^n}{\Gamma(\alpha)\Gamma(n\alpha)}\int_{t_0}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{n\alpha-1}\left(\frac{\tau^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}\frac{d\tau}{\tau^{1-\rho}} \\
&= \frac{C\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{n=1}^{\infty}\frac{CM^n}{\Gamma((n+1)\alpha)}\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{(n+1)\alpha-1} \\
&= \sum_{n=0}^{\infty}\frac{CM^n}{\Gamma(n\alpha+\alpha)}\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{(n+1)\alpha-1} \\
&= C\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(M\rho^{-\alpha}\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha}\right).
\end{aligned}$$

The proof is complete. □

3. Dependence of the solution on parameters for differential equations with generalized fractional derivatives

In this section we will use the Gronwall inequality mentioned in the previous section in order to investigate the dependence of the solution of a certain fractional differential equation with generalized derivatives, on the order and the initial conditions.

Consider the following initial value problem within generalized fractional derivatives:

$$\mathcal{D}_{t_{0+}}^{\alpha,\rho}x(t) = f(t, x(t)), \tag{38}$$

and

$$\mathcal{J}_{t_{0+}}^{1-\alpha,\rho}x(t)\Big|_{t=t_{0+}} = \eta, \tag{39}$$

where $0 < \alpha \leq 1,\ 0 \leq t < T \leq \infty$ and $f : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function with respect to all its arguments. The Volterra integral equations corresponding to the problem (38)-(39) is as follows:

$$x(t) = \eta \frac{\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau, x(\tau)) \frac{d\tau}{\tau^{1-\rho}},\ 0 \leq t_0 \leq t \leq \infty. \tag{40}$$

We prove the equivalence of the Cauchy problem (38)-(39) and the Volterra equation (40) in the sense that, if $x \in C^{\alpha}_{1-\alpha,\rho}[t_0, b]$ satisfies one of them, then it also satisfies the other.

Theorem 3.1. *Let* $1 \geq \alpha > 0$, $(0 < t_0 < b < +\infty)$. *Let G be an open set in $\mathbb{R}$ and let $f :]t_0, b] \times G \longrightarrow \mathbb{R}$ be a function such that $f(t, x) \in C_{\mu,\rho}[t_0, b]$ for any $x \in C_{\mu,\rho}[t_0, b]$ with $1 - \alpha \leq \mu < 1$. if $x \in C^{\alpha}_{1-\alpha,\rho}[t_0, b]$ then x satisfies the relations* (38) *and* (39) *if and only if, x satisfies equation* (40).

Proof. Let $0 < \alpha \leq 1$ and $x \in C^{\alpha}_{1-\alpha,\rho}[t_0, b]$.

(**i**) We prove the necessity.

By hypothesis, $x \in C^{\alpha}_{1-\alpha,\rho}[t_0, b]$ satisfies the relations (38) and (39). Since $f(t, x) \in C_{1-\alpha,\rho}[t_0, b]$ and it follows from (38) that $\mathcal{D}^{\alpha,\rho}_{t_{0+}} x(t) \equiv \gamma \mathcal{J}^{1-\alpha,\rho}_{t_{0+}} x(t) \in C_{1-\alpha,\rho}[t_0, b]$, and hence, usig the definition of the space $C^{n}_{\gamma,\alpha,\rho}[t_0, b]$ and applying Lemma 1.3 and Lemma 1.7, we have

$$\left(\mathcal{J}^{1-\alpha,\rho}_{t_{0+}} x\right)(t) \in C[t_0, b],$$

since

$$\left(\mathcal{J}^{1-\alpha,\rho}_{t_{0+}} x\right)(t) \in C^{1}_{\gamma,1-\alpha,\rho}[t_0, b].$$

By using Lemma 1.6-(iii), we obtain

$$\left(\mathcal{J}^{\alpha,\rho}_{t_{0+}} \mathcal{D}^{\alpha,\rho}_{t_{0+}} x\right)(t) = x(t) - \frac{\eta}{\Gamma(\alpha)} \left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1},\ t \in (t_0, b]. \tag{41}$$

In view of Lemma 1.3, $\mathcal{J}^{\alpha,\rho}_{t_{0+}} f(t, x)$ belongs to the space $C_{\mu-\alpha,\rho}[t_0, b]$. Applying the operator $\left(\mathcal{J}^{\alpha,\rho}_{t_{0+}}\right)$ to the both sides of (38) and using (41), (39), we deduce that there exists an unique solution $x \in C_{1-\alpha,\rho}[t_0, b] \subset C_{\mu,\rho}[t_0, b]$ for $\mu > \alpha$ of the equation (40).

(ii.1) Next, we prove the sufficiency. Let $x \in C^{\alpha}_{1-\alpha,\rho}[t_0, b]$ satisfies the equation (40) then $\mathcal{D}^{\alpha,\rho}_{t_{0+}} x(t) \in C_{1-\alpha,\rho}[t_0, b]$.

Applying the operator $\left(\mathcal{D}^{\alpha,\rho}_{t_{0+}}\right)$ to both sides of (40), taking into account (6), (39), Lemma 1.6 and 7, we obtain

$$\mathcal{D}^{\alpha,\rho}_{t_{0+}} \left(x(t) - \eta \frac{\left(\frac{t^\rho - t_0^\rho}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)} \right)(t) = \mathcal{D}^{\alpha,\rho}_{t_{0+}} \left(\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau, x(\tau)) \frac{d\tau}{\tau^{1-\rho}} \right)(t),$$

because

$$\mathcal{D}^{\alpha,\rho}_{t_{0+}} \left(\left(t^\rho - t_0^\rho\right)^{\alpha-1} \right)(t) = 0 \text{ and } \left(\mathcal{D}^{\alpha,\rho}_{t_{0+}} x\right)(t) = \left(\mathcal{D}^{\alpha,\rho}_{t_{0+}}\right)\left(\mathcal{J}^{\alpha,\rho}_{t_{0+}} f\right)(t) \equiv f(t, x(t)).$$

(ii.2) Now,to show that x satisfies the initial relations (39), we apply the operators $\left(\mathcal{J}_{t_{0+}}^{1-\alpha,\rho}\right)$ both sides of (40) to get

$$\begin{aligned}\mathcal{J}_{t_{0+}}^{1-\alpha,\rho}x(t) &= \mathcal{J}_{t_{0+}}^{1-\alpha,\rho}\left(\eta\frac{\left(\frac{\tau^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}f(\tau,x(\tau))\frac{d\tau}{\tau^{1-\rho}}\right)(t)\\ &= \mathcal{J}_{t_{0+}}^{1-\alpha,\rho}\left(\eta\frac{\left(\frac{\tau^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)}\right)(t)+\mathcal{J}_{t_{0+}}^{1-\alpha,\rho}\left(\mathcal{J}_{t_{0+}}^{\alpha,\rho}f(\tau,x(\tau))\right)(t)\\ &= \mathcal{J}_{t_{0+}}^{1-\alpha,\rho}\left(\eta\frac{\left(\frac{\tau^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)}\right)(t)+\left(\mathcal{J}_{t_{0+}}^{1,\rho}f(\tau,x(\tau))\right)(t).\end{aligned} \tag{42}$$

Using the continuity of f and Lemma 1.4 we have $\mathcal{J}_{t_{0+}}^{1,\rho}f(t,x)\in C[t_0,b]$. Now taking the limit as $t\longrightarrow t_{0+}$ of both sides of (42), we obtain $\mathcal{J}_{t_{0+}}^{1-\alpha,\rho}x(t_0)=\eta$ since by using Lemma 1.5, we have $\mathcal{J}_{t_{0+}}^{1,\rho}f(\tau,x(\tau))(t_0)=0$.
Thus, the sufficiency is proved and this completes the proof of theorem. □

Now let us consider the following Cauchy problem

$$\mathcal{D}_{t_{0+}}^{\alpha-\varepsilon,\rho}y(t)=f(t,y(t)),\ \alpha>0,\ \varepsilon>0, \tag{43}$$

and

$$\mathcal{D}_{t_{0+}}^{\alpha-\varepsilon-1,\rho}y(t)\big|_{t=t_{0+}}=\tilde{\eta}. \tag{44}$$

Theorem 3.2. *Let $\alpha>0,\ \varepsilon>0$ such that $1\geq\alpha>\alpha-\ \varepsilon>0$. Let the function f be continuous and fulfill a Lipschitz condition* (22)*. For $0\leq t\leq h<T$ assume that x and y are the solutions of the initial value problems* (38)-(39) *and* (43)-(44) *respectively. Then, for $0<t\leq h$ the following holds:*

$$\left|y(t)-x(t)\right|\leq a(t)+\int_{t_0}^{t}\sum_{n=1}^{\infty}\left(\frac{L\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\right)^{n}\frac{\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{n(\alpha-\varepsilon)-1}}{\Gamma(n(\alpha-\varepsilon))}a(\tau)\frac{d\tau}{\tau^{1-\rho}},$$

where

$$\begin{aligned}a(t) &= \left[\left|\frac{\left(t^{\rho}-t_0^{\rho}\right)^{\alpha-\varepsilon}}{\rho^{\alpha-\varepsilon}(\alpha-\varepsilon)}\left(\frac{1}{\Gamma(\alpha-\varepsilon)}-\frac{1}{\Gamma(\alpha)}\right)\right|+\left|\frac{\left(t^{\rho}-t_0^{\rho}\right)^{\alpha-\varepsilon}}{\rho^{\alpha-\varepsilon}(\alpha-\varepsilon)\Gamma(\alpha)}-\frac{\left(t^{\rho}-t_0^{\rho}\right)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right|\right]\left\|f\right\|,\\ &\quad+\left|\frac{\tilde{\eta}\rho^{1-(\alpha-\varepsilon)}}{\Gamma(\alpha-\varepsilon)}\left(t^{\rho}-t_0^{\rho}\right)^{\alpha-\varepsilon-1}-\frac{\eta\rho^{1-\alpha}}{\Gamma(\alpha)}\left(t^{\rho}-t_0^{\rho}\right)^{\alpha-1}\right|,\end{aligned} \tag{45}$$

and

$$\left\|f\right\|=\max_{0\leq t\leq h}\left|f(t,x(t))\right|.$$

Proof. The solutions of the initial value problem (38)-(39) and (43)-(44) are given by

$$x(t)=\frac{\eta}{\Gamma(\alpha)}\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}f(\tau,x(\tau))\frac{d\tau}{\tau^{1-\rho}},\ 0\leq t\leq\infty, \tag{46}$$

and

$$y(t)=\frac{\tilde{\eta}}{\Gamma(\alpha-\varepsilon)}\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-\varepsilon-1}+\frac{1}{\Gamma(\alpha-\varepsilon)}\int_{t_0}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-\varepsilon-1}f(\tau,y(\tau))\frac{d\tau}{\tau^{1-\rho}},\ 0\le t\le\infty, \tag{47}$$

respectively. It follows that

$$\begin{aligned}\left|y(t)-x(t)\right| &\le \left|\frac{\tilde{\eta}\rho^{1-(\alpha-\varepsilon)}}{\Gamma(\alpha-\varepsilon)}\left(t^{\rho}-t_0^{\rho}\right)^{\alpha-\varepsilon-1}-\frac{\eta\rho^{1-\alpha}}{\Gamma(\alpha)}\left(t^{\rho}-t_0^{\rho}\right)^{\alpha-1}\right| \\ &\quad+\left|\mathcal{J}_{t_{0+}}^{\alpha-\varepsilon,\rho}f(\tau,y(\tau))(t)-\frac{\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\mathcal{J}_{t_{0+}}^{\alpha-\varepsilon,\rho}f(\tau,y(\tau))(t)\right| \\ &\quad+\left|\frac{\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\mathcal{J}_{t_{0+}}^{\alpha-\varepsilon,\rho}f(\tau,y(\tau))(t)-\frac{\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\mathcal{J}_{t_{0+}}^{\alpha-\varepsilon,\rho}f(\tau,x(\tau))(t)\right| \\ &\quad+\left|\frac{\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\mathcal{J}_{t_{0+}}^{\alpha-\varepsilon,\rho}f(\tau,x(\tau))(t)-\mathcal{J}_{t_{0+}}^{\alpha,\rho}f(\tau,x(\tau))(t)\right| \\ &\le a(t)+\frac{L\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\mathcal{J}_{t_{0+}}^{\alpha-\varepsilon,\rho}\left|y(\tau)-x(\tau)\right|(t).\end{aligned}$$

Therefore,

$$\left|y(t)-x(t)\right|\le a(t)+\frac{L\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\mathcal{J}_{t_{0+}}^{\alpha-\varepsilon,\rho}\left|y(\tau)-x(\tau)\right|(t), \tag{48}$$

where $a(t)$ in given in (45).

Now using Theorem 2.1 yields

$$\left|y(t)-x(t)\right|\le a(t)+\int_{t_0}^{t}\sum_{n=1}^{\infty}\left(\frac{L\Gamma(\alpha-\varepsilon)}{\Gamma(\alpha)}\right)^{n}\frac{\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{n(\alpha-\varepsilon)-1}}{\Gamma(n(\alpha-\varepsilon))}a(\tau)\frac{d\tau}{\tau^{1-\rho}}.$$

□

Corollary 3.3. *Under the hypothesis of Theorem* 3.2*, if* $\alpha>0,\ \varepsilon=0$*, then*

$$\left|y(t)-x(t)\right|\le\left|\tilde{\eta}-\eta\right|\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(L\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha}\right),$$

for $0<t_0\le t\le h$*.*

Proof. If $\varepsilon=0$, then

$$a(t)=\left|\frac{\left(\frac{t^{\rho}-t_0^{\rho}}{\rho}\right)^{\alpha-1}(\tilde{\eta}-\eta)}{\Gamma(\alpha)}\right|.$$

By Theorem 3.2 and Remark 1.2, we obtain

$$\begin{aligned}
\left|y(t)-x(t)\right| &\leq a(t)+\int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{L^n\rho^{1-n\alpha}\left(t^\rho-\tau^\rho\right)^{n\alpha-1}}{\Gamma(n\alpha)}a(\tau)\frac{d\tau}{\tau^{1-\rho}}\\
&\leq a(t)+\frac{(\tilde{\eta}-\eta)}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{L^n}{\Gamma(n\alpha)}\int_{t_0}^{t}\rho^{1-n\alpha}\left(t^\rho-\tau^\rho\right)^{n\alpha-1}\rho^{1-\alpha}\left(\tau^\rho-t_0^\rho\right)^{\alpha-1}\frac{d\tau}{\tau^{1-\rho}}\\
&\leq a(t)+\frac{(\tilde{\eta}-\eta)}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{L^n}{\Gamma(n\alpha)}\frac{\Gamma(n\alpha)\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\rho^{-1-n\alpha-\alpha}\left(t^\rho-t_0^\rho\right)^{n\alpha+\alpha+1}\\
&\leq \frac{\rho^{1-\alpha}\left(t^\rho-t_0^\rho\right)^{\alpha-1}(\tilde{\eta}-\eta)}{\Gamma(\alpha)}+\frac{(\tilde{\eta}-\eta)}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\frac{L^n\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\rho^{-1-n\alpha-\alpha}\left(t^\rho-t_0^\rho\right)^{n\alpha+\alpha+1}\\
&= \rho^{1-\alpha}\left(t^\rho-t_0^\rho\right)^{\alpha-1}(\tilde{\eta}-\eta)\sum_{n=0}^{\infty}\frac{\left(L\rho^{-\alpha}\left(t^\rho-\tau^\rho\right)^{\alpha}\right)^n}{\Gamma(n\alpha+\alpha)}\\
&= \left(\frac{t^\rho-t_0^\rho}{\rho}\right)^{\alpha-1}(\tilde{\eta}-\eta)E_{\alpha,\alpha}\left(L\left(\frac{t^\rho-t_0^\rho}{\rho}\right)^{\alpha}\right),
\end{aligned}$$

for $0 \leq t_0 < t \leq h$. □

4. Conclusion

In this article, we have established the Gronwall inequality in the frame work of the generalized fractional integrals that unify the Riemann-Liouville and Hadamard fractional integrals. Using this inequality we presented the dependence on the order and initial conditions of solutions of differential equations involving the generalized fractional derivatives. It turned out that when $\rho \to 1$ we obtain the Gronwall inequality and the consequent results for Riemann-Liouville fractional operators and $\rho \to 0$, the Gronwall inequality for Hadamrad fractional integrals is obtained. Initial value problems in the frame of generalized fractional operators in certain new weighted spaces of functions have been investigated as well.

5. Aknowlegement

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