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Orthogonal Gabor Systems on Local Fields

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Abstract. The objective of this paper is to provide complete characterizations of orthogonal families, tight frames and orthonormal bases of Gabor systems on local fields of positive characteristic by means of some basic equations in the Fourier domain.

1. Introduction

Gabor systems are the collection of functions

$$\mathcal{G}(a,b,\psi) = \left\{ M_{mb}T_{na}\psi(x) =: e^{2\pi imax}\psi(x-na) : m,n \in \mathbb{Z} \right\} \tag{1.1}$$

which are built by the action of modulations and translations of a single, and hence, can be viewed as the set of time-frequency shifts of $\psi(x) \in L^2(\mathbb{R})$ along the lattice $a\mathbb{Z} \times b\mathbb{Z}$ in $\mathbb{R}^2$. Such systems, also called *Weyl-Heisenberg systems*, were introduced by Gabor [2] with the aim of constructing efficient, time-frequency localized expansions of signals as an infinite linear combinations of elements in (1.1). The system $\mathcal{G}(a,b,\psi)$ is called a *Gabor frame* if there exist constants $A, B > 0$ such that

$$A\left\|f\right\|_2^2 \le \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left|\left\langle f, M_{mb}T_{na}\psi\right\rangle\right|^2 \le B\left\|f\right\|_2^2, \tag{1.2}$$

holds for every $f \in L^2(\mathbb{R})$, and we call the optimal constants A and B the *lower frame bound* and the *upper frame bound*, respectively. A *tight Gabor frame* refers to the case when $A = B$, and a *normalized tight frame* refers to the case when $A = B = 1$. Gabor systems that form frames for $L^2(\mathbb{R})$ have a wide variety of applications. An important problem in practice is therefore to determine conditions for Gabor systems to be frames. In practice, once the window function has been chosen, the first question to investigate for Gabor analysis is to find the values of the time-frequency parameters a, b such that $\mathcal{G}(a,b,\psi)$ is a frame. Therefore, the product ab will decide whether the system $\mathcal{G}(a,b,\psi)$ constitutes a frame or even complete for $L^2(\mathbb{R})$ or not. A useful tool in this context is the Ron and Shen [6] criterion. By using this criterion, Gröchenig et al.[4] have proved that the system $\mathcal{G}(a,b,\psi)$ cannot be a frame for $L^2(\mathbb{R})$ if $|ab| > 1$. In addition to this, they have also shown

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that the system $\mathcal{G}(a, b, \psi)$ will form an orthonormal basis for $L^2(\mathbb{R})$ if $|ab| = 1$. For information on this topic, we refer to the monographs [1, 3].

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in the number theory, but also in the representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics. For more details we refer the reader to the book of Taibleson [10].

The local field K is a natural model for the structure of Gabor frame systems, as well as a domain upon which one can construct Gabor basis functions. There is a substantial body of work that has been concerned with the construction of Gabor frames on K, or more generally, on local fields of positive characteristic. Li and Jiang [5] constructed Gabor frames on local fields of positive characteristic using basic concepts of operator theory and have established a necessary and sufficient conditions for the system $\Big\{M_{u(m)b}T_{u(n)a}\psi =: \chi_m(bx)\psi\big(x - u(n)a\big)\Big\}_{m,n\in\mathbb{N}_0}$ to be a frame for $L^2(K)$. Recently, Shah [7] established a complete characterization of Gabor frames on local fields by virtue of two basic equations in the frequency domain has shown how to construct an orthonormal Gabor basis for $L^2(K)$. Recent results related to Gabor frames on local fields can be found in [8,9] and the references therein.

Motivated by the notion of Gabor systems on local fields, our aim is to give complete characterizations of orthogonal families, tight frames and orthonormal bases of Gabor systems on local fields of positive characteristic by means of some basic equations in the frequency domain.

The outline of the paper is as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and also some results which are required in the subsequent section. Characterizations of orthogonal Gabor systems on local fields of positive characteristic are given in Section 3.

2. Preliminaries on Local Fields

Let K be a fixed local field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimeadean norm $|\cdot| : K \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\,y| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in K : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in K. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle\mathfrak{p}\rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \ldots, q - 1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell\, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal

$\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [10]. In the rest of this paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural numbers, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y,x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [10], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

Definition 2.1. The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\,dx. \tag{2.1}$$

Note that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}\,dx = \int_K f(x)\chi(-\xi x)\,dx.$$

The properties of Fourier transform on local field K are much similar to those of on the real line. In fact, one can prove the following results:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \le \|f\|_1$.
- If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(K)$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\,dx, \tag{2.2}$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^\infty$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\left\{\zeta_j\right\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q,\;\; n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1},\;\; 0 \le a_k < p,\;\; \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\; n \in \mathbb{N}_0,\; 0 \le b_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m + n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.4}$$

We also denote the test function space on K by Ω, that is, each function f in Ω is a finite linear combination of functions of the form $\mathbf{\Phi}_k(x-h), h \in K, k \in \mathbb{Z}$, where $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. This class of functions can also be described in the following way. A function $g \in \Omega$ if and only if there exist integers k, ℓ such that g is constant on cosets of $\mathfrak{B}^k$ and is supported on $\mathfrak{B}^\ell$. It follows that Ω is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $C_0(K)$ as well as in $L^p(K), 1 \leq p < \infty$.

Let a and b be any two fixed elements in K. For fixed positive integer L, let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subset L^2(K)$, define the multi-generator Gabor system

$$\mathcal{G}(a, b, \Psi) := \Big\{ M_{u(m)b} T_{u(n)a} \psi^\ell =: \chi_m(bx) \psi^\ell\Big(x - u(n)a\Big) : n, m \in \mathbb{N}_0, 1 \leq \ell \leq L \Big\}, \tag{2.5}$$

where $M_{u(m)b} f(x) = \chi_m(bx) f(x)$ and $T_{u(n)a} f(x) = f\Big(x - u(n)a\Big)$ are the modulation and translation operators defined on $L^2(K)$, respectively.

We call the Gabor system $\mathcal{G}(a, b, \Psi)$ a *Gabor frame* for $L^2(K)$, if there exist constants C and D, $0 < C \leq D < \infty$ such that

$$C \big\| f \big\|_2^2 \leq \sum_{\ell=1}^{L} \sum_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} \Big| \Big\langle f, M_{u(m)b} T_{u(n)a} \psi^\ell \Big\rangle \Big|^2 \leq D \big\| f \big\|_2^2. \tag{2.6}$$

In order to prove theorems to be presented in next section, we need the following results whose proofs can be found in [1].

Lemma 2.2. Let $\mathcal{H}$ be a Hilbert space and $\{f_k\}_{k=1}^\infty$ be a family of elements of $\mathcal{H}$. Then

$$\sum_{k=1}^{\infty} \Big| \Big\langle f, f_k \Big\rangle \Big|^2 = \big\| f \big\|_2^2, \quad \text{for all } f \in \mathcal{H}$$

if and only if

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad \text{for all } f \in \mathcal{H}.$$

Lemma 2.3. Suppose that $\{f_k\}_{k=1}^\infty$ is a family of elements in a Hilbert space $\mathcal{H}$ such that the first equality of Lemma 2.2 holds for all f in a dense subset $\mathcal{D}$ of $\mathcal{H}$. Then, the same equality is true for all $f \in \mathcal{H}$.

Theorem 2.4. Suppose $\{f_k\}_{k=1}^\infty$ is a tight frame with constant 1 in Hilbert space $\mathcal{H}$. If $\|f_k\|_2 = 1$, for all $k = 1, 2, \dots,$ then $\{f_k\}_{k=1}^\infty$ is an orthonormal basis for $\mathcal{H}$.

We have also the following proposition (See [10]).

Proposition 2.5. *The system* $\Big\{\psi\Big(x - u(k)\Big) : k \in \mathbb{N}_0\Big\}$ *of functions is an orthogonal system in* $L^2(K)$ *if and only if*

$$\sum_{k \in \mathbb{N}_0} \Big| \hat{\psi}\Big(\xi - b^{-1} u(k)\Big) \Big|^2 = |b| \, \big\| \psi \big\|_2^2 \quad a.e. \ \xi \in K.$$

3. Orthogonal Gabor Systems on Local Fields

We shall start this section by characterizing the orthogonality of Gabor systems $\mathcal{G}(a, b, \Psi)$ given by (2.5) in terms of the Fourier transforms of the generators.

Theorem 3.1. *The Gabor system $\mathcal{G}(a, b, \Psi)$ given by (2.5) is orthogonal if and only if*

$$\sum_{n\in\mathbb{N}_0} \left|\hat{\psi}^\ell\left(\xi - b^{-1}u(n)\right)\right|^2 = |b| \left\|\psi^\ell\right\|_2^2, \quad a.e.\ \xi \in K,\ 1 \le \ell \le L, \tag{3.1}$$

$$\sum_{s\in\mathbb{N}_0} \hat{\psi}^\ell\left(\xi + b^{-1}u(s) - u(n)a\right)\overline{\hat{\psi}^\ell\left(\xi + b^{-1}u(s)\right)} = 0, \quad a.e.\ \xi \in K, \tag{3.2}$$

for every $n \in \mathbb{N}, 1 \le \ell \le L$, and

$$\sum_{s\in\mathbb{N}_0} \hat{\psi}^\ell\left(\xi + b^{-1}u(s) - u(n)a\right)\overline{\hat{\psi}^k\left(\xi + b^{-1}u(s)\right)} = 0, \quad a.e.\ \xi \in K, \tag{3.3}$$

for every $n \in \mathbb{N}_0, 1 \le \ell, k \le L, \ell \ne k$.

Proof. We first assume that the system $\mathcal{G}(a, b, \Psi)$ is orthogonal. Then, equation (3.1) is satisfied for every ℓ, by virtue of the Proposition 2.5. Using the Plancherel theorem, we obtain

$$\begin{aligned}
0 &= \left\langle \psi^\ell_{m,n}, \psi^k \right\rangle = \left\langle \hat{\psi}^\ell_{m,n}, \hat{\psi}^k \right\rangle \\
&= \int_K \chi_{mn}(ab)\overline{\chi_m(b\xi)}\, \hat{\psi}^\ell\left(\xi - u(n)a\right)\overline{\hat{\psi}^k(\xi)}\, d\xi \\
&= \sum_{s\in\mathbb{N}_0} \int_{\mathfrak{B}^s} \chi_{mn}(ab)\overline{\chi_m(b\xi)}\, \hat{\psi}^\ell\left(\xi - u(n)a\right)\overline{\hat{\psi}^k(\xi)}\, d\xi \\
&= \chi_{mn}(ab) \sum_{s\in\mathbb{N}_0} \int_{\mathfrak{D}} \overline{\chi_m(b\xi)}\, \hat{\psi}^\ell\left(\xi + b^{-1}u(s) - u(n)a\right)\overline{\hat{\psi}^k\left(\xi + b^{-1}u(s)\right)}\, d\xi.
\end{aligned}$$

Now, if the series

$$\sum_{s\in\mathbb{N}_0} \int_{\mathfrak{D}} \left|\overline{\chi_m(b\xi)}\, \hat{\psi}^\ell\left(\xi + b^{-1}u(s) - u(n)a\right)\overline{\hat{\psi}^k\left(\xi + b^{-1}u(s)\right)}\, d\xi\right|$$

converges, then by Beppo-Levi's theorem, we can interchange the order of integration and summation in the following integral

$$\int_{\mathfrak{D}} \sum_{s\in\mathbb{N}_0} \overline{\chi_m(b\xi)}\, \hat{\psi}^\ell\left(\xi + b^{-1}u(s) - u(n)a\right)\overline{\hat{\psi}^k\left(\xi + b^{-1}u(s)\right)}\, d\xi.$$

However, we observe that

$$\begin{aligned}
&\sum_{s\in\mathbb{N}_0} \int_{\mathfrak{D}} \left|\overline{\chi_m(b\xi)}\, \hat{\psi}^\ell\left(\xi + b^{-1}u(s) - u(n)a\right)\overline{\hat{\psi}^k\left(\xi + b^{-1}u(s)\right)}\, d\xi\right| \\
&\qquad = \int_K \left|\hat{\psi}^\ell\left(\xi - u(n)a\right)\overline{\hat{\psi}^k(\xi)}\right|\, d\xi \\
&\qquad \le \left\{\int_K \left|\hat{\psi}^\ell\left(\xi - u(n)a\right)\right|^2 d\xi\right\}^{1/2} \left\{\int_K \left|\overline{\hat{\psi}^{\ell'}(\xi)}\right|^2 d\xi\right\}^{1/2} < \infty.
\end{aligned}$$

Therefore, for $\ell \neq k$ and $m, n \in \mathbb{N}_0$, we have

$$\int_{\mathfrak{D}} \left\{ \sum_{s\in\mathbb{N}_0} \hat{\psi}^\ell\Big(\xi + b^{-1}u(s) - u(n)a\Big)\overline{\hat{\psi}^k\Big(\xi + b^{-1}u(s)\Big)} \right\} \overline{\chi_m(b\xi)}\, d\xi = 0.$$

Since $\{\chi_m(b\xi) : m \in \mathbb{N}_0\}$ is a basis for $L^2(\mathfrak{D})$ and

$$\sum_{s\in\mathbb{N}_0} \hat{\psi}^\ell\Big(\xi + b^{-1}u(s) - u(n)a\Big)\overline{\hat{\psi}^k\Big(\xi + b^{-1}u(s)\Big)},$$

is a periodic function with period a, so we can conclude that for $\ell = k$ and $n \in \mathbb{N}$, or for $\ell \neq k, n \in \mathbb{N}_0$,

$$\sum_{s\in\mathbb{N}_0} \hat{\psi}^\ell\Big(\xi + b^{-1}u(s) - u(n)a\Big)\overline{\hat{\psi}^k\Big(\xi + b^{-1}u(s)\Big)} = 0, \quad \text{for } a.e.\ \xi \in K.$$

This completes the proof of the first implication.

Now let us assume that all the three conditions are satisfied. It is easy to verify that

$$\Big\langle \psi^\ell_{m,n}, \psi^k_{m',n'} \Big\rangle = \chi\Big(ab\, u(m-m')u(n')\Big) \Big\langle \psi^\ell_{m-m',n+n'}, \psi^k \Big\rangle.$$

Further, Proposition 2.5 implies that the systems $\Big\{\psi^\ell_{0,n} : n \in \mathbb{N}_0\Big\}$ are orthogonal for each ℓ. Therefore to finish the proof, we can invoke Beppo-Levi's theorem, Plancherel formula and, the second and the third conditions to prove, as above, that

$$\Big\langle \psi^\ell_{m,n}, \psi^k \Big\rangle = 0, \quad \text{for } \ell \neq k,\ m, n \in \mathbb{N}_0.$$

This completes the proof. $\square$

To prove the completeness of Gabor systems $\mathcal{G}(a, b, \Psi)$ in $L^2(K)$ when $|ab| = 1$, we set

$$W^\ell_m = \overline{\text{span}}\Big\{\psi^\ell_{m,n} : n \in \mathbb{N}_0\Big\}, \quad 1 \leq \ell \leq L, m \in \mathbb{N}_0. \tag{3.4}$$

Assume that the Gabor systems $\mathcal{G}(a, b, \Psi)$ given by (2.5) are orthogonal in $L^2(K)$ and let P^ℓ_m denote the orthogonal projection onto the space W^ℓ_m, that is;

$$P^\ell_m f(x) = \sum_{n\in\mathbb{N}_0} \Big\langle f, \psi^\ell_{m,n} \Big\rangle \psi^\ell_{m,n}(x), \quad \text{for every } f \in L^2(K). \tag{3.5}$$

Then, we have

$$\begin{aligned}
\Big\langle \hat{f}, \hat{\psi}^\ell_{m,n} \Big\rangle &= \int_K \hat{f}(\xi)\chi_n(b\xi)\, \overline{\chi_{mn}(ab)}\, \overline{\hat{\psi}^\ell\Big(\xi - u(m)a\Big)}\, d\xi \\
&= \sum_{s\in\mathbb{N}_0} \int_{\mathfrak{B}^s} \chi_n(b\xi)\hat{f}(\xi)\, \overline{\hat{\psi}^\ell\Big(\xi - u(m)a\Big)}\, d\xi \\
&= \sum_{s\in\mathbb{N}_0} \int_{\mathfrak{D}} \chi_n\Big(\xi + b^{-1}u(s)\Big)\hat{f}\Big(\xi + b^{-1}u(s)\Big)\overline{\hat{\psi}^\ell\Big(\xi + b^{-1}u(s) - u(m)a\Big)}\, d\xi.
\end{aligned}$$

Note that

$$\begin{aligned}\sum_{s\in\mathbb{N}_0}\int_{\mathfrak{D}}&\left|\chi_n(b\xi)\hat{f}\left(\xi+b^{-1}u(s)\right)\overline{\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)-u(m)a\right)}\right|d\xi\\&=\int_K\left|\hat{f}(\xi)\overline{\hat{\psi}^{\ell}\left(\xi-u(m)a\right)}\right|d\xi\\&\leq\left\{\int_K\left|\hat{f}(\xi)\right|^2d\xi\right\}^{1/2}\left\{\int_K\left|\hat{\psi}^{\ell}\left(\xi-u(m)a\right)\right|^2d\xi\right\}^{1/2}\\&=\left\|\hat{f}\right\|_2\left\|\hat{\psi}\right\|_2<\infty.\end{aligned}$$

Therefore, we can use the Beppo-Levis theorem to obtain:

$$\left\langle\hat{f},\hat{\psi}^{\ell}_{m,n}\right\rangle=\int_{\mathfrak{D}}\left\{\sum_{s\in\mathbb{N}_0}\hat{f}\left(\xi+b^{-1}u(s)\right)\overline{\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)-u(m)a\right)}\right\}\chi_n(b\xi)\,d\xi.$$

Clearly, these are the Fourier coefficients of the a- periodic function

$$\sum_{s\in\mathbb{N}_0}\hat{f}\left(\xi+b^{-1}u(s)\right)\overline{\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)-u(m)a\right)},$$

thus, we can write

$$\sum_{s\in\mathbb{N}_0}\hat{f}\left(\xi+b^{-1}u(s)\right)\overline{\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)-u(m)a\right)}=|b|\sum_{n\in\mathbb{N}_0}\left\langle\hat{f},\hat{\psi}^{\ell}_{m,n}\right\rangle\overline{\chi_n(b\xi)}.$$

Multiplying both sides of the above identity by $\hat{\psi}^{\ell}\left(\xi-u(m)a\right)$, we obtain the desired expression for P^{ℓ}_m as:

$$\widehat{P^{\ell}_m f}(\xi)=\frac{1}{|b|}\sum_{s\in\mathbb{N}_0}\hat{f}\left(\xi+b^{-1}u(s)\right)\overline{\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)-u(m)a\right)}\hat{\psi}^{\ell}\left(\xi-u(m)a\right).\tag{3.6}$$

Theorem 3.2. *Let $\Psi=\{\psi_1,\psi_2,\ldots,\psi_L\}\subseteq L^2(K)$ and $a,b\in K\setminus\{0\}$ be given. If $|ab|=1$ and the functions $\psi_1,\psi_2,\ldots,\psi_L$ satisfy the following three conditions:*

$$\sum_{m\in\mathbb{N}_0}\left|\hat{\psi}^{\ell}\left(\xi-b^{-1}u(m)\right)\right|^2=|b|\left\|\psi^{\ell}\right\|_2^2,\quad a.e.\ \xi\in K,\ 1\leq\ell\leq L,\tag{3.7}$$

$$\sum_{s\in\mathbb{N}_0}\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)-u(m)a\right)\overline{\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)\right)}=0,\quad a.e.\ \xi\in K,\tag{3.8}$$

for every $m\in\mathbb{N},1\leq\ell\leq L$, and

$$\sum_{s\in\mathbb{N}_0}\hat{\psi}^{\ell}\left(\xi+b^{-1}u(s)-u(m)a\right)\overline{\hat{\psi}^{k}\left(\xi+b^{-1}u(s)\right)}=0,\quad a.e.\ \xi\in K,\tag{3.9}$$

for every $m\in\mathbb{N}_0,1\leq\ell,k\leq L,\ell\neq k$. Then, the Gabor system $\mathcal{G}(a,b,\Psi)$ as defined in (2.5) is complete in $L^2(K)$.

Proof. To prove the result, it is sufficient to prove that

$$\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\widehat{P^{\ell}_m f}(\xi)=\left(\sum_{\ell=1}^{L}\left\|\psi^{\ell}\right\|_2^2\right)\hat{f}(\xi)\quad a.e.\ \xi\in K,\tag{3.10}$$

and

$$\lim_{M\to\infty}\left\|\sum_{\ell=1}^{L}\sum_{|m|\le M}\widehat{P_m^\ell f}\right\|_2=\Big(\sum_{\ell=1}^{L}\left\|\psi^\ell\right\|_2^2\Big)\left\|f\right\|_2, \tag{3.11}$$

hold for every $f\in L^2(K)$. Since Ω is dense in $L^2(K)$ and is closed under the Fourier transform, the set

$$\Omega^0=\left\{f\in\Omega:\operatorname{supp}\hat{f}\subset K\setminus\{0\}\right\}.$$

is also dense in $L^2(K)$. Therefore, in view of Lemma 2.2 and 2.3, it is enough to verify that the equalities (3.10) and (3.11) hold for all $f\in\Omega^0$. In Theorem 3.1, we have already shown that the system $\mathcal{G}(a,b,\Psi)$ given by (2.5) is orthogonal in $L^2(K)$; hence by applying (3.6) to the projections P_m^ℓ, we can write

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\widehat{P_m^\ell f}(\xi)\\
&=\frac{1}{|b|}\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\left\{\sum_{s\in\mathbb{N}_0}\hat{f}\Big(\xi+b^{-1}u(s)\Big)\overline{\hat\psi^\ell\Big(\xi+b^{-1}u(s)-u(m)a\Big)}\hat\psi^\ell\Big(\xi-u(m)a\Big)\right\}\\
&=\frac{1}{|b|}\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\hat{f}(\xi)\left|\hat\psi^\ell\Big(\xi-u(m)a\Big)\right|^2+\frac{1}{|b|}\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\sum_{s\in\mathbb{N}}\hat{f}\Big(\xi+b^{-1}u(s)\Big)\overline{\hat\psi^\ell\Big(\xi+b^{-1}u(s)-u(m)a\Big)}\hat\psi^\ell\Big(\xi-u(m)a\Big)\\
&=\frac{1}{|b|}\hat{f}(\xi)\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\left|\hat\psi^\ell\Big(\xi-u(m)a\Big)\right|^2+\frac{1}{|b|}\sum_{\ell=1}^{L}\sum_{s\in\mathbb{N}}\hat{f}\Big(\xi+b^{-1}u(s)\Big)\sum_{m\in\mathbb{N}_0}\overline{\hat\psi^\ell\Big(\xi+b^{-1}u(s)-u(m)a\Big)}\hat\psi^\ell\Big(\xi-u(m)a\Big)\\
&=\hat{f}(\xi)\left(\sum_{\ell=1}^{L}\left\|\psi^\ell\right\|_2^2\right).
\end{aligned}$$

Here we have used our assumption on the functions ψ_ℓ, i.e., equations (3.7)–(3.9) and the fact $|ab|=1$. The change in the order of summation is valid since $f\in\Omega^0$, which implies that the sum over $s\in\mathbb{N}$ is finite.

In order to prove the relation (3.11), we use the fact that P_m^ℓ's are mutually orthogonal, so we have

$$\left\|\sum_{\ell=1}^{L}\sum_{|m|\le M}\widehat{P_m^\ell f}\right\|_2\le\left(\sum_{\ell=1}^{L}\left\|\psi^\ell\right\|_2^2\right)^{1/2}\left\|f\right\|_2,\quad\text{for every }M>0.$$

Moreover, the orthogonality of P_m^ℓ's implies that

$$\left\|\sum_{\ell=1}^{L}\sum_{|m|\le M}\widehat{P_m^\ell f}\right\|_2=\left(\sum_{\ell=1}^{L}\sum_{|m|\le M}\left\|\widehat{P_m^\ell f}\right\|_2^2\right)^{1/2}$$

is an increasing sequence bounded by $\left(\sum_{\ell=1}^{L}\left\|\psi^\ell\right\|_2^2\right)^{1/2}\left\|f\right\|_2$. Therefore, we have

$$\lim_{M\to\infty}\left\|\sum_{\ell=1}^{L}\sum_{|m|\le M}\widehat{P_m^\ell f}\right\|_2\le\left(\sum_{\ell=1}^{L}\left\|\psi^\ell\right\|_2^2\right)^{1/2}\left\|f\right\|_2.$$

On the other hand, by Fatou's lemma we have

$$\lim_{M\to\infty}\left\|\sum_{\ell=1}^{L}\sum_{|m|\le M}\widehat{P_m^\ell f}\right\|_2\ge\left(\sum_{\ell=1}^{L}\left\|\psi^\ell\right\|_2^2\right)^{1/2}\left\|f\right\|_2.$$

Combining the above inequalities, we get the desired result. $\square$

As a consequence of the above theorem, we get following characterization of tight Gabor frames of $L^2(K)$.

Corollary 3.3. *Let $a, b \in K \setminus \{0\}$ be given. Suppose $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subseteq L^2(K)$ be such that $\sum_{\ell=1}^{L} \|\psi^\ell\|_2^2 = 1$. Then with the assumptions of Theorem 3.2, the system $\mathcal{G}(a, b, \Psi)$ constitutes a tight frame with constant 1 for $L^2(K)$.*

By combining Corollary 3.3 with Theorem 2.3, we can obtain the following characterization for Gabor systems generated by a single function.

Theorem 3.4. *Let $\psi \in L^2(K)$ and $a, b \in K \setminus \{0\}$ such that $|ab| = 1$. Then the system $\mathcal{G}(a, b, \Psi)$ is a tight frame with constant 1 if and only if the following equations hold:*

$$\sum_{m\in\mathbb{N}_0} \left|\hat{\psi}\Big(\xi - b^{-1}u(m)\Big)\right|^2 = |b|, \quad a.e.\ \xi \in K,$$

$$\sum_{s\in\mathbb{N}_0} \hat{\psi}\Big(\xi + b^{-1}u(s) - u(m)a\Big)\overline{\hat{\psi}\Big(\xi + b^{-1}u(s)\Big)} = 0, \quad a.e.\ \xi \in K, m \in \mathbb{N}.$$

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