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Solvability of a 2×2 Block Operator Matrix of Chandrasekhar Type on a Bananch Algebra

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Abstract. In this paper, we study the existence of solutions for a system of quadratic integral equations of Chandrasekhar type by applying fixed point theorem of a 2×2 block operator matrix defined on a nonempty bounded closed convex subsets of Banach algebras where the entries are nonlinear operators.

1. Introduction and Preliminaries

Quadratic integral equations have received increasing attention during recent years due to its applications in numerous diverse fields of science and engineering. For example, the theory of radiative transfer, kinetic theory of gases, the theory of neutron transport and the traffic theory. Many authors have studied different kinds of nonlinear quadratic integral equations in different classes (see[1]-[3] and [8]-[18]). Especially, Chandrasekhar's integral equation which has been a subject of much investigation since its appearance around fifty years ago [13].
In this work, we are concerned with the system of two quadratic integral equations of Chandrasekhar type

$$\begin{aligned} x(t) &= f_1(t, x(t)) + (G_1 y)(t) \int_0^t \frac{t}{t+s}\, u_1(s, y(s))\, ds,\ t \in J, \\ y(t) &= f_2(t, y(t)) + (G_2 x)(t) \int_0^t \frac{t}{t+s}\, u_2(s, x(s))\, ds,\ t \in J \end{aligned} \tag{1}$$

which continues the series of publications on the coupled systems ([4]-[6], [26], [27] and [29]). For example, Su [30] studied a two-point boundary value problems for a coupled system of fractional differential equations. Gafiychuk et al. [28] analyzed the solutions of coupled system of nonlinear fractional reaction-diffusion equations. Some existence results for coupled systems of integral equations in reflexive Banach space were proved in [21]-[25]. Also, a comparison between Picard method and Adomian decomposition method of coupled system of quadratic integral equations was proved in [24].
But in this work we apply fixed point theorem of a 2×2 block operator matrix defined on a nonempty bounded closed convex subsets of Banach algebras where the entries are nonlinear operators which differs

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from the technique of proof in all the literature and references therein.
In [19] some fixed point results on Banach algebras of operators defined by a 2×2 block operator matrix

$$L = \begin{pmatrix} A & B.B' \\ C & D \end{pmatrix} \tag{2}$$

were obtained, where the entries of the matrix are in general nonlinear operators dened on Banach algebras.

In [19] the existence of solutions of a system of functional integral equations by using some fixed point theorems on Banach algebras of operators defined by the 2×2 block operator matrix (2) has been established.
Let $X = \mathbb{C}(J, \mathbb{R})$ be the vector of all real-valued continuous functions on $J = [0, b]$. We equip the space X with the norm $\|x\| = \sup_{t \in J} |x(t)|$.
Clearly, $\mathbb{C}(J, \mathbb{R})$ is a complete normed algebra with respect to this supremum norm.

By a solution of the system of the quadratic integral equations of Chandrasekhar type (1) we mean a vector function $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}(J, \mathbb{R}) \times \mathbb{C}(J, \mathbb{R})$ that satisfies (1), where $\mathbb{C}(J, \mathbb{R})$ stands for the space of continuous real-valued functions on J.

Definition 1.1. *[19] A mapping $T : X \to X$ is called totally bounded if $T(S)$ is a totally bounded subset of X for any bounded subset S of X. Again a map $T : X \to X$ is completely continuous if it is continuous and totally bounded on X. It is clearly that every compact operator is totally bounded, but the converse may not be true, however the two notions are equivalent on bounded subsets of a Banach space X.*

Definition 1.2. *[19] Let X be a normed vector space. A mapping $T : X \to X$ is called D-Lipschitzian if there exists a continuous and nondecreasing function ϕ such that*

$$\|Tx - Ty\| \leq \phi_D(\|x - y\|)$$

for all $x, y \in X$ where $\phi(0) = 0$.

Sometimes, we call for the function ϕ_D to be a **D**-function of the mapping T on X. Obsviously, every Lipschitzian mapping is **D**-Lipschitzian. Further, if $\phi(r) < r$, then T is called nonlinear contraction on X.

Theorem 1.3. *[19] Let S be a nonempty convex closed and bounded subset of a Banach algebra X and let S' be a nonempty convex closed and bounded subset of a Banach algebra Y.*
Let $A : S \to X$, $B, B' : S' \to X$, $C : S \to Y$ and $D : S' \to S'$ be five operators such that:

*(i:) The operator B is Lipschitzian with constant β, A and C are **D**-Lipschitzians with the **D**-functions ϕ_A and ϕ_C respectively.*

(ii:) $C(S) \subseteq (I - D)(S')$.

(iii:) D is a contraction with constant k.

(iv:) B' is continuous and C is compact.

(v:) $Ax + TxT'z \in S$ for all $x, z \in S$, where $T = B(I - D)^{-1}C$ and $T' = B'(I - D)^{-1}C$.

Then the operator matrix (2) has a fixed point in $S \times S'$ whenever $\frac{\beta M}{1-k}\phi_C(r) + \phi_A(r) < r$, where $M = \|T'(S)\|$.

2. Existence Theorem

The main aim of this section is to apply Theorem 1.3 to prove the existence of solutions to the coupled system (1).
Consider the following assumptions:

(i) $u_i : J \times \mathbb{R} \to \mathbb{R},\ i = 1,2$ satisfy Carathéodory condition (i.e. measurable in t for all $x \in \mathbb{R}$ and continuous in x for almost all $t \in J$) such that:

$$|u_i(t,x)| \leq m_i(t) \in L^1[J] \quad \forall\ (t,x) \in J \times \mathbb{R}$$

and $k_i = \sup_{t\in J} \int_0^b \frac{1}{t+s} |m_i(s)|\, ds$.

(ii) $f_i : J \times \mathbb{R} \to \mathbb{R}$ are continuous and $M_i = \sup_{(t,x)\in J\times\mathbb{R}} |f_i(t,x)|,\ i = 1,2.$

(iii) There exist constants $l_i,\ \ i = 1,2$ satisfying

$$|f_i(t,x) - f_i(t,y)| \leq l_i\, |x-y|,\ \ i = 1,2$$

for all $t \in J$ and $x,\ y \in \mathbb{R}$.

(iv) $G_i,\ i = 1,2$ are contractions with constants $h_i,\ i = 1,2$. Moreover, $N_i = \|G_i\|$.

Theorem 2.1. *Let the assumptions (i)-(iv) be satisfied. Furthermore, if* $h_1\, \|m_1\|\, h_2\, k_2 < (1-l_1)(1-l_2)$, *then the system of the quadratic integral equations (1) has at least one solution in the space* $\mathbb{C}(J,\mathbb{R}) \times \mathbb{C}(J,\mathbb{R})$.

Proof:
Consider the operators $A,\ B,\ C,\ D$ and B' on $C(J,\mathbb{R})$ defined by:

$$\begin{cases} (Ax)(t) = f_1(t,x(t)) \\ (By)(t) = (G_1y)(t) \\ (Cy)(t) = (G_2y)(t)\ \int_0^t \frac{t}{t+s}\, u_2(s,y(s))\, ds \\ (Dx)(t) = f_2(t,x(t)) \\ (B'y)(t) = \int_0^t \frac{t}{t+s}\, u_1(s,y(s))\, ds \end{cases}$$

In operator form

$$\begin{cases} x(t) = Ax(t) + By(t).B'y(t) \\ y(t) = Dy(t) + Cx(t) \end{cases}$$

and in Matrix form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B.B' \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We shall show that $A,\ B,\ C,\ D$ and B' satisfy all the assumptions of Theorem 1.3.
Let us defined subsets $S,\ S'$ on $\mathbb{C}(J,\mathbb{R}_+)$ by :

$$S = \{x \in \mathbb{C}(J,\mathbb{R}_+),\ \|x\| \leq M_1 + N_1\, k_1\ \}$$

$$S' = \{y \in \mathbb{C}(J,\mathbb{R}_+),\ \|y\| \leq M_2 + N_2\, k_2\ \}.$$

Obviously, S and S' are nonempty, bounded, convex and closed subsets of $\mathbb{C}(J,\mathbb{R}_+)$. First, we begin by showing that A is Lipschitzian on S. To show this, let $x,\ z \in S$. So,

$$\|Ax(t) - Az(t)\| \leq l_1 \|x - z\|$$

and

$$\|Bx(t) - Bz(t)\| = \|G_1x(t) - G_1z(t)\| \le h_1\|x - z\|.$$

Also, we shall show that C is Lipschitzian. To see this, let $x, z \in S$ and set

$$Cx(t) = G_2x(t).Ux(t)$$

where $G_2x(t) = g_2(t, x(t))$ and $Ux(t) = \int_0^t \frac{t}{t+s}\, u_2(s, x(s))\, ds$

$$\|Cx(t) - Cz(t)\| = \|G_2x(t).Ux(t) - G_2z(t).Uz(t)\| \le \|Ux(t)\|.\|G_2x(t) - G_2z(t)\| \le k_2h_2.\|x - z\|$$

We shall show that $C(S)$ is a relatively compact subset in X. For any $x \in S$

$$|Cy(t)| \le |(G_2y)(t))| \int_0^t \frac{t}{t+s}\, |u_2(s, y(s))|\, ds \le N_2\, k_2,$$

for each $t_1,\ t_2 \in I$ (without loss of generality assume that $t_1 < t_2$), we get

$$|(Cy)(t_2) - (Cy)(t_1)| = |(G_2y)(t_2) \int_0^{t_2} \frac{t_2}{t_2+s}\, u_2(s, y(s))\, ds - (G_2y)(t_1) \int_0^{t_1} \frac{t_1}{t_1+s}\, u_2(s, y(s))\, ds$$

$$+ (G_2y)(t_1) \int_0^{t_2} \frac{t_2}{t_2+s}\, u_2(s, y(s))\, ds - (G_2y)(t_1) \int_0^{t_2} \frac{t_2}{t_2+s}\, u_2(s, y(s))\, ds|$$

$$\le |(G_2y)(t_2) - (G_2y)(t_1)| \int_0^{t_2} \frac{t_2}{t_2+s}\, |u_2(s, y(s))|\, ds$$

$$+ |(G_2y)(t_1)|\, | \int_0^{t_2} \frac{t_2}{t_2+s}\, u_2(s, y(s))\, ds - \int_0^{t_1} \frac{t_1}{t_1+s}\, u_2(s, y(s))\, ds\, |,$$

but $t_1 < t_2 \Rightarrow t_1 + s < t_2 + s \Rightarrow \frac{1}{t_1+s} > \frac{1}{t_2+s}$. Then

$$\left| \int_0^{t_2} \frac{t_2}{t_2+s}\, u_2(s, y(s))\, ds - \int_0^{t_1} \frac{t_1}{t_1+s}\, u_2(s, y(s))\, ds \right| \le \int_0^{t_1} \frac{t_2 - t_1}{t_1+s}\, |u_2(s, y(s))|\, ds + \int_{t_1}^{t_2} \frac{t_2}{t_2+s}\, |u_2(s, y(s))|\, ds$$

$$\le |t_2 - t_1| \int_0^b \frac{1}{t_1+s} m_2(s)\, ds + \int_{t_1}^{t_2} m_2(s)\, ds, \qquad (\forall t \in I, t < t+s \Rightarrow \frac{1}{t} > \frac{1}{t+s} \Rightarrow 1 > \frac{t}{t+s})$$

s.

Then we get

$$|\, (Cy)(t_2) - (Cy)(t_1)\, | \le h_2\|y(t_2) - y(t_1)\|.\|m_2\| + N_2[k_2|t_2 - t_1| + \int_{t_1}^{t_2} m_2(s)\, ds.]$$

This means that the functions of US' are equi-continuous on J. Then by the Arzela-Ascoli Theorem [7] the closure of US' is relatively compact.
Next, we show that $C(S) \subseteq (I - D)(S')$. To see that, let $x \in S$ be fixed point.
Define a mapping $\phi_x : \mathbb{C}(J, \mathbb{R}) \to \mathbb{C}(J, \mathbb{R})$ by

$$y \to Cx + Dy.$$

From assumption (ii), it follows that the operator ϕ_x is a contraction with a constant $l_2 + h_2\, k_2,$ then an application of Banach Theorem yields there is a unique point $y \in \mathbb{C}(J, \mathbb{R})$ such that $Cx + Dy = y$ and $Cx = (I - D)y$. Hence,

$$C(S) \subseteq (I - D)C(J, \mathbb{R}).$$

Since $y \in \mathbb{C}(J, \mathbb{R}),$ then there is $t^* \in J$ such that

$$\begin{aligned} \|y\|_\infty = |y(t^*)| &= |Cx(t^*) + Dy(t^*)| \\ &\leq |(G_2x)(t^*) \int_0^{t^*} \frac{t^*}{t^* + s} u_2(s, x(s))\, ds + f_2(t^*, x(t^*))| \\ &\leq N_2\, k_2 + M_2. \end{aligned}$$

Then $C(S) \subseteq (I - D)(S')$.
We claim that B' is continuous on S'. For $x \in S',$ then

$$\begin{aligned} |B'x(t_n) - B'x(t)| &= |\int_0^{t_n} \frac{t_n}{t_n + s}\, u_1(s, x(s))\, ds - \int_0^{t} \frac{t}{t + s}\, u_1(s, x(s))\, ds \\ &+ \int_0^{t_n} \frac{t}{t + s}\, u_1(s, x(s))\, ds - \int_0^{t_n} \frac{t}{t + s}\, u_1(s, x(s))\, ds| \\ &\leq \int_0^{t_n} |\frac{t_n}{t_n + s} - \frac{t}{t + s}|\, |u_1(s, x(s))|\, ds + \int_t^{t_n} \frac{t}{t + s} |u_1(s, x(s))|\, ds \\ &\leq |t_n - t| \int_0^{t_n} |\frac{1}{t + s}\, m_1(s)\, ds + \int_t^{t_n} m_1(s)\, ds \\ &\leq k_1\, |t_n - t| + \int_t^{t_n} m_1(s)\, ds, \end{aligned}$$

since $t_n \to t$ then $B'x(t_n) \to B'x(t)$ in $\mathbb{R},$ so $B'x \in \mathbb{C}(J,\ \mathbb{R})$.
Now from the assumption (ii), it follows that

$$\begin{aligned} M &= \|T'(S)\| = \sup_{t \in J} |T'(S)| \\ &\leq \sup_{t \in J} |\int_0^{t} \frac{t}{t + s}\, u_1(s, y(s))\, ds| \\ &\leq \|m_1\| \end{aligned}$$

and therefore $h_1\, \|m_1\|\, h_2\, k_2 < (1 - l_1)(1 - l_2)$.
Then, it remains to verify that the hypothesis (v) of Theorem 1.3. Let $x, z \in S$ then for all $t \in J$ we have

$$|Ax(t) + B(I - D)^{-1}Cx(t)B'(I - D)^{-1}Cz| \quad \leq \quad M_1 + N_1k_1.$$

This implies that

$$Ax + B(I - D)^{-1}CxB'(I - D)^{-1}Cz \in S \text{ for any } x, z \in S.$$

We conclude that the operators $A,\ B,\ C,\ D$ and B' satisfy all the requirements of Theorem 1.3. Now the results follows from Theorem 1.3.■

3. Special Cases

As particular cases of Theorem 2.1 we can obtain theorems on the existence of solutions belonging to the space $\mathbb{C}(J, \mathbb{R})$ for the following systems of integral equations:

(i) Letting $g_1(t, y(t)) = g_2(t, x(t)) = 0$, then we have the coupled system of functional equations

$$\begin{aligned} x(t) &= f_1(t, x(t)) \\ y(t) &= f_2(t, y(t)). \end{aligned}$$

(ii) Letting $f_1(t, y(t)) = f_2(t, x(t)) = 0$, then we have the coupled system of integral equations

$$\begin{aligned} x(t) &= g_1(t, y(t)) \int_0^t \frac{t}{t+s}\, u_1(s, y(s))\, ds, \ t \in J, \\ y(t) &= g_2(t, x(t)) \int_0^t \frac{t}{t+s}\, u_2(s, x(s))\, ds, \ t \in J. \end{aligned}$$

(iii) Letting $f_1(t, y(t)) = a_1(t)$, $f_2(t, x(t)) = a_2(t)$, then we have the coupled system of integral equations

$$\begin{aligned} x(t) &= a_1(t) + g_1(t, y(t)) \int_0^t \frac{t}{t+s}\, u_1(s, y(s))\, ds, \ t \in J, \\ y(t) &= a_2(t) + g_2(t, x(t)) \int_0^t \frac{t}{t+s}\, u_2(s, x(s))\, ds, \ t \in J. \end{aligned}$$

(v) Letting $f_1(t, y(t)) = a_1(t)$, $f_2(t, x(t)) = a_2(t)$, and $g_1(t, y(t)) = y(t)$, $g_2(t, x(t)) = x(t)$, then we have the coupled system

$$\begin{aligned} x(t) &= a_1(t) + y(t) \int_0^t \frac{t}{t+s}\, u_1(s, y(s))\, ds, \ t \in J, \\ y(t) &= a_2(t) + x(t) \int_0^t \frac{t}{t+s}\, u_2(s, x(s))\, ds, \ t \in J. \end{aligned}$$

(vi) Letting $f_1(t, y(t)) = a_1(t)$, $f_2(t, x(t)) = a_2(t)$, $u_i(t, x) = \lambda_i\, \phi_i(t)\, x(t)$ and $g_1(t, y(t)) = y(t)$, $g_2(t, x(t)) = x(t)$, where ϕ_i are essentially bounded functions need not be continuous. Then we have the coupled system

$$\begin{aligned} x(t) &= a_1(t) + y(t) \int_0^t \frac{t\, \lambda_1\, \phi_1(s)}{t+s}\, y(s)\, ds, \ t \in J, \\ y(t) &= a_2(t) + x(t) \int_0^t \frac{t\, \lambda_2\, \phi_2(s)}{t+s}\, x(s)\, ds, \ t \in J \end{aligned}$$

which is the same result obtained in [29].

(vii) Letting $f_1(t, y(t)) = a_1(t)$, $f_2(t, x(t)) = a_2(t)$, then we have the coupled system

$$\begin{aligned} x(t) &= a_1(t) + \int_0^t \frac{t}{t+s}\, u_1(s, y(s))\, ds, \ t \in J, \\ y(t) &= a_2(t) + \int_0^t \frac{t}{t+s}\, u_2(s, x(s))\, ds, \ t \in J. \end{aligned}$$

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