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Shadowing, Ergodic Shadowing and Uniform Spaces

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Abstract. We introduce and study the topological concepts of ergodic shadowing, chain transitivity and topological ergodicity for dynamical systems on non-compact non-metrizable spaces. These notions generalize the relevant concepts for metric spaces. We prove that a dynamical system with topological ergodic shadowing property is topologically chain transitive, and that topological chain transitivity together with topological shadowing property implies topological ergodicity.

1. Introduction

The notion of *shadowing*, introduced in the 1970s independently by Anosov [1] and Bowen [5], is a well known property in the qualitative theory of dynamical systems [6, 10–12, 14]. This concept is motivated by computer simulations. In fact, let X be a set and $f : X \to X$ be a map. Then in the numerical computation of f with initial value $x_0 \in X$, computer approximates $f(x_0)$ by x_1. To continue the algorithm, it computes the value x_2 as an approximation of $f(x_1)$ and so on. Sometimes this sequence (called a pseudo-orbit) plays the role of a shadow for an orbit $\{f^n(x)\}_{n\in\mathbb{N}_0}$ of some point x in X where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The shadowing problem is related to the following question: Under which conditions, for any pseudo-orbit of f a closed real orbit can be found?

In [8] the authors introduced the concept of *partial* shadowing property formulated in terms of ”large” subsets of $\mathbb{N}$, to answer the following question: Given an ergodic pseudo-orbit, when can we find a real orbit which is close to it part of the time? The main idea of the definition of ergodic pseudo-orbit is to use the concepts of *lower density* and *upper density* for elements of Furstenburg families to measure the set of error times in a pseudo-orbit [13, 15–17]. One can see that shadowing properties depend on metric used in approximations.

A dynamical system on a non-compact metric space may have the shadowing property with respect to one metric but not with respect to another one that induces the same topology. Hence one prefers to have a theory that is independent of any choice of the metric. Recently, many authors have extended various notions of dynamical properties on metric spaces for homeomorphisms on general topological spaces [7, 9, 18]. In [7] the authors have extended the spectral decomposition theorem to dynamical systems on spaces that are not necessarily metrizable and not necessarily compact. Here we give topological definitions of ergodic shadowing property which present a parallel approach to the average shadowing property introduced by Blank [4] for metric spaces.

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The paper is organized as follows: We discuss the preliminaries in the next section. In Section 3 we study the relation between the shadowing property and chain transitivity. Finally, in Section 4 we investigate the properties of topologically ergodic systems.

2. Uniform Spaces and Pseudo-orbits

A *uniform space* is a set with a uniform structure defined on it. A uniform structure $\mathcal{U}$ on a space X is defined by the specification of a system of subsets of the product $X \times X$. The family $\mathcal{U}$ must satisfy the following axioms:

U1) for any $E_1, E_2 \in \mathcal{U}$, the intersection $E_1 \cap E_2$ is also contained in $\mathcal{U}$, and if $E_1 \subset E_2$ and $E_1 \in \mathcal{U}$, then $E_2 \in \mathcal{U}$;

U2) every set $E \in \mathcal{U}$ contains the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$;

U3) if $E \in \mathcal{U}$, then $E^T = \{(y, x) \mid (x, y) \in E\} \in \mathcal{U}$;

U4) for any $E \in \mathcal{U}$ there exists $\hat{E} \in \mathcal{U}$ such that $\hat{E} \circ \hat{E} \subset E$, where

$$\hat{E} \circ \hat{E} = \{(x, y) \mid \text{ there is a } z \in X \text{ with } (x, z) \in \hat{E}, (z, y) \in \hat{E}\}.$$

The elements of $\mathcal{U}$ are called *entourages* of the uniformity defined by $\mathcal{U}$. If $(X, \mathcal{U})$ is a uniform space, then the uniform topology on X is the topology in which a neighborhood base at a point $x \in X$ is formed by the family of sets $E[x]$, where E runs through the entourages of X and $E[x] = \{y \in X | (x, y) \in E\}$ is called the cross section of E at x. By saying that a topological space is uniformizable we mean, of course, that there exists a uniformity such that the associated uniform topology is the given topology. It can be shown that a topological space is uniformizable if and only if it is completely regular. To avoid confusion, we denote subsets of the product space $X \times X$ by D, E, F, etc. , and the subsets of the base space X by $\dot{U}$, $\dot{V}$, $\dot{W}$, etc. and denote

$$\begin{aligned} E^n :=& E \circ E \circ \cdots \circ E \ (n \text{ times}) \\ =& \{(x, y) | \text{there exist } z_0 = x, z_1, \ldots, z_n = y;\ (z_{i-1}, z_i) \in E \text{ for } i = 1, 2, \ldots n\}. \end{aligned}$$

A mapping $f : X \to Y$ from a uniform space X into a uniform space Y is called uniformly continuous if the inverse image $(f \times f)^{-1}(E)$ is an entourage of X for each entourage E of Y.

Throughout the paper, by a dynamical system we mean a pair (X, f) where X is a completely regular topological space and $f : X \to X$ is a uniformly continuous map. For a given dynamical system (X, f), we use the product map $F = f \times f$ on $X \times X$.

Let (X, f) be a dynamical system and let D and E be entourages of X. A (D, f)-chain of length n is a sequence $\xi = \{x_i\}_{i=0}^n$ such that $(f(x_i), x_{i+1}) \in D$ for $i = 0, ..., n-1$. An (f, D)-pseudo-orbit is an infinite (f, D)-chain. If there is no danger of confusion, we write D-chain and D-pseudo-orbit instead of (f, D)-chain and (f, D)-pseudo-orbit, respectively. A D-pseudo-orbit $\xi = \{x_i\}$ is E-traced by a point $y \in X$ if $(f^i(y), x_i) \in E$ for all $i \in \mathbb{N}_0$.

Definition 2.1. ([7]) A dynamical system (X, f) has the *topological shadowing property* if for every entourage E of X , we can find an entourage D of X such that every D-pseudo-orbit is E-shadowed by some point y in X.

Denote by $\mathcal{TS}$ the set of all dynamical systems with the topological shadowing property. It is observed that for general topological spaces metric shadowing and topological shadowing are independent concepts [7]. However, for a compact metric space, topological shadowing and metric shadowing are equivalent.

Definition 2.2. A dynamical system (X, f) is called *topologically chain transitive* if for any entourage D of X and any two points x and y there exists a D-chain from x to y. It is said to be chain recurrent if for any entourage D and any $x \in X$ there is D-chain from x to itself. It is called topologically chain mixing if for any two points $x, y \in X$ and any entourage D of X there is a positive integer N such that for any $n \geq N$ there is a D-chain from x to y of length n.

Denote by $\mathcal{TCT}$ and $\mathcal{TCM}$ the set of all topologically chain transitive and topologically chain mixing dynamical systems, respectively.

Note that in the case of compact metric spaces for any entourage E of X there exists *epsilon* > 0 such that $d^{-1}[0, \epsilon] \subset E$. Therefore the above definitions coincide with the usual ones in compact metric spaces. However, this argument does not hold if X is not compact. For example, consider the entourage $E = \{(x, y) \in \mathbb{R}^2 : |x - y| < e^{x^2}\}$ of $\mathbb{R}^2$. There is no $\epsilon > 0$ with $d^{-1}[0, \epsilon] \subset E$.

3. Topological Ergodic Shadowing Property

A family $\mathcal{F}$ of subsets of $\mathbb{N}_0$ is called a *Furstenburg family* if for any $\mathbb{S} \in \mathcal{F}$, the inclusion $\mathbb{S} \subset \mathbb{S}'$ implies that $\mathbb{S}' \in \mathcal{F}$. For any set $\mathbb{S} \subset \mathbb{N}_0$, the upper density of $\mathbb{S}$ is defined by

$$\overline{d}(\mathbb{S}) := \limsup_{n\to\infty} \frac{1}{n} |\mathbb{S} \cap \{0, 1, \ldots, n-1\}|$$

and the lower density of $\mathbb{S}$ is defined by

$$\underline{d}(\mathbb{S}) := \liminf_{n\to\infty} \frac{1}{n} |\mathbb{S} \cap \{0, 1, \ldots, n-1\}|.$$

If there exists a number $d(\mathbb{S})$ such that $\overline{d}(\mathbb{S}) = \underline{d}(\mathbb{S}) = d(\mathbb{S})$, then we say that the set $\mathbb{S}$ has density $d(\mathbb{S})$.

If $\xi = \{x_i\}_{i=0}^{\infty}$ and $\eta = \{y_i\}_{i=0}^{\infty}$ are sequences in X, then for any entourage E of X we define

$$\Lambda(\xi, \eta, E) = \{i \in \mathbb{N}_0 | (x_i, y_i) \in E\}, \qquad \Lambda^c(\xi, \eta, E) = \{i \in \mathbb{N}_0 | (x_i, y_i) \notin E\}.$$

Here we use the following notations

$$\Lambda(\xi, f, E) := \{i \in \mathbb{N}_0 | (x_{i+1}, f(x_i)) \in E\} \qquad \Lambda^c(\xi, f, E) := \{i \in \mathbb{N}_0 | (x_{i+1}, f(x_i)) \notin E\}$$
$$\Lambda(\xi, z, f, E) := \{i \in \mathbb{N}_0 | (x_i, f^i(z)) \in E\} \qquad \Lambda^c(\xi, z, f, E) := \{i \in \mathbb{N}_0 | (x_i, f^i(z)) \notin E\}.$$

If $\mathbb{S} \subset \mathbb{N}_0$, then we denote by $\mathbb{S}_n$ the set $\mathbb{S} \cap \{0, 1, \ldots, n-1\}$.

A sequence $\xi = \{x_i\}_{i=0}^{\infty}$ is called an *ergodic D-pseudo-orbit* provided that the set $\Lambda^c(\xi, f, D)$ has density zero, that is $\lim_{n\to\infty} 1/n|\Lambda_n^c(\xi, f, D)| = 0$. For an entourage E of X, an ergodic D-pseudo-orbit ξ is said to be ergodically E-shadowed by a point $x \in X$ provided that the set $\Lambda^c(\xi, z, f, E)$ has density zero, that is $\lim_{n\to\infty} 1/n|\Lambda_n^c(\xi, z, f, E)| = 0$.

Definition 3.1. A dynamical system (X, f) has the *topological ergodic shadowing property* provided that for any entourage E of X there exists an entourage D of X such that any ergodic D-pseudo-orbit can be ergodically E-shadowed by some point in X.

Denote by $\mathcal{TES}$ the set of all dynamical systems with the topological ergodic shadowing property.

If (X, d) is a compact metric space, then for any entourage E of X , we can find $\epsilon > 0$ such that $d^{-1}[0, \epsilon] \subset E$. On the other hand, for every $\epsilon > 0$ the set $d^{-1}[0, \epsilon]$ is an entourage of X. Thus, the above definition coincide with the usual notion of ergodic shadowing on compact metric spaces [10].

In the compact case, topological ergodic shadowing property is invariant under topological conjugacy. However, this argument does not hold if we omit the compactness.

Example 3.2. ([7]) Let $X \subset \mathbb{R}^2$ be the subset $\bigcup_{k\in\mathbb{Z}}\{k\} \times [0, 2^{-|k|}]$, with topology inherited from $\mathbb{R}^2$ and define $f : X \to X$ by

$$f(k, x) = \begin{cases} (k + 1, 2^{-1}x) & \text{if } k \geq 0, \\ (k + 1, 2x) & \text{if } k < 0 \end{cases}.$$

Then (X, f) has the topological ergodic shadowing property. Indeed for any $\epsilon > 0$ we obtain diam$(\{k\} \times [0, 2^{-|k|}]) < \epsilon$ for all k with $|k| > \lceil \frac{1}{\epsilon} \rceil$. Also let $Y \subset \mathbb{R}^2$ be the subset $\bigcup_{k\in\mathbb{Z}}\{k\} \times [0, 1]$, with topology inherited from $\mathbb{R}^2$ and define $g : Y \to Y$ by $g(k, y) = (k + 1, y)$. Then (Y, g) does not have the topological shadowing and consequently topological ergodic shadowing property. One can see that the map $h : X \to Y$ defined by $h(k, x) = (k, 2^{|k|}x)$ is a topological conjugacy between (X, f) and (Y, g).

Next we discuss relations between the topological ergodic shadowing property and other dynamical properties. In [10] authors prove that in a compact metric space ergodic shadowing property implies chain transitivity. We use their method to prove the same statement in a non-metrizable space.

Proposition 3.3. *Let (X, f) be a dynamical system with the topological ergodic shadowing property. Then it is topologically chain transitive.*

Proof. Suppose that x and y are two points in X. Let E be an entourage of X. Then there exists an entourage D of X such that every ergodic D-pseudo-orbit can be ergodically shadowed by some point in X. Let $s_k = k(k + 1)$ and $t_k = (k + 1)^2$, $k = 0, 1, 2, \dots$, and choose some point $z_k = f^{-k}(y)$. For $i = 0, 1, 2, \dots$ put

$$x_i = \begin{cases} f^{i-s_k}(x) & \text{if } s_k \leq i < t_k, \\ f^{i-t_k}(y) & \text{if } t_k \leq i < s_{k+1} \end{cases}.$$

We see that

$$x_0 = x, \quad x_1, \dots, x_{s_k-1} = f^k(x), x_{t_k} = f^{-k}(y), \dots, x_{s_{k+1}-1} = f^{-1}(y).$$

Hence $\Lambda^c(\xi, f, D) \subset \{s_k, t_k;\ k = 0, 1, 2, \dots\}$ and

$$\lim_{n\to\infty} \frac{1}{n}|\Lambda^c_n(\xi, f, D)| \leq \lim_{n\to\infty} \frac{1}{n}(2\sqrt{n}) = 0.$$

Thus $\xi = \{x_i\}$ is an ergodic D-pseudo-orbit and can be ergodically E-shadowed by some point z. This means that $\Lambda^c(\xi, z, f, E)$ has density zero and its complement must intersect infinitely many intervals $[s_k, t_k)$ and infinitely many intervals $[t_k, s_{k+1})$. Hence there exist $q, r, s, t \in \mathbb{N}$ such that $(f^q(x), f^r(z)), (f^s(z), w) \in E$ for some $w \in f^{-t}(y)$. Therefore

$$\{x, f(x), \dots, f^{q-1}(x), f^r(z), f^{r+1}(z), \dots, f^{s-1}(z), w, f(w), \dots, f^{t-1}(w), y\}$$

is an E-pseudo-orbit from x to y. This complete the proof. □

Now we show that $\mathcal{TES}$ is a subset of $\mathcal{TS}$.

Proposition 3.4. *Let (X, f) be a dynamical system with the topological ergodic shadowing property. Then it has the topological shadowing property.*

Proof. Suppose that f has the topological ergodic shadowing property. Let E be an entourage. Then there exists an entourage D such that any ergodic D-pseudo-orbit can be ergodically E-shadowed by some point in X. If $\xi = \{x_i;\ i = 1, 2, \dots n\}$ is a finite D-pseudo-orbit, then by Proposition 3.3 there exists a D-chain $\{z_0 = x_n, z_1, \dots, z_k = x_1\}$ from x_n to x_1. Hence the sequence

$$\eta = \{x_1, x_2, \dots, x_n = z_0, z_1, \dots, z_k = x_1, x_2, \dots, x_n = z_0, z_1, \dots\}$$

is an ergodic D-pseudo-orbit and so can be ergodically E-shadowed by some point x in X. Therefore $\Lambda^c(\eta, x, E, f)$ cannot meet every ξ interval (i.e. intervals in η which coincide with ξ). Hence its density is positive. Thus at least one ξ interval is entirely E-shadowed by a piece of the orbit of x. □

Proposition 3.5. *Let (X, f) be a dynamical system. Then the following statements are equivalent.*

1. *(X, f) has the topological ergodic shadowing property;*
2. *(X, f^k) has the topological ergodic shadowing property for any integer $k \geq 2$;*
3. *(X, f^k) has topological ergodic shadowing property for some integer $k \geq 2$.*

Proof. (1 ⇒ 2) Suppose that f has the topological ergodic shadowing property. Let E be any entourage of X and let D be an entourage such that every ergodic (f, D)-pseudo-orbit is ergodically (f, E)-shadowed by some point of X. Assume that $\xi = \{x_i\}_{i=0}^{\infty}$ is an ergodic (f^k, D)-pseudo-orbit, i.e., $\Lambda^c(\xi, f^k, D)$ has density zero. Then

$$\eta = \{x_0, f(x_0), \dots, f^{k-1}(x_0), x_1, f(x_1), \dots, f^{k-1}(x_1), \dots\}$$

is an ergodic (f, D)-pseudo-orbit and hence there exists $y \in X$ such that $\Lambda^c(\eta, y, f, E)$ has density zero. This implies that $\Lambda^c(\xi, y, f^k, E)$ also has density zero. Therefore ξ is ergodically (f^k, E)-shadowed by $y \in X$.

(2 ⇒ 3) is immediate.

(3 ⇒ 1) Let E be any entourage of X. Let U be an entourage of X such that $U^k \subset E$. Then $\hat{E} = \cap_{i-0}^{k-1} F^{-i}(U)$ is an entourage of X, so there exists an entourage D of X such that every ergodic (f^k, D)-pseudo-orbit is $(f^k, \hat{E})$-shadowed by some point y in X. Let V be an entourage of X such that $V^k \subset D$ and define $\hat{D} = \cap_{i=0}^{k-1} F^{-i}(V)$. Now if $\xi = \{x_i\}_{i=0}^{\infty}$ is an ergodic $(f, \hat{D})$-pseudo-orbit, then $\hat{\xi} = \{x_{ik}\}_{i=0}^{\infty}$ is an ergodic (f^k, D)-pseudo-orbit, so it is ergodically $(f^k, \hat{E})$-shadowed by some point y in X. Hence $\Lambda^c(\hat{\xi}, y, f^k, \hat{E})$ has density zero. Thus $\Lambda^c(\xi, y, f, V)$ has density zero. This shows that f has the topological ergodic shadowing property. □

Let $\dot{U}$ and $\dot{V}$ be open subsets of X and let D be an entourage of X. Then we define

$$N_f(\dot{U}, \dot{V}) = \{i \in \mathbb{N} | f^i(\dot{U}) \cap \dot{V} \neq \emptyset\},$$
$$N_f(\dot{U}, D) = \{n \in \mathbb{N} | \text{ there exist } x, y \in \dot{U} \text{ such that } F^n(x, y) \notin D\}.$$

Definition 3.6. A dynamical system (X, f) is said to be *topologically transitive* provided that $N_f(\dot{U}, \dot{V}) \neq \emptyset$ for any pair $(\dot{U}, \dot{V})$ of nonempty open subsets of X. A dynamical system (X, f) is topologically mixing if for every pair $(\dot{U}, \dot{V})$ of nonempty open subsets of X there exists $N \in \mathbb{N}$ such that

$$N_f(\dot{U}, \dot{V}) \supset \{N, N + 1, N + 2, \dots\}.$$

Denote by $\mathcal{TM}$ the set of all topologically mixing dynamical systems.

Example 3.7. ([7]) Let $X \subset \mathbb{R}$ be the subset $X = \{x_k = \sum_{i=1}^{k} 1/i;\ k \in \mathbb{N}\}$, with discrete uniformity. Let $f : X \to X$ be the identity map. Then we obtain the following statements.

1. $(X, f) \in \mathcal{TS}$. Indeed we can choose an entourage D of X such that $D[x] = x$ for all $x \in X$. So an D-pseudo-orbit is a fixed point.
2. $(X, f) \notin \mathcal{TES}$. Indeed we can choose an entourage D of X such that $D[x] = x$ for all $x \in X$. Then the sequence $\{y_i = \sum_{n \in \mathbb{N}} \chi_{[n^2,(n+1)^2)}(i) x_n\}$ is an ergodic D-pseudo-orbit and such an ergodic pseudo-orbit can not be ergodically shadowed by any real orbit.
3. (X, f) *does not have the (metric) shadowing property.* Given $\delta > 0$ we can choose a natural number n with $\frac{1}{n} < \delta$. Then the sequence $\xi = \{\dots, x_n, x_n, x_n, x_{n+1}, x_{n+2}, \dots\}$ is a δ-pseudo-orbit and such a pseudo-orbit can not be shadowed by any real orbit.
4. $(X, f) \notin \mathcal{TM} \cup \mathcal{TCM}$

Next we prove the same result of [10] on the non-metrizable spaces.

Proposition 3.8. *If the dynamical system (X, f) has the topological shadowing property, then the following are equivalent*

(i) (X, f) *is topologically chain mixing;*

(ii) (X, f) *is topologically mixing.*

Proof. Obviously (*ii*) implies (*i*). Suppose that f has the shadowing property, consider $x, y \in X$ and let $\dot{U}$ and $\dot{V}$ be two open sets with $x \in \dot{U}$ and $y \in \dot{V}$. If E is any entourage of X with $E[x] \subset \dot{U}$ and $E[y] \subset \dot{V}$, then there exists an entourage D of X such that any D-pseudo-orbit can be E-shadowed by some point y in X. Taking a sufficiently large positive integer n there is a D-chain of length n from x to y, say $\xi = \{x_i\}_{i=0}^{n}$ with $x_0 = x$ and $x_n = y$. Hence it is E-shadowed by y, i.e. $(f^i(y), x_i) \in E$ for $i = 0, 1, \ldots, n-1$. Therefore $y \in E[x] \subset \dot{U}$ and $f^n(y) \in E[y] \subset \dot{V}$, so $f^n(y) \in f^n(\dot{U}) \cap \dot{V}$. Thus f is topologically mixing. □

4. Ergodic Sensitivity and Topological Ergodicity

Sensitive dependence on initial conditions is a key ingredient of chaos for dynamical systems [2]. A dynamical system (X, f) is sensitive if any open region of the phase space contains two points for which there is an integer $i \in N$ such that the ith iterates of these points under the map f are significantly separated. The largeness of the set of all discrete times i where this significant separation or sensitivity happens can be thought of as a measure of how sensitive the dynamical system is. Here we study an stronger form of sensitivity formulated in terms of subsets of $\mathbb{N}$ with positive density.

Definition 4.1. A dynamical system (X, f) is *ergodically sensitive* if there exists an entourage D such that $N_f(\dot{V}, D)$ has positive upper density for any nonempty open subset $\dot{V}$ of X.

Definition 4.2. A dynamical system (X, f) is *topologically ergodic* if $N_f(\dot{U}, \dot{V})$ has positive upper density for all nonempty open subsets $\dot{U}$ and $\dot{V}$ of X.

Denote by $\mathcal{TE}$ the set of all topologically ergodic dynamical systems.

Example 4.3. Let $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$. Suppose that $\mathbf{a} = \{a_i\}_{i\in\mathbb{Z}} \subset \mathbb{P}$ is an increasing bi-sequence for which there exists a positive integer k such that $a_i + 1 = a_{i+k}$ for all $i \in \mathbb{Z}$. Put

$$U_a = \Cup_{i\in\mathbb{Z}}[(a_i, a_i) \cup (a_i, a_{i+1}) \times (a_i, a_{i+1})].$$

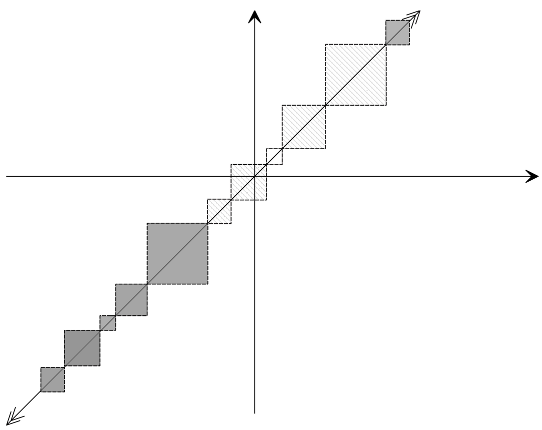

Figure 1: Filter base for the uniformity

(See Figure 1). Then the family $\mathcal{B} = \{U_{\mathbf{a}}\}$ is a filter base and the uniformity generated by this filter base generate a topology in which any point of $\mathbb{P}$ is isolated. Let S^1 be the unit circle. Consider the uniformity $\mathcal{U}$ on S^1 -just taking the projection modulo 1- and let $f : S^1 \to S^1$ be a map defined by $f(x) = x + \alpha$, $\alpha \in \mathbb{P}$. Then one can see that $(S^1, f) \in \mathcal{TCT}$ while $(S^1, f) \notin \mathcal{TS} \cup \mathcal{TES}$.

Example 4.4. The product of uniform spaces $(X_t, \mathcal{U}_t)$, $t \in T$, is the uniform space $(\prod X_t, \prod \mathcal{U}_t)$, where $\prod \mathcal{U}_t$ is the uniformity on $\prod X_t$ with as base for the entourages sets of the form

$$\{(\{x_t\}, \{y_t\}) \mid (x_{t_i}, y_{t_i}) \in V_{t_i}, i = 1, \dots, n\}, \qquad t_i \in T, \quad V_{t_i} \in \mathcal{U}_{t_i}, \quad i = 1, 2, \dots$$

The topology induced on $\prod X_t$ by the uniformity $\prod \mathcal{U}_t$ coincides with the Tikhonov product of the topologies of the spaces X_t. The projections of the product onto the components are uniformly continuous. Consider $\{0, 1\}^{\mathbb{Z}}$ as product of countably infinite copies of the discrete space $\{0, 1\}$ with the product uniformity which is homeomorphic to the Cantor set. Consider the map $T : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ induced by the map $f : \{0, 1\}^3 \to \{0, 1\}$ with

$$f(x, y, z) = \begin{cases} (x + 1) \bmod 1 & \text{if } (y, z) = (1, 0), \\ x & \text{otherwise} \end{cases}.$$

Then $(\{0, 1\}^{\mathbb{Z}}, T) \in \mathcal{TCM}$ while $(\{0, 1\}^{\mathbb{Z}}, T) \notin \mathcal{TS} \cup \mathcal{TES}$(see [3]).

This example shows that $\mathcal{TCT} \not\subset \mathcal{TE}$. If the dynamical system has the topological shadowing property then we have the following proposition. Next we generalize the result of [12] to non-metrizable case.

Proposition 4.5. *If a dynamical system (X, f) is topologically chain transitive with topological shadowing property, then it is topologically ergodic.*

Proof. Suppose that $\dot{U}$ and $\dot{V}$ are nonempty open subsets of X. For points $x \in \dot{U}$ and $y \in \dot{V}$, there exists an entourage E such that $E[x] \in \dot{U}$ and $E[x] \in \dot{V}$. Let D be the entourage that corresponds to the entourage E in the definition of shadowing property. If $\xi = \{x_0 = x, x_1, \dots, x_l = y\}$ and $\eta = \{y_0 = y, y_1, \dots, y_k = x\}$ are chains between x and y and $\xi' = \xi - \{x_0\}$ and $\eta' = \eta - \{y_0\}$, then the sequence $\xi\overline{\eta'\xi'} = \xi\eta'\xi'\eta'\xi' \dots$ is a D-pseudo-orbit and it is E-shadowed by some point z in X. Thus

$$\{(z, x), (f^l(z), y), (f^{l+k}(z), x), (f^{2l+k}(z), y), \dots, (f^{i(l+k)+l}(z), y)\} \in E$$

for $i = 1, 2, \dots$. Hence $N_f(\dot{U}, \dot{V}) \geq (l + k)\mathbb{N} + l$, so

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} |N_f(\dot{U}, \dot{V}) \cap \{0, 1, \dots, n - 1\} &\geq \limsup_{n\to\infty} \frac{1}{(l + k)n + l} |((l + k)\mathbb{N} + l) \cap \{0, 1, \dots, (l + k)n + l - 1\}| \\ &\geq \limsup_{n\to\infty} \frac{n}{(l + k)n + l} = \frac{1}{l + k} > 0, \end{aligned}$$

that is, $N_f(\dot{U}, \dot{V})$ has positive upper density. This complete the proof. □

Recall that a dynamical system (X, f) is called *minimal* if

$$\overline{\mathcal{O}_f(x)} = \overline{\{f^i(x) | \, i \in \mathbb{N}_0\}} = X$$

for all $x \in X$.

Theorem 4.6. *Let (X, f) be a dynamical system. If F is topologically ergodic and f is not minimal, then f is ergodically sensitive.*

Proof. Since f is not minimal, there exists $a \in X$ such that $\overline{\mathcal{O}_f(a)} \neq X$. Hence $X \setminus \overline{\mathcal{O}_f(a)}$ is a nonempty open set. If $b \in X \setminus \overline{\mathcal{O}_f(a)}$, then there exists an entourage D such that $D[b] \in X \setminus \overline{\mathcal{O}_f(a)}$. Hence there exists an entourage $\hat{D}$ such that $\hat{D} \circ \hat{D} \circ \hat{D} \circ \hat{D} \subset D$. Put $\dot{V} = \hat{D}[a]$ and $\dot{W} = \hat{D}[b]$. We claim that $\hat{D}[a] \cap \hat{D}[b] = \emptyset$. Indeed, if $x \in \hat{D}[a] \cap \hat{D}[b]$ then there exist $v \in \dot{V}$ and $w \in \dot{W}$ such that $(a, v), (v, x), (b, w), (w, x) \in \hat{D}$. Hence $(a, b) \in \hat{D} \circ \hat{D} \circ \hat{D} \circ \hat{D} \subset D$ which is a contradiction. We claim that $N_F(\dot{U} \times \dot{U}, \dot{V} \times \dot{W}) \subseteq N_f(\dot{U}, \hat{D})$ for any nonempty open subset $\dot{U}$ of X. To see this let $n \in N_F(\dot{U} \times \dot{U}, \dot{V} \times \dot{W})$. Then $f^n(\dot{U}) \times f^n(\dot{U}) \cap \dot{V} \times \dot{W} \neq \emptyset$. If $x \in f^n(\dot{U}) \cap \dot{V}$ and $y \in f^n(\dot{U}) \cap \dot{W}$, then $(a, x), (b, y) \in \hat{D}$ and there exist $u_1, u_2 \in \dot{U}$ such that $x = f^n(u_1)$ and $y = f^n(u_2)$. Hence $(a, f^n(u_1)), (b, f^n(u_2)) \in \hat{D}$. We see that $F^n(u_1, u_2) \notin \hat{D}$. Indeed, if $F^n(u_1, u_2) \in \hat{D}$ then $(b, f^n(u_1)), (a, f^n(u_2)) \in \hat{D} \circ \hat{D}$. Thus $(a, b) \in \hat{D} \circ \hat{D} \circ \hat{D} \subseteq D$ which is not possible. This proves the desired inclusion. Since F is topologically ergodic, the set $N_F(\dot{U} \times \dot{U}, \dot{V} \times \dot{W})$ and hence the set $N_f(\dot{U}, \hat{D})$ has positive upper density. Therefore f is ergodically sensitive. □

5. Conclusion

In this paper we studied the notions of topological shadowing, topological ergodic shadowing, topological ergodicity and topologically chain mixing and using the mentioned notations we proved the following inclusions

$$\mathcal{TES} \cap \mathcal{TCM} = \mathcal{TES} \cap \mathcal{TM} \qquad \mathcal{TES} \subsetneq \mathcal{TS}$$
$$\mathcal{TES} \subset \mathcal{TCT} \qquad \mathcal{TCT} \cap \mathcal{TS} \subset \mathcal{TE}.$$

We know that any mapping in a compact metric space with the ergodic shadowing property is chaotic in the sense of Li-Yorke and Auslander-Yorke [10]. Considering this problem in non-metrizable spaces is a topic for further research.

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References

[1] D.V. Anosov, Geodesic flows on closed riemannian manifolds of negative curvature, Trudy Mat. Inst. V.A. Steklova 90 (1967) 3–210.
[2] N. Aoki, K. Hiraide, Topological Theory of Dynamical Systems: Recent Advances, vol. 52, Elsevier, 1994.
[3] F. Blanchard, A. Maass, Dynamical behaviour of coven's aperiodic cellular automata, Theor. Comput. Sci. 163 (1996) 291–302.
[4] M. Blank, L. Blank, Metric properties of ε-trajectories of dynamical systems with stochastic behaviour, Ergodic Theory Dynam. Syst. 8 (1988) 365–378.
[5] R. Bowen, ω-limit sets for axiom a diffeomorphisms, J. Differential Equat. 18 (1975) 333–339.
[6] B. Carvalho, D. Kwietniak, On homeomorphisms with the two-sided limit shadowing property, J. Math. Anal. Appl. 420 (2014) 801–813.
[7] T. Das, K. Lee, D. Richeson, J. Wiseman, Spectral decomposition for topologically anosov homeomorphisms on noncompact and non-metrizable spaces, Topology Appl. 160 (2013) 149–158.
[8] D.A. Dastjerdi, M. Hosseini, Sub-shadowings, Nonlinear Anal.: Theory, Methods Appl. 72 (2010) 3759–3766.
[9] D. Dikranjan, M. Sanchis, S. Virili, New and old facts about entropy in uniform spaces and topological groups, Topology Appl. 159 (2012) 1916–1942.
[10] A. Fakhari, F.H. Ghane, On shadowing: Ordinary and ergodic, J. Math. Anal. Appl. 364 (2010) 151–155.
[11] M. Kulczycki, A unified approach to theories of shadowing, Regular Chaotic Dynam. 19 (2014) 310–317.
[12] R. Li, A note on shadowing with chain transitivity, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 2815–2823.
[13] Y. Niu, S. Su, On strong ergodicity and chaoticity of systems with the asymptotic average shadowing property, Chaos, Solitons Fractals 44 (2011) 429–432.
[14] P. Oprocha, Shadowing, thick sets and the ramsey property, Ergodic Theory Dynam. Syst. (2016) 1–14.
[15] P. Oprocha, D.A. Dastjerdi, M. Hosseini, On partial shadowing of complete pseudo-orbits, J. Math. Anal. Appl. 404 (2013) 47–56.
[16] P. Oprocha, D. A. Dastjerdi, M. Hosseini, On partial shadowing of complete pseudo-orbits, J. Math. Anal. Appl. 411 (2014) 454–463.
[17] X. Wu, P. Oprocha, G. Chen, On various definitions of shadowing with average error in tracing, Nonlinearity 29 (2016) 1942–1972.
[18] K. Yan, F. Zeng, Topological entropy, pseudo-orbits and uniform spaces, Topology Appl. 210 (2016) 168–182.