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Maximality, Fixed Points and Variational Principles for Mappings on Quasi-Uniform Spaces

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Abstract. By making use of maximality on some appropriate preorderings, some classical results stated in the context of metric spaces are extended to spaces endowed with quasi-uniform structures. Indeed, various results on fixed point theory and variational principles have been proved by arguments using order relations in metric spaces. In this work, some of the mentioned results are extended to spaces having a quasi-uniform structure, by means of appropriate preorderings. The concept of w-distance is used to this purpose. Moreover, equivalences of maximality are stated for general preorderings.

1. Introduction

Several results about existence of fixed point and variational principles, for certain mappings, can be obtained by means of existence of maximal elements for suitable preorderings on the spaces where the mappings are defined. This is the case, for instance, in the articles by Brøndsted [3, 4], Ekeland [8, 9], Jachymski [13] and Takahashi [18], among others. Also, some variants of the Caristi-Kirk theorem [6] have been proved through an argument of maximality. An interesting work for obtaining maximality, under quite general assumptions and without topological conditions, is that by Brézis and Browder [2], where a number of results are unified.

Park in [17] states five equivalent conditions to maximality with respect to a specific preordering defined on a metric space. In this paper, we prove that these equivalences hold for arbitrary preorderings, without metric considerations, and two additional conditions are added to this set of equivalences. Moreover, when the set is endowed with a quasi-uniform topological structure, we state conditions under which maximality hold. Also, some Brøndsted type preorderings are defined, by means of w-distances, and conditions for maximality are stated. In particular, extensions of results such as Mizoguchi [14] are obtained from these preorderings, which are used to extend the Ekeland variational principle [8], the nonconvex minimization theorem according to Takahashi [19] and the Caristi theorem [5] to our setting. Also, following Weston [21], the completeness of a quasi-uniform space, with respect to a w-distances, is characterized.

We are mainly interested in results based on uniform spaces. Most of the results of this paper are based on assumptions related with w-distances and every w-distance, with respect to a quasi-uniformity, is also

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a w-distance with respect to a uniformity generating a finer topology than that generated by the quasi-uniformity. For this reason, we have preferred to state the results in terms of quasi-uniform spaces, even though much of the results based on quasi-uniform spaces are slight extensions of the corresponding results based on uniform spaces. Our interest in starting this research in the context of uniform spaces comes from the necessity of unifying some results related with mappings defined on metric spaces and others defined on topological vector spaces. Indeed, even though the completeness concept there exists both for metric and topological vector spaces, some metric spaces are not linear and some linear spaces are not metrizable. This fact, along with the convenience of developing fixed point theory and variational principles into a structure containing both of these class of spaces, have motivated us to consider the framework of the quasi-uniform structures, which includes uniformities, as an appropriate scenario to carry out our study. Other spaces such as the Menger (probabilistic) metric spaces are included in this setting. Furthermore, Fang in [10] proved that these classes are F-type topological spaces, which, according to Hamel [12] (see also [11]) coincide with the category of uniform spaces.

Including this introduction, the paper is divided in five sections. In Section 2, the above-mentioned equivalences of maximality are included and general conditions under which these equivalences hold are stated. In Section 3, some particular Brøndsted type preorderings, defined by means of w-distances, are considered. Also, an old result by Weston and other by Oettli and Théra are included in the setting of quasi-uniform spaces. Section 4 is devoted for some fixed point theorems, where extended versions of Caristi's theorem are presented. Finally, we devote Section 5 to extensions to quasi-uniform spaces of the Ekeland variational principle and of Takahashi's nonconvex minimization theorem.

2. Preliminaries

Let $\preceq$ be a preordering on a nonempty set X, i.e. for all $x \in X$, $x \preceq x$ ($\preceq$ is reflexive) and for all $x, y, z \in X$, $x \preceq y$ and $y \preceq z$ imply $x \preceq z$, ($\preceq$ is transitive). For each $x \in X$, we denote $S(x, \preceq) = \{y \in X : x \preceq y\}$.

Theorem 2.1. *Let $x_0 \in X$. The following eight conditions are equivalent:*

(2.1.1) *there exists a maximal element $x^* \in X$ such that $x_0 \preceq x^*$;*

(2.1.2) *there exists $x_1 \in S(x_0, \preceq)$ such that for each chain C in $S(x_1, \preceq)$, $\bigcap_{x \in C} S(x, \preceq) \neq \emptyset$;*

(2.1.3) *there exist $x_1 \in S(x_0, \preceq)$ and a maximal chain C^* in $S(x_1, \preceq)$ such that $\bigcap_{x \in C^*} S(x, \preceq) \neq \emptyset$;*

(2.1.4) *for each $T : S(x_0, \preceq) \to 2^X$ such that, for each $x \in S(x_0, \preceq) \setminus Tx$, there exists $y \in X \setminus \{x\}$ satisfying $x \preceq y$, there exists $z \in S(x_0, \preceq)$ such that $z \in Tz$;*

(2.1.5) *any function $f : S(x_0, \preceq) \to X$ such that $x \preceq f(x)$, for all $x \in S(x_0, \preceq)$, has a fixed point;*

(2.1.6) *for each $T : S(x_0, \preceq) \to 2^X \setminus \{\emptyset\}$ such that $x \preceq y$, for all $x \in S(x_0, \preceq)$ and $y \in Tx$, there exists $z \in S(x_0, \preceq)$ such that $Tz = \{z\}$;*

(2.1.7) *any family $\mathcal{F}$ of functions $f : S(x_0, \preceq) \to X$ such that $x \preceq f(x)$, for all $x \in S(x_0, \preceq)$, has a common fixed point;*

(2.1.8) *for any subset Y of X such that $S(x_0, \preceq) \cap Y = \emptyset$, there exists $x \in S(x_0, \preceq) \setminus Y$ satisfying $S(x, \preceq) = \{x\}$.*

Proof. Suppose (2.1.1) holds. By choosing $x_1 = x^*$, we have $C = \{x^*\}$ is the unique chain in $S(x_1, \preceq)$ and $\bigcap_{x \in C} S(x, \preceq) = S(x^*, \preceq) \neq \emptyset$, which proves (2.1.2).

Let x_1 be as in (2.1.2). By Hausdorff maximal principle, there exists a maximal chain C^* in $S(x_1, \preceq)$ and from (2.1.2), $\bigcap_{x \in C^*} S(x, \preceq) \neq \emptyset$. Thus (2.1.2) implies (2.1.3).

Let us assume (2.1.3), T be as in (2.1.4), $x_1 \in S(x_0, \preceq)$ and C^* be a maximal chain in $S(x_1, \preceq)$, and $z \in \bigcap_{x \in C^*} S(x, \preceq)$. Suppose that $z \notin Tz$. From assumption, there exists $y \in X \setminus \{z\}$ such that $z \preceq y$. Thus y is an upper bound of C^* and $C^* \cup \{y\}$ is a chain in $S(x_1, \preceq)$, which contradicts the maximality of C^*. Hence, $z \in Tz$ and condition (2.1.4) holds.

By assuming (2.1.4) and taking f as in (2.1.5), we define $T : S(x_0, \preceq) \to 2^X$ such that $Tx = \{f(x)\}$. Suppose $f(x) \neq x$, for all $x \in S(x_0, \preceq)$. Since for each $x \in S(x_0, \preceq) \setminus Tx$, $f(x) \in X \setminus \{x\}$ and $x \preceq f(x)$, condition (2.1.4) implies that there exists $z \in S(x_0, \preceq)$ such that $z \in Tz$, which is a contradiction. Consequently, f has a fixed point.

Let us assume condition (2.1.5) and let $T : S(x_0, \preceq) \to 2^X \setminus \{\emptyset\}$ be a function such that $x \preceq y$, for all $x \in S(x_0, \preceq)$ and all $y \in Tx$. Suppose that $Tz \setminus \{z\} \neq \emptyset$, for all $z \in S(x_0, \preceq)$. Let $f : S(x_0, \preceq) \to X$ be a selection of the set-valued function $F : S(x_0, \preceq) \to 2^X \setminus \{\emptyset\}$ defined as $Fz = Tz \setminus \{z\}$. Hence, f has no fixed point and $f(x) \preceq x$, for all $x \in S(x_0, \preceq)$. However, condition (2.1.5) implies that f has a fixed point, which is a contradiction. Consequently, there exists $z \in S(x_0, \preceq)$ such that $Tz \setminus \{z\} = \emptyset$ and condition (2.1.6) holds.

Let $\mathcal{F}$ be a family of functions $f : S(x_0, \preceq) \to X$ such that $x \preceq f(x)$, for all $x \in S(x_0, \preceq)$. Define $T : S(x_0, \preceq) \to 2^X \setminus \{\emptyset\}$ as $Tx = \{f(x) : f \in \mathcal{F}\}$ and suppose condition (2.1.6) holds. Notice that, for all $x \in S(x_0, \preceq)$ and $y \in Tx$, we have $x \preceq y$. Hence, condition (2.1.6) implies that there exists $z \in S(x_0, \preceq)$ such that $Tz = \{z\}$, which implies that any $f \in \mathcal{F}$ has a fixed point. Consequently, condition (2.1.7) holds.

Assume condition (2.1.7) and suppose that there exists a subset Y of X such that $S(x_0, \preceq) \cap Y = \emptyset$ and that for each $x \in S(x_0, \preceq) \setminus Y$, $S(x, \preceq) \neq \{x\}$. Since $S(x_0, \preceq) \setminus Y = S(x_0, \preceq)$, there exists a nonempty function $f : S(x_0, \preceq) \setminus Y \to X$ such that $x \preceq f(x)$ and $f(x) \neq x$, for all $x \in S(x_0, \preceq)$. Hence condition (2.1.7) implies that there exists $z \in S(x_0, \preceq)$ such that $f(z) = z$ and thus, $z \in S(x_0, \preceq) \cap Y$, which is a contradiction. Therefore, there exists $x \in S(x_0, \preceq) \setminus Y$ such that $S(x, \preceq) \setminus \{x\} = \emptyset$ and consequently condition (2.1.7) implies condition (2.1.8).

Next, suppose condition (2.1.8) holds and let $Y = \{x \in X : S(x, \preceq) = \{x\}\}$. Notice that for all $x \in S(x_0, \preceq) \setminus Y$, there exists $y \in S(x_0, \preceq) \setminus \{x\}$. Consequently, condition (2.1.8) implies that there exists $x^* \in S(x_0, \preceq) \cap Y$. Since $S(x^*, \preceq) = \{x^*\}$, x^* is maximal and $x^* \in S(x_0, \preceq)$, which prove condition (2.1.1) and the proof is complete. □

In this section, $(X, \mathcal{U})$ is a Hausdorff quasi-uniform space, i.e. $\mathcal{U}$ is a filter on $X \times X$ satisfying the axioms of a uniformity, with the possible exception of the symmetry axiom. For each $x \in X$, we denote

$$U[x] = \{y \in X : (x, y) \in U\}, \quad U \in \mathcal{U}.$$

These sets form a neighborhood basis of x, which induces a topology on X. In the sequel, we consider the space X endowed with this topology.

A filter base $\mathcal{B}$ on X is said to be $\mathcal{U}$-Cauchy, if for each $U \in \mathcal{U}$, there exist $x = x_U \in X$ and $B \in \mathcal{B}$ such that $B \subseteq U[x]$. The pair $(X, \mathcal{U})$ is said to be complete (respectively, sequentially complete), if every $\mathcal{U}$-Cauchy filter base on X (respectively, countable filter base) converges to some $y \in X$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathcal{U}$-Cauchy, if the filter base $\{B_n\}_{n\in\mathbb{N}}$ is $\mathcal{U}$-Cauchy, where $B_n = \{x_m; n \leq m\}$. Notice that, for uniform spaces, this definition of Cauchy filter base coincides with the usual definition. For general definitions about quasi-uniform spaces, we cite [15].

Lemma 2.2. *Let $x_0 \in X$, $\preceq$ be a preordering on X and suppose the following two conditions hold:*

(2.2.1) *for each $x \in S(x_0, \preceq)$, $S(x, \preceq)$ is closed; and*

(2.2.2) *for each totally ordered subset C of $S(x_0, \preceq)$, the filter base $\mathcal{B}_C = \{S(x, \preceq) \cap C : x \in C\}$ converges.*

Then, there exists a maximal element $x^* \in X$ such that $x_0 \preceq x^*$.

Proof. Let C be a totally ordered subset of $S(x_0, \preceq)$. From (2.2.1) and (2.2.2), $\mathcal{B}_C$ converges to some $v \in S(x_0, \preceq)$ and, since for each $x \in S(x_0, \preceq)$, $S(x, \preceq)$ is closed, we have

$$\{v\} = \bigcap_{x \in C} \overline{S(x, \preceq) \cap C} \subseteq \bigcap_{x \in C} S(x, \preceq),$$

which implies that v is an upper bound of C. Therefore, by Zorn's Lemma, there exists a maximal element $x^* \in S(x_0, \preceq)$, concluding de proof. □

Theorem 2.3 below states conditions under which conditions (2.1.1)-(2.1.8) hold.

Theorem 2.3. *Let $x_0 \in X$, $\preceq$ be a preordering on X and suppose the following two conditions hold:*

(2.3.1) *for each $y_0 \in S(x_0, \preceq)$, $S(y_0, \preceq)$ is complete; and*

(2.3.2) *for each $U \in \mathcal{U}$ and $y_0 \in S(x_0, \preceq)$, there exists $x_U \in S(y_0, \preceq)$ such that, $S(x_U, \preceq) \subseteq U[x_U]$.*

Then, there exists a maximal element $x^* \in X$ such that $x_0 \preceq x^*$ and hence conditions (2.1.2)-(2.1.8) hold for the preordering $\preceq$.

Proof. Let C be a totally ordered subset of $S(x_0, \preceq)$ and define $\mathcal{B}_C = \{S(x, \preceq) \cap C : x \in C\}$. Hence $\mathcal{B}$ is a filter base in $S(x_0, \preceq)$ and from (2.3.2), $\mathcal{B}_C$ is Cauchy. From (2.3.1), $\mathcal{B}_C$ converges to some $v \in S(x_0, \preceq)$ and since for each $x \in S(x_0, \preceq)$, $S(x, \preceq)$ is closed, it follows from Lemma 2.2 that there exists a maximal element $x^* \in S(x_0, \preceq)$, concluding de proof. □

When $\mathcal{U}$ is a uniformity on X, condition (2.3.2) can be weakened by mean of condition (2.4.2) below, in order to obtain the same conclusion in Theorem 2.3. This fact is expressed in the following theorem.

Theorem 2.4. *Let $x_0 \in X$, $\preceq$ be a preordering on X and suppose the following two conditions hold:*

(2.4.1) *$\mathcal{U}$ is a uniformity on X;*

(2.4.2) *for each $y_0 \in S(x_0, \preceq)$, $S(y_0, \preceq)$ is complete; and*

(2.4.3) *for each $U \in \mathcal{U}$ and $y_0 \in S(x_0, \preceq)$, there exists $x_U \in S(y_0, \preceq)$ such that, $x, y \in S(x_U, \preceq) \setminus \{x_U\}$ implies $(x, y) \in U$.*

Then, there exists a maximal element $x^* \in X$ such that $x_0 \preceq x^*$ and hence conditions (2.1.2)-(2.1.8) hold for the preordering $\preceq$.

Proof. Let C be a totally ordered subset of $S(x_0, \preceq)$ and define $\mathcal{B}_C = \{S(x, \preceq) \cap C : x \in C\}$. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ such that $V \circ V^{-1} \subseteq U$ and choose $y_0 \in S(x_0, \preceq) \cap C$. From (2.4.3), there exists $x_V \in S(y_0, \preceq)$, such that for any $x, y \in S(x_V, \preceq) \cap C$, we have $(x_V, x) \in V$ and $(x_V, y) \in V$. Hence $(x, y) \in U$ and from (2.4.1), $\mathcal{B}_C$ is a Cauchy filter base in $S(x_0, \preceq)$. By (2.4.2), $\mathcal{B}_C$ converges to some $v \in S(x_0, \preceq)$ and since for each $x \in S(x_0, \preceq)$, $S(x, \preceq)$ is closed, it follows from Lemma 2.2 that there exists a maximal element $x^* \in S(x_0, \preceq)$, which concludes the proof. □

For each $\phi : X \times X \to (-\infty, \infty]$ and each preordering $\preceq$ on X, we denote

$$D(\phi, \preceq) = \{x \in X : \inf_{y \in S(x, \preceq)} \phi(x, y) < \infty\}. \tag{1}$$

Corollary 2.5 below generalizes the Mizoguchi Lemma in [14].

Corollary 2.5. *Let $\preceq$ be a preordering on X, $\{\phi_U\}_{U \in \mathcal{U}}$ be a family of functions from $X \times X$ to $(-\infty, \infty]$ and $x_0 \in \bigcap_{U \in \mathcal{U}} D(\phi_U, \preceq)$. Suppose for each $y_0 \in S(x_0, \preceq)$ and $U \in \mathcal{U}$, $S(y_0, \preceq)$ is complete and the following two conditions hold:*

(2.5.1) *there exists $\delta_U > 0$ such that, $x_0 \preceq x \preceq y$ and $\phi_U(x_0, x) - \phi_U(x_0, y) < \delta_U$ imply $(x, y) \in U$; and*

(2.5.2) $\inf_{y \in S(y_0, \preceq)} \phi_U(x_0, y) > -\infty$.

Then, conditions (2.1.1)-(2.1.8) hold for the preordering $\preceq$.

Proof. Let $U \in \mathcal{U}$, $y_0 \in S(x_0, \preceq)$ and $\delta_U > 0$ as in (2.5.2). From (2.5.2), $L =: \inf_{y \in S(y_0, \preceq)} \phi_U(x_0, y) > -\infty$ and from assumption, $L < \infty$. Hence there exists $x_U \in S(y_0, \preceq)$ such that $L \leq \phi_U(x_0, x_U) < L + \delta_U$. Let $x \in S(x_U, \preceq)$. Hence $\phi_U(x_0, x_U) - \phi_U(x_0, x) < \delta_U$ and by (2.5.1), $(x_U, x) \in U$. Therefore, Theorem 2.3 implies that conditions (2.1.1)-(2.1.8) hold for the preordering $\preceq$ and the proof is complete. □

3. Brøndsted Type Preorderings

Let $p : X \times X \to [0, \infty)$ be a w-distance on $(X, \mathcal{U})$, i.e. p is a function satisfying the following three conditions:

(C1) $p(x, y) \leq p(x, z) + p(z, y)$, for any $x, y, z \in X$,

(C2) $p(x, \cdot)$ is lower semicontinuous, for each $x \in X$, and

(C3) for any $U \in \mathcal{U}$, there exists $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $(x, y) \in U$.

In the sequel, p stands for a w-distance on $(X, \mathcal{U})$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be p-Cauchy, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ satisfying $p(x_m, x_n) < \epsilon$, for all $m, n \geq N$. We say X is p-complete whenever for any p-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Given a preordering $\preceq$ on X, the pair $(X, \mathcal{U})$ is said to be $\preceq$-sequentially complete, if any nondecreasing $\mathcal{U}$-Cauchy sequence converges. The space X is said to be $(\preceq, p)$-complete, if for any nondecreasing p-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$.

Remark 3.1. *Let $\mathcal{U} \vee \mathcal{U}^{-1} = \{U \cap U^{-1} : U \in \mathcal{U}\}$. It is easy to see that, the w-distance p on $(X, \mathcal{U})$ is also a w-distance p on the uniform space $(X, \mathcal{U} \vee \mathcal{U}^{-1})$. Consequently, for statements based on w-distances, it is not a great extension to assume $\mathcal{U}$ is a quasi-uniformity, instead of a uniformity.*

Lemma 3.2. *Any p-Cauchy sequence is a $\mathcal{U}$-Cauchy sequence.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a p-Cauchy sequence in X and $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be the filter base defined as $B_n = \{x_m; n \leq m\}$. Let $U \in \mathcal{U}$. Hence, there exists $\delta > 0$ such that $p(z, u) < \delta$ and $p(z, v) < \delta$ imply $(u, v) \in U$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a p-Cauchy sequence, there exists $N \in \mathbb{N}$ such that $p(x_m, x_n) < \delta$ for all $m, n \geq N$. Hence, $B_N \subseteq U[x_N]$, due to $p(x_N, x_N) < \delta$ and $p(x_N, x_m) < \delta$ for $m \geq N$. Thus, $\mathcal{B}$ is a Cauchy filter base in X and the proof is complete. □

The following proposition states the relation between completeness with respect to the quasi-uniformity and p-completeness.

Proposition 3.3. *Suppose $(X, \mathcal{U})$ is sequentially complete. Then, X is p-complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a p-Cauchy sequence in X. From Lemma, $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ is a $\mathcal{U}$-Cauchy sequence, where $B_n = \{x_m; n \leq m\}$. Since $(X, \mathcal{U})$ is sequentially complete, there exists $x \in X$ such that $\mathcal{B}$ converges to x. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $p(x_m, x_n) < \epsilon$ whenever $m, n \geq N$. From the lower semicontinuity of $p(x_m, \cdot)$, we have $p(x_m, x) \leq \epsilon$, for each $m \geq N$, i.e. $\lim_{n\to\infty} p(x_n, x) = 0$. Therefore, X is p-complete, which concludes the proof. □

Corollary 3.4. *Let $\preceq$ be a preordering on X and suppose $(X, \mathcal{U})$ is $\preceq$-sequentially complete. Then, X is $(\preceq, p)$-complete.*

Proof. It follows from Lemma 3.2 and Proposition 3.3. □

Although a p-complete subspace of X need not to be closed, the following proposition holds.

Proposition 3.5. *Suppose X is p-complete and let F be a closed subset of X. Then, F is p-complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a p-Cauchy sequence in F. From assumption, there exists $x \in X$ such that $p(x_n, x) \to 0$. Suppose $x \notin F$. Hence there exists $U \in \mathcal{U}$ such that $U[x] \cap F = \emptyset$. From (C3), there exists $\delta > 0$ such that $p(z, u) < \delta$ and $p(z, v) < \delta$ implies $(u, v) \in U$. Choose $N \in \mathbb{N}$ such that $p(x_N, x_m) < \delta$ and $p(x_m, x) < \delta$ whenever $m \geq N$. Hence $p(x_N, x_N) < \delta$ and $p(x_N, x) < \delta$. Consequently, $(x, x_N) \in U$, which is a contradiction due to $x_N \in F$. Therefore, $x \in F$ and the proof is complete. □

Let $\Phi[X, \mathcal{U}]$ be the set of all functions $\phi : X \times X \to (-\infty, \infty]$ such that the following two conditions hold:

(C4) $\phi(x, y) \leq \phi(x, z) + \phi(z, y)$, for any $x, y, z \in X$; and

(C5) $\phi(x,\cdot) : X \to (-\infty,\infty]$ is lower semicontinuous, for any $x \in X$.

Given $\phi \in \Phi[X,\mathcal{U}]$, we consider the preordering $\preceq_\phi$ defined as

$$x \preceq_\phi y, \quad \text{if and only if,} \quad x = y \text{ or } \phi(x,y) + p(x,y) \leq 0.$$

Remark 3.6. *Let $x \in X$, $\phi \in \Phi[X,\mathcal{U}]$ and notice that, for all $u,v \in X$ such that $u \preceq_\phi v$ and $u \neq v$, we have $\phi(u,x) \leq \phi(u,v) + \phi(v,x) \leq \phi(v,x)$ and $\phi(x,v) \leq \phi(x,u) + \phi(u,v) \leq \phi(x,u)$. Consequently, $\phi(\cdot,x)$ is increasing and $\phi(x,\cdot)$ is decreasing. In particular, for each $x \in X$,*

$$\inf_{y\in S(x,\preceq_\phi)} \phi(x,y) = \phi(x,x). \tag{2}$$

and from (1)*, we have*

$$D(\phi,\preceq_\phi) = \{x \in X : \phi(x,x) < \infty\}.$$

Theorem 3.7. *Let $\phi \in \Phi[X,\mathcal{U}]$, $x_0 \in D(\phi,\preceq_\phi)$ and suppose $S(x_0,\preceq_\phi)$ is $(\preceq_\phi,p)$-complete. Then, conditions (2.1.1)-(2.1.8) hold for the preordering $\preceq_\phi$.*

Proof. The proof is similar to that in Theorem 2.1 in [1]. If for some $x \in S(x_0,\preceq_\phi)$, $\phi(x,y) + p(x,y) > 0$ for all $y \in S(x_0,\preceq_\phi)\setminus\{x\}$, we have x is maximal and we are done. Hence, we assume that for each $x \in S(x_0,\preceq_\phi)$, there exists $y \in S(x,\preceq_\phi) \setminus \{x\}$. Since from (2), $\inf_{y\in S(x_0,\preceq_\phi)} \phi(x_0,y) > -\infty$, we define recursively an nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by means of

$$x_n \in S(x_{n-1},\preceq_\phi) \setminus \{x_{n-1}\} \quad \text{with} \quad \phi(x_{n-1},x_n) < L_n + 1/n,$$

where $L_n = \inf\{\phi(x_{n-1},y) : y \in S(x_{n-1},\preceq_\phi) \setminus \{x_{n-1}\}\}$. Consequently, for each $n,q \in \mathbb{N} \setminus \{0\}$ and $y \in S(x_{n+q-1},\preceq_\phi) \setminus \{x_{n+q-1}\} \subseteq S(x_{n-1},\preceq_\phi) \setminus \{x_{n-1}\}$, we have

$$p(x_n,y) \leq -\phi(x_n,y) \leq \phi(x_{n-1},x_n) - \phi(x_{n-1},y) \leq \phi(x_{n-1},x_n) - L_n < \frac{1}{n}.$$

In particular, $p(x_n,x_{n+q}) < 1/n$ and hence $\{x_n\}_{n\in\mathbb{N}}$ is a p-Cauchy sequence. Since $S(x_0,\preceq_\phi)$ is $(\preceq_\phi,p)$-complete there exists $x^* \in S(x_0,\preceq_\phi)$ such that $\lim_{n\to\infty} p(x_n,x^*) = 0$. Moreover, for $m \geq n$, $p(x_n,x_m) + \phi(x_n,x_m) \leq 0$ and $p(x_n,\cdot) + \phi(x_n,\cdot)$ is lower semicontinuous. Hence $x_n \preceq_\phi x^*$, for all $n \in \mathbb{N}$. Suppose $y \in S(x_0,\preceq_\phi)$ satisfies $x^* \preceq_\phi y$. If $x^* = y$, we are done. Otherwise $x_n \preceq_\phi x^* \preceq_\phi y$, for all $n \in \mathbb{N}$, i.e. $p(x_n,y) + \phi(x_n,y) \leq 0$ and thus $p(x_n,y) \leq -\phi(x_n,y) < 1/n$ and $\lim_{n\to\infty} p(x_n,y) = 0$. But from (C3), the limit respect to p is unique and thus $x^* = y$. Therefore $x^* \in S(x_0,\preceq_\phi)$ is a maximal element, which concludes the proof. □

Due to Corollary 3.4, when (X,d) is a quasi-metric space, Theorem 1 in [17] follows in the more general form. This result is stated as follows.

Corollary 3.8. *Let (X,d) be a quasi-metric space, $\phi \in \Phi[X,\mathcal{U}]$, $x_0 \in D(\phi,\preceq_\phi)$ and suppose $S(x_0,\preceq_\phi)$ is $(\preceq_\phi,p)$-complete. Then, conditions (2.1.1)-(2.1.8) hold for the preordering $\preceq_\phi$.*

Proof. It directly follows due to any quasi-metric space is quasi-uniformizable. □

Theorem 3.9 below extends Theorem 4 by Oettli and Théra in [16], in the setting of uniform spaces and with weaker assumptions.

Theorem 3.9. *Let $\phi \in \Phi[X,\mathcal{U}]$, $x_0 \in X$, $Y \subseteq X$ and suppose the following two conditions hold:*

(3.9.1) *$S(x_0,\preceq_\phi)$ is p-complete; and*

(3.9.2) *for each $x \in S(x_0,\preceq_\phi) \setminus Y$, there exists $y \in X$ such that $x \preceq_\phi y$ and $x \neq y$.*

Then, $S(x_0,\preceq_\phi) \cap Y \neq \emptyset$.

Proof. Suppose $S(x_0, \preceq_\phi) \cap Y = \emptyset$. Hence $x_0 \in S(x_0, \preceq_\phi) = S(x_0, \preceq_\phi) \setminus Y$ and from (3.9.2), there exists $y \in X$ such that $\phi(x_0, y) \leq 0$. Thus, $x_0 \in D(\phi, \preceq_\phi)$ and Theorem 3.7 implies condition (2.1.8) in Theorem 2.1 holds, and hence, there exists $x \in S(x_0, \preceq_\phi) \setminus Y$ such that $S(x, \preceq_\phi) \setminus \{x\} = \emptyset$. But this fact contradicts condition (3.9.2) and therefore $S(x_0, \preceq_\phi) \cap Y \neq \emptyset$, which concludes the proof. □

An old result by Weston in [21] admits an extension to quasi-uniform spaces as follows.

Theorem 3.10. *Suppose p is symmetric and for each lower semicontinuous and bounded below function $f : X \to (-\infty, \infty)$, there exists a maximal element $x^* \in X$ with respect to $\preceq_\phi$, where $\phi(x, y) = f(y) - f(x)$. Then, X is p-complete.*

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ be a p-Cauchy sequence in X. Since p is symmetric, for each $x \in X$ and $m, n \in \mathbb{N}$, we have

$$|p(x, x_m) - p(x, x_n)| \leq p(x_m, x_n).$$

Hence, there exists $\lim_{n\to\infty} p(x, x_n)$. Let $f : X \to (-\infty, \infty)$ such that $f(x) = 2\lim_{n\to\infty} p(x, x_n)$. For each $\alpha \in \mathbb{R}$, we have

$$\{x \in X : f(x) > \alpha\} = \bigcup_{n=0}^{\infty} \bigcup_{\gamma > \alpha} \bigcup_{m=n}^{\infty} \{x \in X : p(x, x_m) > \gamma\}$$

and since for each $m \in \mathbb{N}$, $p(\cdot, x_m)$ is lower semicontinuous, f so is. The maximality of x^* implies that

$$f(x_n) + p(x^*, x_n) \geq f(x^*), \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Observe that $\lim_{m\to\infty} f(x_m) = 0$. Hence by taking limit in (3), we obtain $f(x^*) \geq 2f(x^*)$ and consequently $f(x^*) = 0$. The symmetry of p implies that $\lim_{n\to\infty} p(x_n, x^*) = 0$. Therefore, X is p-complete, which concludes the proof. □

For a family $\{\phi_U\}_{U \in \mathcal{U}}$ of functions belonging to $\Phi[X, \mathcal{U}]$, a more general preordering $\preceq_{\mathcal{U}}$ on X is defined as follows:

$$x \preceq_{\mathcal{U}} y, \quad \text{if and only if,} \quad \text{for each } U \in \mathcal{U}, x = y \text{ or } \phi_U(x, y) + p(x, y) \leq 0.$$

Notice that for each $x \in X$ and $U \in \mathcal{U}$, $\phi(x, \cdot)$ is decreasing with respect to $\preceq_{\mathcal{U}}$. Hence,

$$\inf_{y \in S(x, \preceq_{\mathcal{U}})} \phi_U(x, y) = \phi_U(x, x) > -\infty. \tag{4}$$

Theorem 3.11 below improves the conclusion of Theorem 3.7, however its assumptions are stronger.

Theorem 3.11. *Let $\{\phi_U\}_{U \in \mathcal{U}}$ be a family in $\Phi[X, \mathcal{U}]$ and $x_0 \in \bigcap_{U \in \mathcal{U}} D(\phi_U, \preceq_{\mathcal{U}})$. Suppose $S(x_0, \preceq_{\mathcal{U}})$ is complete and one of the following two conditions hold:*

(3.11.1) *for each $x \in X$, $p(x, x) = 0$; and*

(3.11.2) *$\mathcal{U}$ is a uniformity.*

Then, conditions (2.1.1)-(2.1.8) hold for the preordering $\preceq_{\mathcal{U}}$.

Proof. Let $y_0 \in S(x_0, \preceq_{\mathcal{U}})$. Since for each $U \in \mathcal{U}$, $p(y_0, \cdot) + \phi_U(y_0, \cdot)$ is lower semicontinuous, $S(y_0, \preceq_{\mathcal{U}})$ is closed and thus $S(y_0, \preceq_{\mathcal{U}})$ is complete. Hence, condition (2.3.1) holds for each function ϕ_U. Consequently, it suffices to verify that $\preceq_{\mathcal{U}}$ satisfies condition (2.3.2). Let $U \in \mathcal{U}$, $y_0 \in S(x_0, \preceq_{\mathcal{U}})$ and $\delta_U > 0$ such that, $p(z, x) < \delta_U$ and $p(z, y) < \delta_U$ imply $(x, y) \in U$. Let $L_U := \inf_{y \in S(y_0, \preceq_{\mathcal{U}})} \phi_U(x_0, y)$. Since $x_0 \in D(\phi_U, \preceq_{\mathcal{U}})$ and (4) holds, we have $-\infty < L_U < \infty$. Consequently, there exists $x_U \in S(y_0, \preceq_{\mathcal{U}})$ such that $L_U \leq \phi_U(x_0, y_0) \leq \phi_U(x_0, x_U) < L_U + \delta_U$. Assume $S(x_U, \preceq_{\mathcal{U}}) \setminus \{x_U\} \neq \emptyset$ and let $x \in S(x_U, \preceq_{\mathcal{U}}) \setminus \{x_U\}$. Hence

$$p(x_U, x) \leq -\phi_U(x_U, x) \leq \phi_U(x_0, x_U) - \phi_U(x_0, x) < \delta_U. \tag{5}$$

In case condition (3.11.1) holds, we have $p(x_U, x_U) < \delta_U$ and from (C3), $(x_U, x) \in U$. Thus it follows from Theorem 2.3 that conditions (2.1.1)-(2.1.8) hold for the ordering $\preceq_{\mathcal{U}}$. Otherwise, condition (3.11.2) holds and $x, y \in S(x_U, \preceq_{\mathcal{U}}) \setminus \{x_U\}$ implies that $\max\{p(x_U, x), p(x_U, y)\} < \delta_U$. By (5) and (C3), $(x, y) \in U$ and, accordingly, condition (2.4.2) holds. Hence, Theorem 2.4 implies that conditions (2.1.1)-(2.1.8) hold for the ordering $\preceq_{\mathcal{U}}$ and the proof is complete. □

Corollary 3.12. *Let* $\{\phi_U\}_{U\in\mathcal{U}}$ *be a family in* $\Phi[X,\mathcal{U}]$*,* $x_0 \in \bigcap_{U\in\mathcal{U}} D(\phi_U, \preceq_{\mathcal{U}})$ *and suppose* (X, d) *is a complete quasi-metric space such that, for each* $x \in X$*,* $p(x, x) = 0$*. Then, conditions (2.1.1)-(2.1.8) hold for the preordering* $\preceq_{\mathcal{U}}$*.*

Corollary 3.13. *Let* $\{\phi_U\}_{U\in\mathcal{U}}$ *be a family in* $\Phi[X,\mathcal{U}]$*,* $x_0 \in \bigcap_{U\in\mathcal{U}} D(\phi_U, \preceq_{\mathcal{U}})$ *and suppose* $(X,\mathcal{U})$ *is a complete uniform space. Then, conditions (2.1.1)-(2.1.8) hold for the preordering* $\preceq_{\mathcal{U}}$*.*

4. Fixed Point Results

Let 2^X be the family of all nonempty subsets of X. Two extensions of the Caristi fixed point theorem are stated as Theorem 4.3 and Corollary 4.4 in this section.

Theorem 4.1. *Let* $\phi \in \Phi[X,\mathcal{U}]$*,* $x_0 \in D(\phi, \preceq_\phi)$*,* $T : X \to 2^X$ *be a set-valued mapping and suppose* $S(x_0, \preceq_\phi)$ *is* $(\preceq_\phi, p)$*-complete. The following two propositions hold:*

(4.1.1) *If for each* $x \in S(x_0, \preceq_\phi)$*, there exists* $y \in Tx$ *such that* $\phi(x, y) + p(x, y) \le 0$*, then there exists* $x^* \in S(x_0, \preceq_\phi)$ *such that* $x^* \in Tx^*$*.*

(4.1.2) *If for each* $x \in S(x_0, \preceq_\phi)$ *and each* $y \in Tx$*,* $\phi(x, y) + p(x, y) \le 0$*, then there exists* $x^* \in S(x_0, \preceq_\phi)$ *such that* $\{x^*\} = Tx^*$*.*

Proof. From assumptions and Theorem 3.7, conditions (2.1.4) and (2.1.6) hold. Let $x \in S(x_0, \preceq)$ and $y \in Tx$ such that $\phi(x, y) + p(x, y) \le 0$. Suppose $x \notin Tx$. Hence $x \in S(x_0, \preceq) \setminus Tx$, $y \in X \setminus \{x\}$ and $x \preceq_\phi y$. Consequently, condition (2.1.4) implies that there exists $x^* \in S(x_0, \preceq_\phi)$ such that $x^* \in Tx^*$, which proves (4.1.1).

Next, let $x \in S(x_0, \preceq)$ and suppose for each $y \in Tx$, we have $\phi(x, y) + p(x, y) \le 0$. Accordingly, $x \preceq_\phi y$ and from (2.1.6) there exists $x^* \in S(x_0, \preceq_\phi)$ such that $\{x^*\} = Tx^*$. Therefore, the proof is complete. □

Theorem 4.2. *Let* $\{\phi_U\}_{U\in\mathcal{U}}$ *be a family of functions belonging to* $\Phi[X,\mathcal{U}]$*,* $x_0 \in \bigcap_{U\in\mathcal{U}} D(\phi_U, \preceq_{\mathcal{U}})$ *and* $T : X \to 2^X$ *be a set-valued mapping. Suppose* $S(x_0, \preceq_{\mathcal{U}})$ *is complete and one of the following two conditions hold:*

(4.2.1) *for each* $x \in X$*,* $p(x, x) = 0$*; and*

(4.2.2) $\mathcal{U}$ *is a uniformity.*

The following two propositions hold:

(4.2.3) *If for each* $x \in S(x_0, \preceq_{\mathcal{U}})$*, there exists* $y \in Tx$ *such that for each* $U \in \mathcal{U}$*,* $\phi_U(x, y) + p(x, y) \le 0$*, then there exists* $x^* \in S(x_0, \preceq_{\mathcal{U}})$ *such that* $x^* \in Tx^*$*.*

(4.2.4) *If for each* $x \in S(x_0, \preceq_{\mathcal{U}})$ *and each* $y \in Tx$*,* $\phi_U(x, y) + p(x, y) \le 0$*, for each* $U \in \mathcal{U}$*, then there exists* $x^* \in S(x_0, \preceq_{\mathcal{U}})$ *such that* $\{x^*\} = Tx^*$*.*

Proof. From assumptions and Theorem 3.11, conditions (2.1.4) and (2.1.6) hold. Consequently, this proof concludes in a similar way to the proof of Theorem 4.1. □

It is well-known that a uniformity $\mathcal{U}$ is generated by a family of pseudo metrics $\{d_\lambda\}_{\lambda\in\Lambda}$ on X. Let Φ_0 be the set of all $\phi \in \Phi[X,\mathcal{U}]$ such that $\phi(x, x) = 0$, for all $x \in X$. In the sequel, $\{\phi_\lambda\}_{\lambda\in\Lambda}$ and $\{d_\lambda\}_{\lambda\in\Lambda}$ stand for a family of functions in Φ_0 and a family of pseudo metrics, respectively. An ordering $\preceq_\Lambda$ on X is defined as

$$x \preceq_\Lambda y, \quad \text{if and only if,} \quad \phi_\lambda(x, y) + d_\lambda(x, y) \le 0, \text{ for all } \lambda \in \Lambda.$$

Theorem 4.3 below follows from Theorem 3.11 and is an extension of the well-known theorem by Caristi [7], for single valued functions.

Theorem 4.3. *Let* $f : X \to X$ *be an arbitrary function and* $x_0 \in \bigcap_{\lambda\in\Lambda} D(\phi_\lambda, \preceq_\Lambda)$ *such that* $S(x_0, \preceq_\Lambda)$ *is complete. Suppose for each* $\lambda \in \Lambda$*, the following two conditions hold:*

(4.3.1) $\inf_{y\in S(x_0,\preceq_\Lambda)} \phi_\lambda(x_0, y) > -\infty$ *and*

(4.3.2) *for each* $x \in S(x_0, \preceq_\Lambda)$, $\phi_\lambda(x, f(x)) + d_\lambda(x, f(x)) \leq 0$.

Then, there exists $x^* \in S(x_0, \preceq_\Lambda)$ *such that* $f(x^*) = x^*$.

Proof. Fix $\lambda \in \Lambda$. As before, $\phi_\lambda(x_0, \cdot)$ is decreasing with respect to $\preceq_\Lambda$ and hence

$$L_\lambda := \inf_{y \in S(x_0, \preceq_\Lambda)} \phi_\lambda(x_0, y) > -\infty.$$

Let $\epsilon > 0$ and choose $x_\lambda \in S(x_0, \preceq_\Lambda)$ such that $\phi_\lambda(x_\lambda) < L_\lambda + \epsilon$. For any $x, y \in X$ such that $x_0 \preceq x \preceq y$, we have

$$d_\lambda(x, y) \leq -\phi_\lambda(x, y) \leq \phi_\lambda(x_0, x) - \phi_\lambda(x_0, y).$$

Consequently, from Corollary 2.5, condition (2.1.5) holds. On the other hand, from (4.3.2), for each $x \in S(x_0, \preceq_\Lambda)$, we have $x \preceq_\lambda f(x)$. Therefore, condition (2.1.5) implies that there exists $x^* \in S(x_0, \preceq_\Lambda)$ such that $f(x^*) = x^*$, which concludes the proof. □

Let $\mathcal{F} = \{k_\lambda\}_{\lambda \in \Lambda}$ be a family of constants such that for each $\lambda \in \Lambda$, $0 \leq k_\lambda < 1$ and $f : X \to X$. We say f is an $\mathcal{F}$-contractive mapping, if for any $x, y \in X$, $d_\lambda(f(x), f(y)) \leq k_\lambda d_\lambda(x, y)$.

The following corollary is an old result by Tarafdar in [20].

Corollary 4.4. *Suppose* $(X, \mathcal{U})$ *is complete and let* $\mathcal{F} = \{k_\lambda\}_{\lambda \in \Lambda}$ *be a family of constants such that for each* $\lambda \in \Lambda$, $0 \leq k_\lambda < 1$ *and* $f : X \to X$ *be an* $\mathcal{F}$*-contractive mapping. Then, there exists a unique* $x^* \in X$ *such that* $f(x^*) = x^*$.

Proof. For each $x \in X$ and $\lambda \in \Lambda$, we have $(1 - k_\lambda)d_\lambda(x, f(x)) \leq d_\lambda(x, f(x)) - d_\lambda(f(x), f^2(x))$ and hence $\phi_\lambda(x, f(x)) + d_\lambda(x, f(x)) \leq 0$, where

$$\phi_\lambda(x, y) = [d_\lambda(y, f(y)) - d_\lambda(x, f(x))]/(1 - k_\lambda).$$

Since ϕ_λ is continuous and bounded below, Theorem 4.3 applies and thus, f has a fixed point.

The existence of two fixed points x and y of f, implies that for each $\lambda \in \Lambda$,

$$d_\lambda(x, y) = d_\lambda(f(x), f(y)) \leq k_\lambda d_\lambda(x, y)$$

and hence $d_\lambda(x, y) = 0$, for all $\lambda \in \Lambda$. Consequently, uniqueness follows and the proof is complete. □

5. Variational Principles

A function $f : X \to (-\infty, \infty]$ is said to be proper, whenever there exists $x \in X$ such that $f(x) < \infty$.

The following results are extensions of Theorem 1.1 by Ekeland in [8] to quasi-uniform spaces.

Theorem 5.1. *Let* $\epsilon > 0$, $\lambda > 0$ *and* $\phi \in \Phi[X, \mathcal{U}]$. *Moreover, suppose the following two conditions:*

(5.1.1) $S(x_0, \preceq_\phi)$ *is* $(\preceq_\phi, p)$*-complete; and*

(5.1.2) $\inf_{y \in X} \phi(x_0, y) < 0 < \inf_{y \in X} \phi(x_0, y) + \epsilon$.

Then, there exists $x^* \in X$ *such that the following three conditions hold:*

(5.1.3) $\phi(x_0, x^*) \leq 0$, *whenever* $x_0 \neq x^*$;

(5.1.4) $p(x_0, x^*) < \lambda$, *whenever* $x_0 \neq x^*$; *and*

(5.1.5) *for every* $x \in X \setminus \{x^*\}$, $(\epsilon/\lambda)p(x^*, x) > -\phi(x^*, x)$.

Proof. Let $\psi = (\lambda/\epsilon)\phi$. Clearly, ψ satisfies assumptions in Theorem 3.7 and hence there exists a maximal element $x^* \in S(x_0, \preceq_\psi)$. Accordingly, $x_0 = x^*$ or $(\epsilon/\lambda)p(x_0, x^*) \leq -\phi(x_0, x^*) < \epsilon$ and consequently (5.1.3) and (5.1.4) hold. Finally, since x^* is maximal, for every $x \in X \setminus \{x^*\}$, $x^* \not\preceq_\psi x$ and consequently (5.1.5) holds. This concludes the proof. □

Corollary 5.2. *Suppose* $(X, \mathcal{U})$ *is sequentially complete and let* $\epsilon > 0$, $\lambda > 0$ *and* $f : X \to (-\infty, \infty]$ *be a proper and bounded below lower semicontinuous function. Then, for every* $x_0 \in X$ *satisfying* $\inf_{y \in X} f(y) < f(x_0) < \inf_{y \in X} f(y) + \epsilon$, *there exists* $x^* \in X$ *such that the following three conditions hold:*

(5.2.1) $f(x^*) \leq f(x_0)$;

(5.2.2) $p(x_0, x^*) < \lambda$, *whenever* $x_0 \neq x^*$; *and*

(5.2.3) *for every* $x \in X \setminus \{x^*\}$, $f(x) + (\epsilon/\lambda)p(x^*, x) > f(x^*)$.

Proof. Fix $x_0 \in X$ such that $\inf_{y \in X} f(y) < f(x_0) < \inf_{y \in X} f(y) + \epsilon$ and let $\phi(x, y) = f(y) - f(x)$. Hence (5.1.2) holds and by Propositions 3.3 and 3.5, $S(\phi, \preceq_\phi)$ is p-complete. Consequently, ϕ satisfies assumptions in Theorem 5.1 and thus (5.2.1)-(5.2.3) hold, which completes the proof. □

An extension of the nonconvex minimization theorem according to Takahashi [19] can be stated as follows.

Theorem 5.3. *Let* $\phi \in \Phi_0$ *and* $x_0 \in D(\phi, \preceq)$ *such that the following conditions hold:*

(5.3.1) *X is* $(\preceq_\phi, p)$*-complete;*

(5.3.2) *there exists* $x_0 \in X$ *such that* $\inf_{y \in X} \phi(x_0, y) > -\infty$; *and*

(5.3.3) *for each* $x \in X$ *such that* $\inf_{y \in X} \phi(x, y) < 0$, *there exists* $y \in X \setminus \{x\}$ *satisfying* $\phi(x, y) + p(x, y) \leq 0$.

Then, there exists $x \in X$ *such that* $\inf_{y \in X} \phi(x, y) = 0$.

Proof. Suppose for every $x \in X$, $\inf_{y \in X} \phi(x, y) < 0$. Hence, there exists $y_0 \in X$ such that $\phi(x_0, y_0) < 0$ and by Theorem 3.7, $\preceq_\phi$ has a maximal element $x^* \in X$ such that $y_0 \preceq_\phi x^*$. We have $\phi(x_0, x^*) \leq \phi(x_0, y_0)$ and hence $\phi(x_0, x^*) < 0$. Consequently, condition (5.3.3) implies that there exists $x \in X \setminus \{x^*\}$ such that $x^* \preceq_\phi x$. But, the maximality of x^* implies $x = x^*$, which is a contradiction. Therefore, there exists $x \in X$ such that $\inf_{y \in X} \phi(x, y) \geq 0$. But $\phi(x, x) = 0$ and therefore $\inf_{y \in X} \phi(x, y) = 0$, which completes the proof. □

Below we state the nonconvex minimization theorem according to Takahashi [19], for an objective function defined on a quasi-uniform space.

Corollary 5.4. *Let* $f : X \to (-\infty, \infty]$ *a lower semicontinuous function,* $\not\equiv +\infty$, *bounded below. Suppose that for each* $x \in X$ *with* $\inf_{y \in X} f(y) < f(x)$ *there exists* $y \in X \setminus \{x\}$ *and* $f(y) + d(x, y) \leq f(x)$. *Then, there exists* $x \in X$ *such that* $\inf_{y \in X} f(y) = f(x)$.

Proof. Let $\phi(x, y) = f(y) - f(x)$ and fix a point $x_0 \in X$ such that $f(x_0) < \infty$. It is easy to see that ϕ satisfies assumptions of Theorem 5.3. Consequently, there exists $z \in X$ such that $\inf_{y \in X} \phi(x_0, y) = \phi(x_0, z)$ and the proof is complete. □

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