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Generalized Sherman-Morrison-Woodbury Formula for the Generalized Drazin Inverse in Banach Algebra

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Abstract. In this article, we investigate the generalization of Sherman-Morrison-Woodbury formula for the generalized Drazin inverse for elements in Banach algebra.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. The sets of all invertible, nilpotent and quasi-nilpotent elements ($\sigma(a) = \{0\}$) of $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively.

The generalized Drazin inverse of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of a) is the element $b \in \mathcal{A}$ which satisfies

$$bab = b, \qquad ab = ba, \qquad a - a^2b \in \mathcal{A}^{qnil}.$$

If the generalized Drazin inverse of a exists, it is unique and denoted by a^d. Then, we say that the element a is generalized Drazin invertible. The set of all generalized Drazin invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^d$.

The Drazin inverse is a special case of the generalized Drazin inverse for which it holds $a - a^2b \in \mathcal{A}^{nil}$ instead of $a - a^2b \in \mathcal{A}^{qnil}$, i.e. the Drazin inverse of a is the element b which satisfies $bab = b$, $ab = ba$ and $a^{k+1}b = a^k$, for some nonnegative integer k. The least such k is called the Drazin index of a and it is denoted by $i(a)$. The Drazin inverse of a is denoted by a^D. Obviously, if a is Drazin invertible, then it is generalized Drazin invertible.

The group inverse is the Drazin inverse for which the condition $a - a^2b \in \mathcal{A}^{nil}$ is replaced with $a = aba$, i.e. $i(a) = 1$. We use $a^\#$ to denote the group inverse of a, and we use $\mathcal{A}^\#$ and $\mathcal{A}^D$ to denote the sets of all group invertible and Drazin invertible elements of $\mathcal{A}$, respectively.

Recall that $a \in \mathcal{A}$ is generalized Drazin invertible if and only if there exists an idempotent $p = p^2 \in \mathcal{A}$ such that

$$ap = pa \in \mathcal{A}^{qnil}, \qquad a + p \in \mathcal{A}^{-1}.$$

Then $p = 1 - aa^d$ is the *spectral idempotent* of a corresponding to the set $\{0\}$, and it will be denoted by a^π. The generalized Drazin inverse a^d double commutes with a, that is, $ax = xa$ implies $a^dx = xa^d$.

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We use the following lemma.

Lemma 1.1. [1, Lemma 2.4] *Let $p, q \in \mathcal{A}^{qnil}$ and let $pq = 0$. Then $p + q \in \mathcal{A}^{qnil}$.*

Sherman and Morrison [4] and Woodbury [5] discovered the formula for the inverse of matrices. The original Sherman-Morrison-Woodbury (for short SMW) stands

$$(A + YGZ^*)^{-1} = A^{-1} - A^{-1}Y(G^{-1} + Z^*A^{-1}Y)^{-1}Z^*A^{-1},$$

where A, G, Y, and Z are matrices of the appropriate size such that A, G and $G^{-1} + Z^*A^{-1}Y$ are invertible.

The SMW formula is also valid for the elements of Banach algebra. The following theorem proves it.

Theorem 1.2. *Let $a, g, y, z \in \mathcal{A}$ such that $a, g \in \mathcal{A}^{-1}$ and let $g^{-1} + za^{-1}y$ be also invertible. Then $a + ygz$ is invertible and*

$$(a + ygz)^{-1} = a^{-1} - a^{-1}y(g^{-1} + za^{-1}y)^{-1}za^{-1}. \tag{1}$$

Proof. Let $b = a + ygz$ and $t = g^{-1} + za^{-1}y$. Note that $za^{-1}y = t - g^{-1}$. If the right hand side of (1) is denoted by m, we obtain

$$\begin{aligned} bm &= (a + ygz)(a^{-1} - a^{-1}yt^{-1}za^{-1}) \\ &= aa^{-1} - aa^{-1}yt^{-1}za^{-1} + ygza^{-1} - ygza^{-1}yt^{-1}za^{-1} \\ &= 1 - yt^{-1}za^{-1} + ygza^{-1} - yg(t - g^{-1})t^{-1}za^{-1} \\ &= 1 - yt^{-1}za^{-1} + ygza^{-1} - ygtt^{-1}za^{-1} + ygg^{-1}t^{-1}za^{-1} \\ &= 1 \end{aligned}$$

Analogously,

$$\begin{aligned} mb &= (a^{-1} - a^{-1}yt^{-1}za^{-1})(a + ygz) \\ &= 1 + a^{-1}ygz - a^{-1}yt^{-1}z - a^{-1}yt^{-1}za^{-1}ygz \\ &= 1 + a^{-1}ygz - a^{-1}yt^{-1}z - a^{-1}yt^{-1}(t - g^{-1})gz \\ &= 1 \end{aligned}$$

Now, we can conclude that b is invertible and it's inverse is m. So, we proved that (1) holds. □

In this paper, we will consider the generalized case of SMW formula. We will generalize the SMW formula to the cases when a and $a + ygz$ are not invertible, but generalized Drazin invertible.

2. Results

The first result is the generalization of the Theorem 2.3 [2], which is proved for the operators on the Hilbert space. We will generalize the SMW formula for the elements of Banach algebra. The generalization is up to the case of generalized Drazin inverse.

Theorem 2.1. *Let $a, g \in \mathcal{A}^d$, $y, z \in \mathcal{A}$ and $b = a + ygz$, $t = g^d + za^dy$ such that $b, t \in \mathcal{A}^d$. If*

$$b^\pi a^d = 0, \quad b^d a^\pi = 0, \quad yg^\pi = 0, \quad gt^\pi = 0,$$

then

$$(a + ygz)^d = a^d - a^dy(g^d + za^dy)^dza^d. \tag{2}$$

Proof. The condition $b^\pi a^d = 0$ gives us $b^d ba^d = a^d$. Analogously, from the other three conditions, we obtain $b^d aa^d = b^d$, $ygg^d = y$ and $gtt^d = g$. Using these equations, we get

$$\begin{aligned} b^d ygt &= b^d yg(g^d + za^d y) = b^d ygg^d + b^d ygza^d y \\ &= b^d y + b^d(b-a)a^d y = b^d y + b^d ba^d y - b^d aa^d y \\ &= b^d y + a^d y - b^d y = a^d y, \end{aligned}$$

and then

$$b^d yg = b^d ygtt^d = a^d yt^d.$$

Now, we can conclude

$$b^d = b^d aa^d = b^d(b - ygz)a^d = b^d ba^d - b^d ygza^d = a^d - a^d yt^d za^d.$$

The theorem has been proved. □

Analogously to the Theorem 2.1, we can prove the following theorem.

Theorem 2.2. *Let $a, g \in \mathcal{A}^d$, $y, z \in \mathcal{A}$ and $b = a + ygz, t = g^d + za^d y$ such that $b, t \in \mathcal{A}^d$. If*

$$a^d b^\pi = 0, \quad a^\pi b^d = 0, \quad g^\pi z = 0, \quad t^\pi g = 0,$$

then

$$(a + ygz)^d = a^d - a^d y(g^d + za^d y)^d za^d. \tag{3}$$

Proof. The conditions $a^d b^\pi = 0, a^\pi b^d = 0, g^\pi z = 0, t^\pi g = 0$, imply the equations $b^d = a^d ab^d$, $a^d = a^d bb^d$, $z = gg^d z$ and $g = t^d tg$. Using these equations, we get

$$\begin{aligned} tgzb^d &= (g^d + za^d y)gzb^d = g^d gzb^d + za^d ygzb^d \\ &= zb^d + za^d(b-a)b^d = zb^d + za^d bb^d - za^d ab^d \\ &= zb^d + za^d - zb^d = za^d, \end{aligned}$$

and then

$$gzb^d = t^d tgzb^d = t^d za^d.$$

Now, we have

$$b^d = a^d ab^d = a^d(b - ygz)b^d = a^d bb^d - a^d ygzb^d = a^d - a^d yt^d za^d.$$

The theorem has been proved. □

The next theorem is the generalization of the Theorem 2.1 [3] which is proved for the modified matrices. We will prove the following result for the elements of Banach algebra and generalize the SMW formula to the case of generalized Drazin inverse.

Theorem 2.3. *Let $a, g \in \mathcal{A}^d$, $y, z \in \mathcal{A}$, $b = a + ygz$, $t = g^d + za^d y$, $t \in \mathcal{A}^d$ and*

$$a^\pi y = 0, \quad yg^\pi = 0, \quad g^\pi z = 0, \quad yt^\pi = 0, \quad t^\pi z = 0.$$

Then $b \in \mathcal{A}^d$ and

$$b^d = (a^d - a^d yt^d za^d)\left(1 + \sum_{n=0}^{+\infty}(a^d - a^d yt^d za^d)^n a^d yt^d za^n a^\pi\right). \tag{4}$$

Proof. Denote with $m = a^d - a^d yt^d za^d$ and with $s = \sum_{n=0}^{+\infty} (a^d - a^d yt^d za^d)^n a^d yt^d za^n a^\pi$. Also, let D be the right-hand side of the equation (6). So, $D = m(1 + s)$.

Since $a^d a^\pi = a^\pi a^d = 0$, we have $sa^d = 0$ and $s^2 = 0$. Also, it holds $aa^d m = a^d aa^d - a^d aa^d yt^d za^d = m = ma^d a$, which implies

$$aa^d s = aa^d \sum_{n=0}^{+\infty} m^n a^d yt^d za^n a^\pi = aa^d a^d yt^d za^\pi + \sum_{n=1}^{+\infty} aa^d m^n a^d yt^d za^n a^\pi = s.$$

Since the conditions $a^\pi y = 0$, $t^\pi z = 0$ and $yg^\pi = 0$ imply, respectively, the equations $y = aa^d y$, $z = tt^d z$ and $y = ygg^d$, we have

$$\begin{aligned}
bD &= bm(1+s) = (ba^d - ba^d yt^d za^d)(1+s) \\
&= (aa^d + ygza^d - aa^d yt^d za^d - ygza^d yt^d za^d)(1+s) \\
&= (aa^d + ygza^d - yt^d za^d - yg(t - g^d)t^d za^d)(1+s) \\
&= (aa^d + ygza^d - yt^d za^d - ygtt^d za^d + ygg^d t^d za^d)(1+s) \\
&= (aa^d + ygza^d - yt^d za^d - ygza^d + yt^d za^d)(1+s) \\
&= aa^d(1+s) = aa^d + s.
\end{aligned}$$

Analoguosly, the conditions $yt^\pi = 0$ and $g^\pi z = 0$ imply, respectively, the equations $y = yt^d t$ and $z = gg^d z$. Since $a^\pi b = a^\pi(a + ygz) = a^\pi a$, we have

$$\begin{aligned}
Db &= mb + msb = (a^d - a^d yt^d za^d)(a + ygz) + \sum_{n=0}^{+\infty} m^{n+1} a^d yt^d za^n a^\pi b \\
&= a^d a + a^d ygz - a^d yt^d za^d a - a^d yt^d za^d ygz + \sum_{n=0}^{+\infty} m^{n+1} a^d yt^d za^{n+1} a^\pi \\
&= a^d a + a^d ygz - a^d yt^d za^d a - a^d yt^d (t - g^d) gz + \sum_{n=1}^{+\infty} m^n a^d yt^d za^n a^\pi \\
&= a^d a + a^d ygz - a^d yt^d za^d a - a^d yt^d tgz + a^d yt^d g^d gz + s - a^d yt^d za^\pi \\
&= a^d a + a^d ygz - a^d yt^d z(aa^d + a^\pi) - a^d ygz + a^d yt^d z + s \\
&= a^d a + s.
\end{aligned}$$

Since a and a^d commute, we have $bD = Db$.

Further, we have

$$DbD = m(1+s)(aa^d + s) = maa^d + ms + msaa^d + ms^2 = m + ms = D.$$

Finally, since we already calculate that $bm = aa^d$, we have

$$\begin{aligned}
b - b^2 D &= b - b(aa^d + s) = ba^\pi - bs \\
&= ba^\pi - b\left(a^d yt^d za^\pi + \sum_{n=1}^{+\infty} m^n a^d yt^d za^n a^\pi\right) \\
&= ba^\pi - (a + ygz)a^d yt^d za^\pi - bm \sum_{n=1}^{+\infty} m^{n-1} a^d yt^d za^n a^\pi \\
&= ba^\pi - aa^d yt^d za^\pi - ygza^d yt^d za^\pi - aa^d sa \\
&= ba^\pi - yt^d za^\pi - yg(t - g^d)t^d za^\pi - sa
\end{aligned}$$

$$\begin{aligned}
&= ba^\pi - yt^dza^\pi - ygtt^dza^\pi + ygg^dt^dza^\pi - sa\\
&= ba^\pi - yt^dza^\pi - ygza^\pi + yt^dza^\pi - sa\\
&= ba^\pi - ygza^\pi - sa\\
&= aa^\pi - sa
\end{aligned}$$

Since $a^\pi a^d = 0$ and $a^\pi m = 0$, we get

$$(sa)^2 = \left(\sum_{n=0}^{+\infty} m^n a^d yt^d za^{n+1}a^\pi\right)\left(\sum_{n=0}^{+\infty} m^n a^d yt^d za^{n+1}a^\pi\right) = 0,$$

which implies that sa is the nilpotent element, so it is also the quasi-nilpotent element.

Now, we have $aa^\pi, sa \in \mathcal{A}^{qnil}$ and

$$aa^\pi sa = asa - a^2a^dsa = asa - asa = 0,$$

so, we can apply the Lemma 1.1 and we get that $b - b^2D = aa^\pi - sa \in \mathcal{A}^{qnil}$.

We have proved that $D \in \mathcal{A}$ is such that $bD = Db$, $DbD = D$ and $b - b^2D \in \mathcal{A}^{qnil}$. Thus, D is generalized Drazin inverse of b. □

As a corollary of the Theorem 2.3, we will prove the following result. The obtained result in the next corollary generalize the Theorem 2.4 [2] proved for the operators on Hilbert space.

Corollary 2.4. *Let $a, g \in \mathcal{A}^d$, $y, z \in \mathcal{A}$, $b = a + ygz$, $t = g^d + za^dy$, $t \in \mathcal{A}^d$ and*

$$a^\pi y = 0, \quad za^\pi = 0, \quad yg^\pi = 0, \quad g^\pi z = 0, \quad yt^\pi = 0, \quad t^\pi z = 0.$$

Then $b \in \mathcal{A}^d$ and

$$b^d = a^d - a^dyt^dza^d.$$

Proof. The conditions from the Theorem 2.3 are satisfied, so we have the equation

$$b^d = (a^d - a^dyt^dza^d)\left(1 + \sum_{n=0}^{+\infty}(a^d - a^dyt^dza^d)^n a^dyt^dza^na^\pi\right). \tag{5}$$

Since $za^\pi = 0$, we have $\sum_{n=0}^{+\infty}(a^d - a^dyt^dza^d)^n a^dyt^dza^na^\pi = 0$ and $b^d = a^d - a^dyt^dza^d$. □

Further, as the corollary of the Theorem 2.3, we can get the formula for the generalized Drazin inverse for the sum of elements in Banach algebra. We will apply Theorem 2.3 in the case when $g = 1$, $z = 1$ and when the element $1 + a^dy$ is invertible.

Corollary 2.5. *Let $a \in \mathcal{A}^d$, $y \in \mathcal{A}$, $t = 1 + a^dy \in \mathcal{A}^{-1}$ and $a^\pi y = 0$. Then $a + y \in \mathcal{A}^d$ and*

$$(a + y)^d = t^{-1}a^d\left(1 - t^{-1}\sum_{n=0}^{+\infty}(a^dt^{-1})^n a^na^\pi\right). \tag{6}$$

Proof. Since the conditions of the Theorem 2.3 are satisfied, we get that $a+y$ is generalized Drazin invertible and it holds

$$\begin{aligned}
(a+y)^d &= (a^d - a^d y t^{-1} a^d)\left(1 + \sum_{n=0}^{+\infty} (a^d - a^d y t^{-1} a^d)^n a^d y t^{-1} a^n a^\pi\right) \\
&= (a^d - (t-1)t^{-1} a^d)\left(1 + \sum_{n=0}^{+\infty} (a^d - (t-1)t^{-1} a^d)^n (t-1) t^{-1} a^n a^\pi\right) \\
&= (a^d - a^d + t^{-1} a^d)\left(1 + \sum_{n=0}^{+\infty} (t^{-1} a^d)^n (t-1) t^{-1} a^n a^\pi\right) \\
&= t^{-1} a^d \left(1 + \sum_{n=0}^{+\infty} (t^{-1} a^d)^n a^n a^\pi - \sum_{n=0}^{+\infty} (t^{-1} a^d)^n t^{-1} a^n a^\pi\right) \\
&= t^{-1} a^d \left(1 + a^\pi + \sum_{n=1}^{+\infty} (t^{-1} a^d)^n a^\pi a^n - t^{-1} a^\pi - \sum_{n=1}^{+\infty} (t^{-1} a^d)^n t^{-1} a^n a^\pi\right) \\
&= t^{-1} a^d \left(1 + a^\pi - t^{-1} a^\pi - \sum_{n=1}^{+\infty} t^{-1} (a^d t^{-1})^n a^n a^\pi\right) \\
&= t^{-1} a^d \left(1 + a^\pi - \sum_{n=0}^{+\infty} t^{-1} (a^d t^{-1})^n a^n a^\pi\right) \\
&= t^{-1} a^d \left(1 - t^{-1} \sum_{n=0}^{+\infty} (a^d t^{-1})^n a^n a^\pi\right).
\end{aligned}$$

□

Analogously to the proof of the Theorem 2.3, we have proved the following result.

Theorem 2.6. *Let $a, g \in \mathcal{A}^d$, $y, z \in \mathcal{A}$, $b = a + ygz$, $t = g^d + za^d y$, $t \in \mathcal{A}^d$ and*

$$za^\pi = 0, \quad yg^\pi = 0, \quad g^\pi z = 0, \quad yt^\pi = 0, \quad t^\pi z = 0.$$

Then $b \in \mathcal{A}^d$ and

$$b^d = \left(1 + \sum_{n=0}^{+\infty} a^\pi a^n y t^d z a^d (a^d - a^d y t^d z a^d)^n\right)(a^d - a^d y t^d z a^d). \tag{7}$$

Proof. The proof of this theorem is similar to the proof of the Theorem 2.3. Because of that, we will write only the results after calculations. Denote the right-hand side of the equation (7) with D. So, we have

$$Db = bD = aa^d + \sum_{n=0}^{+\infty} a^\pi a^n y t^d z a^d (a^d - a^d y t^d z a^d)^n,$$

$$DbD = D,$$

$$b - b^2 D = aa^\pi - a\left(\sum_{n=0}^{+\infty} a^\pi a^n y t^d z a^d (a^d - a^d y t^d z a^d)^n\right) \in \mathcal{A}^{qnil}.$$

□

As a corollary of the previous Theorem 2.6, we can get the formula for the generalized Drazin inverse for the sum $a+z$ under conditions that $za^\pi = 0$, $a \in \mathcal{A}^d$ and $1 + za^d \in \mathcal{A}^{-1}$.

Corollary 2.7. *Let $a \in \mathcal{A}^d$, $z \in \mathcal{A}$, $t = 1 + za^d \in \mathcal{A}^{-1}$ and $za^\pi = 0$. Then $a + z \in \mathcal{A}^d$ and*

$$(a+z)^d = \left(1 - a^\pi \sum_{n=0}^{+\infty} a^n (t^{-1}a^d)^n t^{-1}\right) a^d t^{-1}. \tag{8}$$

Proof. Notice that $za^d = t - 1$, so it holds $a^d - a^d t^{-1} z a^d = a^d - a^d t^{-1}(t-1) = a^d t^{-1}$. Now, appling the Theorem 2.6 to the elements a, z, $y = 1$, $g = 1$, we have

$$\begin{aligned}
(a+z)^d &= \left(1 + \sum_{n=0}^{+\infty} a^\pi a^n t^{-1} z a^d (a^d - a^d t^{-1} z a^d)^n\right)(a^d - a^d t^{-1} z a^d) \\
&= \left(1 + \sum_{n=0}^{+\infty} a^\pi a^n t^{-1} z a^d (a^d t^{-1})^n\right) a^d t^{-1} \\
&= \left(1 + a^\pi t^{-1} z a^d + \sum_{n=1}^{+\infty} a^\pi a^n t^{-1} z a^d (a^d t^{-1})^n\right) a^d t^{-1} \\
&= \left(1 + a^\pi t^{-1}(t-1) + \sum_{n=1}^{+\infty} a^\pi a^n t^{-1} (t-1) (a^d t^{-1})^n\right) a^d t^{-1} \\
&= \left(1 + a^\pi - a^\pi t^{-1} + \sum_{n=1}^{+\infty} a^\pi a^n (a^d t^{-1})^n - \sum_{n=1}^{+\infty} a^\pi a^n t^{-1} (a^d t^{-1})^n\right) a^d t^{-1} \\
&= \left(1 + a^\pi - a^\pi t^{-1} - a^\pi \sum_{n=1}^{+\infty} a^n (t^{-1} a^d)^n t^{-1}\right) a^d t^{-1} \\
&= \left(1 - a^\pi \sum_{n=0}^{+\infty} a^n (t^{-1} a^d)^n t^{-1}\right) a^d t^{-1}
\end{aligned}$$

□

The next theorem generalize SMW formula and holds under some new conditions.

Theorem 2.8. *Let $a, g, y, z \in \mathcal{A}$ such that $a, g \in \mathcal{A}^d$ and $b = a + ygz$, $t = g^d + za^d y$ such that $t \in \mathcal{A}^d$. If*

$$gt^\pi = g^\pi t^d, \qquad t^\pi g = t^d g^\pi, \qquad a^\pi y t^d z a^d = a^d y t^d z a^\pi, \qquad ba^\pi \in \mathcal{A}^{qnil},$$

then $b \in \mathcal{A}^d$ and

$$b^d = a^d - a^d y t^d z a^d.$$

Proof. Let $m = a^d - a^d y t^d z a^d$. Using the conditions $gt^\pi = g^\pi t^d$ and $t^\pi g = t^d g^\pi$, we have

$$\begin{aligned}
bm &= aa^d + ygza^d - aa^d y t^d z a^d - ygza^d y t^d z a^d \\
&= aa^d + ygza^d - (1 - a^\pi) y t^d z a^d - yg(t - g^d) t^d z a^d \\
&= aa^d + ygza^d - (1 - a^\pi) y t^d z a^d - ygtt^d z a^d + ygg^d t^d z a^d \\
&= aa^d + ygza^d - y t^d z a^d + a^\pi y t^d z a^d - yg(1 - t^\pi) z a^d + y(1 - g^\pi) t^d z a^d \\
&= aa^d + ygza^d - y t^d z a^d + a^\pi y t^d z a^d - ygza^d + ygt^\pi z a^d + y t^d z a^d - y g^\pi t^d z a^d \\
&= aa^d + a^\pi y t^d z a^d + y(gt^\pi - g^\pi t^d) z a^d \\
&= aa^d + a^\pi y t^d z a^d
\end{aligned} \tag{9}$$

and

$$\begin{aligned} mb &= a^d a + a^d ygz - a^d yt^d za^d a - a^d yt^d za^d ygz \\ &= a^d a + a^d ygz - a^d yt^d z(1 - a^\pi) - a^d yt^d (t - g^d) gz \\ &= a^d a + a^d ygz - a^d yt^d z(1 - a^\pi) - a^d yt^d tgz + a^d yt^d g^d gz \\ &= a^d a + a^d ygz - a^d yt^d z + a^d yt^d za^\pi - a^d y(1 - t^\pi) gz + a^d yt^d (1 - g^\pi) z \\ &= a^d a + a^d ygz - a^d yt^d z + a^d yt^d za^\pi - a^d ygz + a^d yt^\pi gz + a^d yt^d z - a^d yt^d g^\pi z \\ &= a^d a + a^d yt^d za^\pi + a^d y(t^\pi g - t^d g^\pi) z \\ &= a^d a + a^d yt^d za^\pi. \end{aligned} \tag{10}$$

Since $a^\pi yt^d za^d = a^d yt^d za^\pi$, we can conclude that $bm = mb$.

Notice that the condition $a^\pi yt^d za^d = a^d yt^d za^\pi$ and the fact that a^π is idempotent imply

$$a^\pi yt^d za^d = a^\pi a^\pi yt^d za^d = a^\pi a^d yt^d za^\pi = 0,$$

so it holds $a^\pi yt^d za^d = a^d yt^d za^\pi = 0$ and we have $bm = mb = aa^d$.

Further, we have

$$mbm = a^d aa^d - a^d yt^d za^d aa^d = m$$

Finally,

$$\begin{aligned} b - b^2 m &= a + ygz - a^2 a^d - ygzaa^d \\ &= aa^\pi + ygza^\pi \\ &= ba^\pi \in \mathcal{A}^{qnil}. \end{aligned}$$

From all proved above, we have that $b \in \mathcal{A}^d$ and $b^d = m = a^d - a^d yt^d za^d$. □

Since $b - b^2 m = b - mb^2 = (1 - mb)b = a^\pi b$, the condition $ba^\pi \in \mathcal{A}^{qnil}$ in the Theorem 2.8 can be replaced by $a^\pi b \in \mathcal{A}^{qnil}$.

Lemma 2.9. *Let $a, g, y, z \in \mathcal{A}$ such that $a, g \in \mathcal{A}^d$ and $t = g^d + za^d y$ such that $t \in \mathcal{A}^d$. If $gt^\pi = g^\pi t^d$, $t^\pi g = t^d g^\pi$ and $a^\pi yt^d za^d = a^d yt^d za^\pi$, then $ygza^\pi = yt^d za^\pi$ and $a^\pi ygz = a^\pi yt^d z$.*

Proof. As we have proved in the Theorem 2.8, the equation $a^\pi yt^d za^d = a^d yt^d za^\pi$ implies that $a^\pi yt^d za^d = a^d yt^d za^\pi = 0$. Further, we have $a^d yt^d za^\pi = a^d yt^d z - a^d yt^d zaa^d = 0$. So, $a^d yt^d z = a^d yt^d zaa^d$.

The condition $gt^\pi = g^\pi t^d$ gives us $g - gtt^d = t^d - gg^d t^d$. So, we have

$$g - t^d = g(t - g^d) t^d = gza^d yt^d.$$

Multiplying the last equation with y on the left side and with z on the right side, we obtain $ygz - yt^d z = ygza^d yt^d z$. So, $ygz - yt^d z = ygza^d yt^d zaa^d$ and we have $ygza^\pi - yt^d za^\pi = ygza^d yt^d zaa^d a^\pi = 0$. It holds $ygza^\pi = yt^d za^\pi$.

Analogously, $a^\pi yt^d za^d = 0$ implies $yt^d za^d = a^d ayt^d za^d$. The condition $t^\pi g = t^d g^\pi$ implies $g - t^d = t^d (t - g^d) g = t^d za^d yg$, so we get $a^\pi ygz - a^\pi yt^d z = a^\pi a^d ayt^d za^d ygz = 0$. □

Under assumption that the first three conditions in the Theorem 2.8 hold, by Lemma 2.9, we have the following equivalence:

$$ba^\pi \in \mathcal{A}^{qnil} \Leftrightarrow (a + yt^dz)a^\pi \in \mathcal{A}^{qnil}.$$

So, the fourth condition in Theorem 2.8 can be replaced by $(a + yt^dz)a^\pi \in \mathcal{A}^{qnil}$. Analogously, it can be also replaced by $a^\pi(a + yt^dz) \in \mathcal{A}^{qnil}$.

Lemma 2.10. *Let $a, g, y, z \in \mathcal{A}$ such that $a, g \in \mathcal{A}^d$ and $t = g^d + za^dy$ such that $t \in \mathcal{A}^d$. If $yt^dza^d - a^dyt^dz = a^da$, then $a^\pi yt^dza^d = a^dyt^dza^\pi = 0$.*

Proof. We have $a^\pi yt^dza^d = a^\pi(a^da + a^dyt^dz) = 0$ and $a^dyt^dza^\pi = (yt^dza^d - aa^d)a^\pi = 0$. □

Using the Lemma 2.10, the following corollary of Theorem 2.8 holds.

Corollary 2.11. *Let $a, g, y, z \in \mathcal{A}$ such that $a, g \in \mathcal{A}^d$ and $b = a + ygz$, $t = g^d + za^dy$ such that $t \in \mathcal{A}^d$. If*

$$gt^\pi = g^\pi t^d, \qquad t^\pi g = t^dg^\pi, \qquad yt^dza^d - a^dyt^dz = 1 - a^\pi, \qquad ba^\pi \in \mathcal{A}^{qnil},$$

then $b \in \mathcal{A}^d$ and

$$b^d = a^d - a^dyt^dza^d.$$

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