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On Submersions of the Complex Indicatrix

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Abstract. In this paper we study the complex indicatrix associated to a complex Finsler space as an embedded CR - hypersurface of the holomorphic tangent bundle, considered in a fixed point. Following the study of CR - submanifolds of a Kähler manifold, there are investigated some properties of the complex indicatrix as a real submanifold of codimension one, using the submanifold formulae and the fundamental equations. As a result, the complex indicatrix is an extrinsic sphere of the holomorphic tangent space in each fibre of a complex Finsler bundle. Also, submersions from the complex indicatrix onto an almost Hermitian manifold and some properties that can occur on them are studied. As application, an explicit submersion onto the complex projective space is provided.

1. Introduction

Many geometers have investigated the relationships between the geometric properties of a Riemannian or Finsler manifold M and those of its unit tangent sphere bundle, or indicatrix ([3–5, 11, 16, 22], etc.). This represents a well-known and interesting research field, mainly because the indicatrix is a compact and strictly convex set surrounding the origin, which is used, for example, in the volume definition of a Finsler space or in the Hodge theories.

However, in the present paper, we extend the study to unit sphere bundle of a complex Finsler manifold (M, F). The complex indicatrix will be treated as an embedded real hypersurface in a complex space, i.e. a CR submanifold, and its properties will be described intrinsically by studying the properties of the holomorphic vector fields which are tangent to the indicatrix. Thus, having in mind the similarity (in term of distributions) between the total space of a Riemannian submersion and a CR-submanifold of a Kähler manifold, considered by S. Kobayashi [14], it comes naturally to study the submersions of the complex indicatrix.

Firstly, in Section 1, we recall some basic notions about complex Finsler geometry and the geometry of CR-manifolds. Then, in the second Section, we analyse the complex indicatrix in a fixed point z_0 as a CR-hypersurface of the holomorphic tangent bundle, which can be locally viewed as a Kähler manifold. Moreover, we obtain the fundamental equations of the complex indicatrix as a real submanifold of codimension one. The properties of submersions from the complex indicatrix as CR-hypersurface of the Kähler manifold T'_zM onto an almost Hermitian manifold M' are studied in Section 3 (e.g. the link between the holomorphic sectional curvatures of T'_zM and M' in Theorem 3, the Kähler-Einstein properties in Theorem 4). Also, a submersion on the complex projective space is emphasized by Theorem 5.

Now, we will make a short overview of the concepts and terminology used in the geometries of complex Finsler manifolds (as in [1, 17]) and CR-submanifolds (see [6–8, 12–14]). Let M be a complex manifold,

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$\dim_{\mathbb{C}} M = n$, and $z := (z^k)$, $k = 1,..,n$, the complex coordinates on a local chart (U, φ). The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The bundle $T'M$ is in its turn a $2n$-dimensional complex manifold, of local coordinates in a local chart as $(z^k, \eta^k) \in T'M$, $k = 1,..,n$.

Definition 1.1. *A* complex Finsler space *is a pair* (M, F), *with* $F : T'M \to \mathbb{R}^+$, $F = F(z, \eta)$, *is a continuous function satisfying the following conditions:*

i. F is a smooth function on $\widetilde{T'M} := T'M \setminus \{0\}$;

ii. $F(z, \eta) \geq 0$, *the equality holds if and only if* $\eta = 0$;

iii. $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, $\forall \lambda \in \mathbb{C}$;

iv. the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ *is positive definite, where* $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ *is the fundamental metric tensor, with* $L := F^2$ *the complex Lagrangian associated to the complex Finsler function F.*

The positivity of $(g_{i\bar{j}})$ from condition iv. ensures the existence of the inverse $(g^{\bar{j}i})$, with $g^{\bar{j}i} g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}$. Moreover, it is equivalent to the convexity of L and to the strongly pseudoconvex property of the complex indicatrix $I_z M = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$ in a fixed point $z \in M$.

Moreover, the third condition implies that L is homogeneous with respect to the complex norm $L(z, \lambda\eta) = \lambda\bar{\lambda}L(z, \eta)$, $\forall \lambda \in \mathbb{C}$, and by applying Euler's formula we get that:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i \bar{\eta}^j. \tag{1}$$

An immediate consequence of the above homogeneity conditions concerns the following Cartan complex tensors: $C_{i\bar{j}k} := \frac{\partial g_{i\bar{j}}}{\partial \eta^k}$ and $C_{i\bar{j}\bar{k}} := \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}$. They have the following properties:

$$C_{i\bar{j}k} = C_{k\bar{j}i} \;;\; C_{i\bar{j}\bar{k}} = C_{i\bar{k}\bar{j}} \;;\; C_{i\bar{j}k} = \overline{C_{j\bar{i}\bar{k}}} \text{ and} \tag{2}$$

$$C_{i\bar{j}k}\eta^k = C_{i\bar{j}\bar{k}}\bar{\eta}^j = C_{i\bar{j}k}\eta^i = C_{i\bar{j}\bar{k}}\bar{\eta}^k = 0 \tag{3}$$

In the geometry of a complex Finsler space are studied the geometric objects of the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. A first step is the analyze of the sections of complexified tangent bundle of $T'M$, $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''_u(T'M) = \overline{T'_u(T'M)}$. Let $V(T'M) \subset T'(T'M)$ be the vertical bundle, locally spanned by $(1,0)-$ vector fields $\{\frac{\partial}{\partial \eta^k}\}$. Fundamental in "linearization" of the complex Finsler geometry is the complex nonlinear connection, briefly (c.n.c.), which is the supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$ [17]. The horizontal distribution $H_u(T'M)$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\}$, where $N^j_k(z, \eta)$ are the coefficients of the (c.n.c.). Then, the pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ represents the adapted frame of the (c.n.c.), having the dual adapted base $\{dz^k, \delta\eta^k := d\eta^k + N^k_j dz^j\}$. Further we will use the following notation $\bar{\eta}^j := \eta^{\bar{j}}$ to denote a conjugate object.

Cauchy-Riemann (or CR) submanifolds of almost Hermitian or Kähler manifolds were introduced by A. Bejancu, [6–8]. This notion was generalized to the Finsler geometry by S. Dragomir in [12, 13]. A real n-dimensional submanifold $\tilde{M}$ of the $2m$-dimensional almost Hermitian Finsler space (M, g), is said to be a *CR-submanifold* if it carries a pair of complementary Finslerian distributions (with respect to the restriction of g to $\tilde{M}$), $\mathcal{D} : u \to \mathcal{D}_u \subset T_u\tilde{M}$ and $\mathcal{D}^\perp : u \to \mathcal{D}^\perp_u \subset T_u\tilde{M}$, such that $\mathcal{D}$ is invariant, $J(\mathcal{D}_u) = \mathcal{D}_u$, and $\mathcal{D}^\perp$ is anti-invariant, $J(\mathcal{D}^\perp_u) \subset (T_u\tilde{M})^\perp$, for each $u \in \tilde{M}$, Here J is an almost complex structure on $\tilde{M}$. A CR-submanifold is called *holomorphic* or *complex* if $\dim\mathcal{D}^\perp_u = 0$, *totally-real* if $\dim\mathcal{D}_u = 0$ and *proper* if it is neither holomorphic nor totally-real. Any real hypersurface $\tilde{M}$ of M is a CR-submanifold, where we define $\mathcal{D}^\perp : u \to \mathcal{D}^\perp_u = J(T_u\tilde{M})^\perp$ and take $\mathcal{D}$ to be the complementary orthogonal distribution of $\mathcal{D}^\perp$ in $T\tilde{M}$.

The study of Riemannian submersion $\phi : \tilde{M} \to M'$ of a Riemannian manifold $\tilde{M}$ onto a Riemannian manifold M' was initiated by O'Neill [19]. Considering that for a CR-submanifold $\tilde{M}$ of a Kähler manifold M the distribution $\mathcal{D}^{\perp}$ is integrable [10], S.Kobayashi observed in [14] the similarity between the total space of a submersion ϕ and the CR-submanifold $\tilde{M}$ of a Kähler manifold M, in terms of distributions. Thus, Kobayashi considered that a submersion from a CR-submanifold $\tilde{M}$ of a Kähler manifold M onto an almost Hermitian manifold M' is a *Riemannian submersion* $\phi : \tilde{M} \to M'$ with the following conditions (given for the case $(T\tilde{M})^{\perp} = J(\mathcal{D}^{\perp})$):

(i) $\mathcal{D}^{\perp}$ is the kernel of ϕ_*;

(ii) $\phi_* : \mathcal{D}_u \to T_{\phi(u)}M'$ is complex isometry for every $u \in \tilde{M}$.

2. The Complex Indicatrix as a CR-hypersurface

Given a complex Finsler manifold (M, F), we consider T'_zM the corresponding holomorphic tangent space of M and F_z the Finsler metric in an arbitrary fixed point $z \in M$. Thus, (T'_zM, F_z) is a complex Minkowski space, of complex coordinate system (η^i), where $\eta = (\eta^i) = \eta^i \frac{\partial}{\partial z^i}|_z$. The Hermitian metric on $\widetilde{T'_zM}$, associated to F_z, is defined by $\mathcal{G}$ and has the explicit form:

$$\mathcal{G} := \frac{\partial^2 F_z^2}{\partial \eta^i \partial \bar{\eta}^j} \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k = g_{j\bar{k}}(z, \eta) \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k.$$

Clearly, $\mathcal{G}$ is smooth at $\eta = 0$ if and only if F_z is a Hermitian norm.

A linear connection ∇ on M extends by linearity to $T_{\mathbb{C}}M$ [17], which is isomorphic to $V_{\mathbb{C}}(T'M)$ via vertical lift, and it is well defined by the next set of coefficients $\overline{\Gamma^i_{jk}} = \Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}$, $\overline{\Gamma^{\bar{\imath}}_{\bar{j}k}} = \Gamma^i_{j\bar{k}}$, $\overline{\Gamma^{\bar{\imath}}_{jk}} = \Gamma^i_{\bar{j}\bar{k}}$, $\overline{\Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}} = \Gamma^i_{\bar{j}k}$. We require ∇ to be a compatible complex connection with respect to the natural complex structure J

$$J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k, \; J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}, \text{ with } \mathrm{i} := \sqrt{-1}, \tag{4}$$

i.e. $\nabla J = 0$. So, it results that ∇ conserves the holomorphic tangent space.

We can choose ∇ to be the Levi-Civita connection, which is a metrical and symmetric connection and, using (2), we get the following non-zero components of the Levi-Civita connection:

$$C^i_{jk} := \Gamma^i_{jk} = \overline{\Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}} = g^{\bar{h}i} C_{j\bar{h}k} = g^{\bar{h}i} \dot{\partial}_k g_{j\bar{h}},$$

with $C^i_{jk} = C^i_{kj}$ and $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$. Since $\Gamma^i_{\bar{j}k} = \Gamma^{\bar{\imath}}_{\bar{j}k} = 0$, it takes that the Levi-Civita connection is Hermitian. Moreover, it is equivalent to the linear Chern connection on pull-back tangent bundle $\pi^* T'M = span\{\frac{\partial}{\partial z^i}\}$, with $\pi : T'M \to M$ the natural projection (as in [2]), and since $C^i_{jk} - C^i_{kj} = 0$, we get that $(\widetilde{T'_zM}, F_z)$ is Kählerian and ∇ is a Kählerian connection, i.e. $\nabla_X(JY) = J\nabla_X Y$.

By direct calculation we obtain $\frac{\partial F}{\partial \eta^i} = \frac{1}{2} l_i := \frac{\eta_i}{2F}$ and $\frac{\partial F}{\partial \bar{\eta}^i} = \frac{1}{2} l_{\bar{\imath}} := \frac{\eta_{\bar{\imath}}}{2F}$.

So, for an arbitrary fixed point z on M, the unit sphere in (T'_zM, F_z) is the so-called *complex indicatrix* in z as:

$$I_zM = \{\eta \in T'_zM \mid F(z, \eta) = 1\}.$$

Since the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ is positive definite, then $L = F^2$ is convex and the complex indicatrix I_zM is a strictly pseudo convex submanifold. Moreover, as we have only one defining equation which involves the real valued Finsler function F, the complex indicatrix I_zM is a real hypersurface of the holomorphic tangent bundle, and thus a CR-hypersurface in T'_zM, $\forall z \in M$.

Let $(u^1, ..., u^{2n-1})$ be local real coordinates on I_zM and let

$$\eta^j = \eta^j(u^1, .., u^{2n-1}), \ \ \forall j \in \{1, .., n\}$$

be the equations of the inclusion map $i : I_zM \hookrightarrow \widetilde{T'_zM}$. Set $l^j = \frac{1}{F}\eta^j$ and $l_j = g_{j\bar{k}} l^{\bar{k}}$. Since $L(z, \eta(u)) = 1$, by differentiation after u, we obtain

$$l_j \frac{\partial \eta^j}{\partial u^\alpha} + l_{\bar{j}} \frac{\partial \eta^{\bar{j}}}{\partial u^\alpha} = 0, \quad \alpha = \overline{1, 2n-1}, \ j = \overline{1, n}. \tag{5}$$

Tangent map i_* acts on the tangent vectors of the complex indicatrix as

$$i_*\left(\frac{\partial}{\partial u^\alpha}\right) = X_\alpha := \frac{\partial \eta^k}{\partial u^\alpha}\frac{\partial}{\partial \eta^k} + \frac{\partial \bar{\eta}^k}{\partial u^\alpha}\frac{\partial}{\partial \bar{\eta}^k},$$

where X_α is a real tangent vector of the indicatrix expressed in terms of tangent vectors of the complexified tangent bundle of $T'M$. From (5), we can set (cf. [13])

$$N = l^j \dot{\partial}_j + l^{\bar{j}} \dot{\partial}_{\bar{j}} \tag{6}$$

and thus we obtain $G_R(X_\alpha, N) = 0$, where by G_R we have denoted the Riemannian metric applied to real vector fields given by

$$G_R(X, Y) = \mathrm{Re}\, \mathcal{G}(X', \overline{Y'}), \tag{7}$$

where X' and $\overline{Y'}$ are the holomorphic and the anti-holomorphic part, respectively, of tangent vectors $X, Y \in T_C(T'M)$, given by $X' = \frac{1}{2}(X - \mathrm{i}JX)$ and $\overline{Y'} = \frac{1}{2}(Y + \mathrm{i}JY)$, where $\mathrm{i} = \sqrt{-1}$. Consequently $N \in T_R(I_zM)^\perp$, so that N is the normal vector of the indicatrix bundle. Also, the normal vector has unit length, i.e. $G_R(N, N) = 1$.

If we apply the theory of submanifolds and denote by $\tilde{\nabla}$ and $\nabla^\perp$ the induced tangent and normal connection on I_zM of the Levi-Civita connection ∇ of T'_zM, the Gauss-Weingarten formulae are

$$\nabla_X Y = \tilde{\nabla}_X Y + h(X, Y) \quad \text{and} \quad \nabla_X W = -A_W X + \nabla^\perp_X W, \tag{8}$$

for any $X, Y \in \Gamma(T_R(I_zM))$ and $W \in \Gamma(T_R(I_zM)^\perp)$, with h and A the *second fundamental form* and the *shape operator* (or *Weingarten operator*) of I_zM, respectively. Since $T_R(I_zM)^\perp = span\{N\}$, these maps are defined by the following set of local coefficients $h_{\alpha\beta}$, A^α_β, regarded as:

$$h(X_\beta, X_\alpha) = h_{\alpha\beta} N, \qquad A_N(X_\beta) = A^\alpha_\beta X_\alpha, \qquad \forall X_\alpha, X_\beta \in T_R(I_zM).$$

The Riemannian metric on T'_zM obtained from the Hermitian metric $\mathcal{G}$ in (7) and the induced metric on the complex indicatrix will be denoted by the same symbol G_R, the last one representing the restriction of G_R to the real tangent vector fields of I_zM. Making the notations $B^i_\alpha := \frac{\partial \eta^i}{\partial u^\alpha}$, $B^{\bar{i}}_\alpha := \frac{\partial \bar{\eta}^i}{\partial u^\alpha}$ and $\mathrm{Re}(\tau) = \frac{1}{2}(\tau + \bar{\tau})$, for any form $\tau \in \mathcal{A}^{p,q}(M)$, we obtain:

Proposition 2.1. *Let (M, F) be a complex Finsler manifold and $z \in M$ an arbitrary fixed point. With respect to the Levi-Civita connection ∇, we have:*

$$\nabla_{X_\alpha} N = \frac{1}{F} X_\alpha, \qquad \nabla_{X_\beta} X_\alpha = 2\mathrm{Re}\,\{[X_\beta(B^i_\alpha) + B^k_\beta B^j_\alpha C^i_{jk}]\dot{\partial}_i\}, \qquad g_{\alpha\beta} := G_R(X_\alpha, X_\beta) = \mathrm{Re}(g_{i\bar{j}} B^i_\alpha B^{\bar{j}}_\beta).$$

Since we have $G_R(\nabla_X Y, N) = G_R(h(X, Y), N)$ and $G_R(\nabla_X N, X_\beta) = -G_R(A_N X, X_\beta)$, it results:

Proposition 2.2. *The local coefficients of the second fundamental form and Weingarten operator are given by*

$$h_{\alpha\beta} = \mathrm{Re}\left(l_i X_\beta(B^i_\alpha) + l_i B^k_\beta B^j_\alpha C^i_{jk}\right) \quad \text{and} \quad A^\alpha_\beta = -\frac{1}{F}\delta^\alpha_\beta.$$

Since $\nabla^{\perp}_{X_\alpha} N = 0$, the Weingarten formula becomes

$$\nabla_{X_\alpha} N = -A_N X_\alpha, \quad \forall X_\alpha \in T\left(I_z M\right). \tag{8'}$$

Using Proposition 2.2, it results that

$$A_N X_\alpha = -\frac{1}{F} X_\alpha. \tag{9}$$

Moreover, using following relation

$$G_R(h(X_\alpha, X_\beta), N) = G_R(A_N X_\alpha, X_\beta), \quad \forall X_\alpha, X_\beta \in \Gamma(T_R(I_z M)), \tag{10}$$

which holds with respect to any metric connection, we obtain

Proposition 2.3. *The local coefficients of the second fundamental form of the indicatrix hypersurface satisfy*

$$h_{\alpha\beta} = -\frac{1}{F} g_{\alpha\beta}.$$

Corollary 2.4. *The second fundamental form considered with respect to the complex indicatrix hypersurface is symmetric, i.e. locally* $h_{\alpha\beta} = h_{\beta\alpha}$.

Remark 2.5. ***a)*** *Corollary 2.4 assures the torsion free property of* $\tilde{\nabla}$, *i.e.* $\tilde{T}(X, Y) = 0$ *for any tangent vector fields* X, Y *of* $I_z M$. *Also,* $\tilde{\nabla}$ *is a metric connection, thus the induced connection* $\tilde{\nabla}$ *is a Levi-Civita connection.*
b) *From Proposition 2.3 it is obtained that the complex indicatrix can* not *be a* totally geodesic *manifold of the holomorphic tangent bundle* $\widetilde{T'_z M}$, *since the condition* $h(X, Y) = 0, \forall X, Y \in \Gamma(T_R(I_z M)))$ *(cf. [8]) is not fulfilled.*

Having in mind that a CR-submanifold is *totally umbilical* if its first and second fundamental forms are proportional, where the proportionality factor is a normal vector field H, called the field of curvature vectors (see [8]), and taking into account Proposition 2.3, we obtain that:

Theorem 2.6. *The complex indicatrix* $I_z M$ *of a complex Finsler manifold* (M, F) *is a totally umbilical manifold, i.e.* $h(X_\alpha, X_\beta) = H G_R(X_\alpha, X_\beta)$, *with constant field of curvature vectors* $H = -\frac{1}{F} N$.

Remark 2.7. ***a)*** *An equivalent condition of totally umbilical property of a subspace is* $A_N X_\alpha = G_R(N, H) X_\alpha$, *which can be verified for the complex indicatrix case by relation (9) and the above Theorem.*
b) *If we take into account that* $F = 1$ *on the complex indicatrix* $I_z M$, *we can ommit the* $\frac{1}{F}$ *factor from the above results.*

In the Riemannian theory, a submanifold $\tilde{M}$ of a Riemannian manifold M is said to be an *extrinsic sphere* if it is totally umbilical and it has non-zero parallel mean curvature vector (cf. Nomizu-Yano [18]). Basic results concerning extrinsic spheres in Riemannian and Kählerian geometry were obtained by B.Y.Chen [9, 10]. An orientable hypersurface $\tilde{M}$ is an *extrinsic hypersphere* of a Kähler manifold M, if it is satisfied $h(X, Y) = H g(X, Y)$, for any X, Y vector fields on $\tilde{M}$. Here H denotes the mean curvature vector field of $\tilde{M}$ and its norm is a non zero constant function on the extrinsic hypersphere $\tilde{M}$. So, considering this definition and the result from Proposition 2.3, we obtain a result which can be found in [13] as well:

Theorem 2.8. *Let* (M, F) *be a complex Finsler manifold. Then, for an arbitrary fixed point* $z \in M$, *the complex indicatrix* $I_z M$ *is an extrinsic sphere of the Kähler manifold* $T'_z M$.

Considering as above $\tilde{\nabla}$ and $\nabla^{\perp}$ the induced tangent and normal connection on $I_z M$ of the Levi-Civita connection of $T'_z M$, we can obtain the link between the $R(X, Y)Z$ and $\tilde{R}(X, Y)Z$ curvatures of ∇ and $\tilde{\nabla}$

connections, respectively. Following similar steps as in [20], the Gauss, H-Codazzi, A-Codazzi and Ricci equations of the indicatrix bundle are:

$$\begin{aligned} G_R\left(R(X,Y)Z,U\right) &= G_R(\tilde{R}(X,Y)Z,U) + G_R\left(A_{h(X,Z)}Y,U\right) - G_R\left(A_{h(Y,Z)}X,U\right), && (11)\\ G_R\left(R(X,Y)Z,N\right) &= G_R\left(\left(\nabla_X h\right)(Y,Z) - \left(\nabla_Y h\right)(X,Z),N\right), && (12)\\ G_R\left(R(X,Y)N,Z\right) &= G_R\left(\left(\nabla_Y A\right)(N,X) - \left(\nabla_X A\right)(N,Y),Z\right), && (13)\\ G_R\left(R(X,Y)N,N\right) &= G_R\left(h\left(Y,A_N X\right) - h\left(X,A_N Y\right),N\right), && (14) \end{aligned}$$

for any vector fields X, Y, Z, U tangent to I_zM.

If we choose to use the curvature tensor of $(0, 4)$ type given by the Riemannian curvature tensor $R(X,Y;U,Z) = G_R(R(X,Y)Z,U)$ and we use relation (10), the Gauss equation can be rewritten as

$$R(X,Y;U,Z) = \tilde{R}(X,Y;U,Z) + G_R(h(X,Z),h(Y,U)) - G_R(h(Y,Z),h(X,U)). \qquad (11')$$

Moreover, if we take into consideration the totally umbilicality condition from Theorem 2.6, the above relation becomes

$$R(X,Y;U,Z)\tilde{R}(X,Y;U,Z) + \frac{1}{L}G_R(X,Z)G_R(Y,U) - \frac{1}{L}G_R(Y,Z)G_R(X,U). \qquad (11'')$$

If we denote by $[R(X,Y)Z]^{\perp}$ the normal component of the curvature, the H-Codazzi equation has the following form

$$[R(X,Y)Z]^{\perp} = \left(\nabla_X h\right)(Y,Z) - \left(\nabla_Y h\right)(X,Z).$$

Using (10) for the Ricci equation, we obtain that $G_R\left(R(X,Y)N,N\right) = 0$, and thus, $R(X,Y)N$ has only tangent component to the complex indicatrix. Then, the A-Codazzi equation becomes

$$R(X,Y)N = \left(\nabla_Y A\right)(N,X) - \left(\nabla_X A\right)(N,Y). \qquad (13')$$

Locally, using the Gauss equation (11′) and Proposition 2.3 we obtain

Proposition 2.9. *The Riemannian curvature tensor with respect to the induced Levi-Civita tangent connection of the complex indicatrix I_zM is*

$$\tilde{R}(X_\alpha,X_\beta;X_\gamma,X_\delta) = 2\mathrm{Re}\{B^j_\delta B^{\bar{l}}_\gamma[B^k_\beta X_\alpha(C^i_{jk}) - B^k_\alpha X_\beta(C^i_{jk})]g_{i\bar{l}}\} - \frac{1}{L}g_{\delta\alpha}g_{\gamma\beta} + \frac{1}{L}g_{\delta\beta}g_{\gamma\alpha}.$$

3. Submersions of the Complex Indicatrix I_zM

Since each real orientable hypersurface $\tilde{M}$ of a Kähler manifold is a CR-submanifold with $\mathcal{D}^{\perp}_x = J((T_x\tilde{M})^{\perp})$ (cf. [8]), we get that $\mathcal{D}^{\perp} = span\{JN\}$ for the indicatrix of a complex Finsler space case, where N is the unit normal vector field to I_zM given in (6) and J is the complex structure defined on T'_zM in (4). Then we take

$$\xi = JN = \mathrm{i}\left(l^k\dot{\partial}_k - l^{\bar{k}}\dot{\partial}_{\bar{k}}\right), \quad \mathrm{i} := \sqrt{-1},$$

which is a tangent unit vector of I_zM with $\xi = \bar{\xi}$, such that $\mathcal{D}^{\perp} = span\{\xi\}$ and $N = -J\xi$. Then, let $\mathcal{D}$ be the maximal J-invariant subspace of the tangent space of I_zM, orthogonal to the one dimensional anti-invariant distribution $\mathcal{D}^{\perp}$, such that $T_R(I_zM) = \mathcal{D} \oplus \mathcal{D}^{\perp}$. Since M is an n dimensional complex manifold, then $\dim_R I_zM = 2n-1$ and $\dim_R \mathcal{D} = 2n-2$.

Theorem 3.1. *[20] Let (M,F) be a complex Finsler manifold, $z \in M$ an arbitrary fixed point and I_zM the complex indicatrix. Then the following affirmations take place:*

(a) *the anti-invariant distribution $\mathcal{D}^{\perp}$ is integrable;*

(b) *even though the complex CR-structures $\mathcal{D}'$, $\mathcal{D}''$ of $\mathcal{D}$, $\mathcal{D}\otimes\mathbb{C}=\mathcal{D}'\oplus\mathcal{D}''$, are integrable, the real invariant distribution $\mathcal{D}$ is no involutive, nor integrable.*

For the complex indicatrix case I_zM, we consider the submersion $\phi : I_zM \to M'$, with M' an almost Hermitian manifold, such that the map $\phi_* : T_R(I_zM) \to T_RM'$ fulfills $\phi_*(\xi)=0$, i.e. $T_RM'=\phi_*(\mathcal{D})$, and $\phi_* : \mathcal{D}_\eta \to T_{\phi(\eta)}M'$ is a complex isometry for every $\eta \in I_zM$.

The sections of these two complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are called the *horizontal*, respectively the *vertical vector fields* of the submersion $\phi : I_zM \to M'$. Since we have already mentioned the horizontal and vertical bundles of a Finsler space, denoted by $H(T'M)$ and $V(T'M)$ in Section 1, not to make any confusion with $\mathcal{D}$ and $\mathcal{D}^\perp$, we use for the last ones the notations from [7] and we suppose that the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are given by the projectors P and Q, respectively. So, for any $E \in \chi(I_zM)$, PE and QE denote the horizontal and the vertical components of E, respectively. Then, since the Levi-Civita connection ∇ on T'_zM is a Kählerian connection, we have $\nabla_X(JY)=J\nabla_XY$ and from the Gauss formula (8), we obtain

$$\nabla_X(JPY)+\nabla_X(JQY)=J\tilde{\nabla}_XY+Jh(X,Y), \quad \forall X,Y\in\Gamma(T_R(I_zM)).$$

Considering that $JPY\in\mathcal{D}$, $JQY\in\mathcal{D}^\perp$, $J(T(I_zM)^\perp)=\mathcal{D}^\perp$, using the Gauss and the Weingarten formulae (8) and by comparing horizontal, vertical and normal parts, we obtain respectively

$$\begin{aligned} JP(\tilde{\nabla}_XY) &= P(\tilde{\nabla}_XJPY)-P(A_{JQY}X),\\ Jh(X,Y) &= Q(\tilde{\nabla}_XJPY)-Q(A_{JQY}X),\\ JQ(\tilde{\nabla}_XY) &= h(X,JPY), \end{aligned}$$

for any vector fields X,Y tangent to I_zM. In particular, taking $Y=\xi$ in the above relations, we get that

$$JP(\tilde{\nabla}_X\xi)=P(A_NX), \quad Jh(X,\xi)=Q(A_NX) \quad \text{and} \quad JQ(\tilde{\nabla}_X\xi)=0.$$

A horizontal vector field X of I_zM is said to be *basic* if it induces (or comes from) a vector field $\underline{X}'$ on the base manifold M', i.e. $\phi_*X=\underline{X}'$. Clearly, the map $X\mapsto\underline{X}'$ gives a one-to-one correspondence between the basic vector fields of I_zM and the vector fields of M'. The following lemma is adapted from O'Neill [19]:

Lemma 3.2. *Let X and Y be basic vector fields on I_zM. Then*

(i) $G_R(X,Y)=G'_R(\underline{X}',\underline{Y}')\circ\phi$, *where G'_R is the Hermitian metric on M';*

(ii) *the horizontal part $P[X,Y]$ of $[X,Y]$ is a basic vector field corresponding to $[\underline{X}',\underline{Y}']$, i.e. $\phi_*(P[X,Y])=[\underline{X}',\underline{Y}']$;*

(iii) $[\xi,X]$ *is vertical;*

(iv) $P(\tilde{\nabla}_XY)$ *is a basic vector field corresponding to $\nabla'_{\underline{X}'}\underline{Y}'$, where ∇' is the covariant differentiation on M'.*

For the Levi-Civita connection ∇' on M', we define the corresponding connection $\bar{\nabla}'$ for basic vector fields on I_zM by

$$\bar{\nabla}'_XY=P(\tilde{\nabla}_XY), \quad \forall X,Y\in\Gamma(\mathcal{D}) \text{ basic}. \tag{15}$$

So, $\bar{\nabla}'_XY$ is a basic vector field and by Lemma 3.2 we have $\phi_*(\bar{\nabla}'_XY)=\nabla'_{\underline{X}'}\underline{Y}'$. By direct calculus, we obtain

$$\nabla_\xi\xi=-\frac{1}{F}N, \quad \nabla_NN=0, \quad \nabla_N\xi=0, \quad \nabla_\xi N=\frac{1}{F}\xi,$$

and so, from the Gauss formula (8), we obtain $\tilde{\nabla}_\xi\xi=0$ and $h(\xi,\xi)=-\frac{1}{F}N$.

If we apply now the Gauss and Weingarten's formulae of the leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$ in I_zM, we define the tensor fields C, $\tilde{A}$ and $\tilde{T}$ by

$$\tilde{\nabla}_XY = \bar{\nabla}'_XY+C(X,Y), \tag{16}$$

$$\tilde{\nabla}_X\xi = Q(\tilde{\nabla}_X\xi)+\tilde{A}_X\xi, \tag{17}$$

$$\tilde{\nabla}_\xi X = P(\tilde{\nabla}_\xi X)+\tilde{T}_\xi X,$$

for any basic vector fields X, Y, where $C(X,Y) := Q(\tilde{\nabla}_X Y)$, $\tilde{A}_X\xi := P(\tilde{\nabla}_X\xi)$ and $\tilde{T}_\xi X := Q(\tilde{\nabla}_\xi X)$. By taking into consideration that $\tilde{\nabla}_\xi\xi = 0$, the Gauss formula for the leaves of $\mathcal{D}^\perp$ assures us that the leaves of the anti-invariant distribution $\mathcal{D}^\perp$ are totally geodesic in I_zM, but not in T'_zM.

In [14] is proved that the second fundamental form C of the immersion of M' in I_zM defines a bilinear mapping $\mathcal{D}\times\mathcal{D}\mapsto\mathcal{D}^\perp$ and using the properties

$$[X,Y] = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X \quad \text{and} \quad \tilde{\nabla}_X G_R(Y,Z) = G_R(\tilde{\nabla}_X Y, Z) + G_R(Y, \tilde{\nabla}_X Z)$$

of the Levi-Civita connection $\tilde{\nabla}$, it is obtained that $C(X,X) = 0$, i.e. $C(X,Y) = -C(Y,X)$, which means that C has a skew-symmetric property, and it satisfies $C(X,Y) = \frac{1}{2}Q([X,Y])$ for any basic vector fields X, Y.

Also, it is easy to check that $\tilde{A} : \mathcal{D}\times\mathcal{D}^\perp \mapsto \mathcal{D}$ is a bilinear map and since $[\xi, X]$ is vertical for any basic vector field X, we have $\tilde{A}_X\xi = P(\tilde{\nabla}_X\xi) = P(\tilde{\nabla}_\xi X)$. Using this, we can relate the two tensor fields $\tilde{A}$ and C by

$$G_R(\tilde{A}_X\xi, Y) = -G_R(\xi, C(X,Y)), \tag{18}$$

for any basic vector fields X, Y and the vertical vector field ξ. Moreover, since $\tilde{\nabla}$ is a metrical connection and $\tilde{\nabla}_\xi\xi = 0$, we get

$$G_R(\tilde{T}_\xi X, \xi) = 0. \tag{19}$$

By straightforward calculation, considering R' the curvature tensor corresponding to the connection ∇' of the base manifold M', making use of (15), (16), (17) and (18), we obtain that R' and $\tilde{R}$ are related by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R'(\underline{X}',\underline{Y}',\underline{Z}',\underline{W}') - G_R(C(X,Z),C(Y,W)) + G_R(C(Y,Z),C(X,W)) \\ &\quad -2G_R(C(X,Y),C(Z,W)).\end{aligned} \tag{20}$$

Using the same considerations, the Jacobi identity and (19), we also have

$$\begin{aligned}\tilde{R}(X,Y,\xi,Z) &= -G_R((\tilde{\nabla}_Z C)(X,Y),\xi) - \sum_{(X,Y,Z)} G_R(C(X,[Y,Z]),\xi), \\ \tilde{R}(X,\xi,Y,\xi) &= -G_R(\tilde{A}_{\tilde{A}_X\xi}\xi, Y), \\ \tilde{R}(X,Y,\xi,\xi) &= 0,\end{aligned} \tag{21}$$

where $\sum_{(X,Y,Z)}$ denotes the cyclic sum over the basic vector fields X, Y, Z.

Further, we will find some relations between the second fundamental form h studied in the previous Section and the bilinear map C. Since $T_R(I_zM) = \mathcal{D}\oplus\mathcal{D}^\perp$ and $\mathcal{D}^\perp = span\{\xi\}$, we have the following orthogonal decomposition of $T_R(T'_zM)$

$$T_R(T'_zM) = \mathcal{D}\oplus span\{\xi\}\oplus span\{N\}.$$

By (8) and (16), for basic vector fields X, Y, we get

$$\nabla_X Y = \bar{\nabla}'_X Y + C(X,Y) + h(X,Y).$$

Since $\phi_* : \mathcal{D}_\eta \to T_{\phi(\eta)}M'$ preserves complex structure J, we have JY a basic vector field and if we apply J to the above relation and we use $\nabla_X(JY) = J\nabla_X Y$, $J(\mathcal{D}) = \mathcal{D}$, considering that J interchanges $\mathcal{D}^\perp$ and $T(I_zM)^\perp$, by comparing horizontal, vertical and normal parts, we obtain respectively

$$\bar{\nabla}'_X JY = J\bar{\nabla}'_X Y, \tag{22}$$

$$C(X,JY) = Jh(X,Y), \tag{23}$$

$$h(X,JY) = JC(X,Y). \tag{24}$$

From (22) we obtain that the almost complex structure of M' is parallel and, hence M' is Kähler. Thus, we can state

Theorem 3.3. *Let $\phi : I_zM \to M'$ be a CR-submersion of the indicatrix I_zM of a complex Finsler space M, considered in a fixed point $z \in M$, on an almost Hermitian manifold M'. Then M' is a Kähler manifold.*

From (23), (24), Corollary 2.4, the skew-symmetric property of C, we get

$$h(JX, JY) = h(X, Y) \quad \text{and} \quad C(JX, JY) = C(X, Y), \tag{25}$$

for any basic vector fields X, Y. Similarly, we get for basic vector fields

$$h(X, JY) = -h(JX, Y) \quad \text{and} \quad C(X, JY) = -C(JX, Y). \tag{26}$$

Since $\mathcal{D}$ is not integrable, i.e. $h(X, JY) \neq h(JX, Y)$, then $h(X, JY) \neq 0$. Thus, from (24), $C(X, Y) \neq 0$ and $\tilde{\nabla}_X Y \notin \mathcal{D}$, i.e. $\mathcal{D}$ is not parallel. Moreover, using Theorem 4.4, we have

Proposition 3.4. *Let $\phi : I_zM \to M'$ be a CR-submersion of the indicatrix I_zM of a complex Finsler space M, considered in a fixed point $z \in M$, on an almost Hermitian manifold M'. Since $\mathcal{D}$ is not integrable, $\mathcal{D}$ is not parallel. Moreover, we can not express I_zM as the product $M_1 \times M_2$, where M_1 is a complex submanifold and M_2 is a totally real submanifold of T'_zM.*

In order to compare the holomorphic bisectional curvature

$$\varkappa(X, Y) = \frac{R(X, JX, Y, JY)}{\|X\|^2 \, \|Y\|^2}, \ \forall X, Y \neq 0$$

of the Kähler manifold T'_zM with that of M', where $\|X\|^2 := G_R(X, X)$, we set $Y = JX$, $U = Y$ and $Z = JY$ in (11′) and we have

$$R(X, JX, Y, JY) \;=\; \tilde{R}(X, JX, Y, JY) + G_R(h(X, JY), h(JX, Y)) - G_R(h(JX, JY), h(X, Y))$$

for any basic vector fields X, Y. Using (25), (26), it can be rewriten as

$$R(X, JX, Y, JY) = \tilde{R}(X, JX, Y, JY) - \|h(X, JY)\|^2 - \|h(X, Y)\|^2 . \tag{27}$$

Now, if we set in (20) $Y = JX$, $Z = Y$ and $W = JY$, for any basic vector fields X and Y we have

$$\begin{aligned}\tilde{R}(X, JX, Y, JY) \;=\;& R'(\underline{X}', J\underline{X}', \underline{Y}', J\underline{Y}') - G_R(C(X, Y), C(JX, JY)) + G_R(C(JX, Y), C(X, JY)) \\ &-2G_R(C(X, JX), C(Y, JY))\end{aligned}$$

and using again relations (25), (26), we get

$$\tilde{R}(X, JX, Y, JY) \;=\; R'(\underline{X}', J\underline{X}', \underline{Y}', J\underline{Y}') - \|C(X, Y)\|^2 - \|C(X, JY)\|^2 - 2G_R(C(X, JX), C(Y, JY)). \tag{28}$$

Now, using (23) and (24), we combine (27) and (28) to obtain

$$R(X, JX, Y, JY) \;=\; R'(\underline{X}', J\underline{X}', \underline{Y}', J\underline{Y}') - 2\,\|C(X, Y)\|^2 - 2\,\|C(X, JY)\|^2 - 2G_R(C(X, JX), C(Y, JY)).$$

Considering now the formula of the sectional holomorphic curvature of a Kähler manifold $K(X) = R(X, JX, X, JX)$, for any basic unit vector X, we set in the above relation $Y = X$ and we use $C(X, X) = 0$. Thus, we obtain

$$R(X, JX, X, JX) = R'(\underline{X}', J\underline{X}', \underline{X}', J\underline{X}') - 4\,\|C(X, JX)\|^2 ,$$

or equivalently,

$$R(X, JX, X, JX) = R'(\underline{X}', J\underline{X}', \underline{X}', J\underline{X}') - 4\,\|h(X, X)\|^2 .$$

Now, using the totally umbilicality condition $h(X, Y) = HG_R(X, Y)$ of I_zM, with H given in Theorem 2.6 and condition $F = 1$ on I_zM, we can state

Theorem 3.5. *Let $\phi : I_zM \to M'$ be a CR-submersion of the indicatrix I_zM of a complex Finsler space M, considered in a fixed point $z \in M$, on an almost Hermitian manifold M'. If we denote by K and K' the holomorphic sectional curvature of the Kähler manifolds T'_zM and M', then, for any basic vector field X of I_zM, we have*

$$K(X) = K'(\underline{X}') - 4\|X\|^2, \quad \text{where } \underline{X}' = \phi_* X.$$

Next, we will study the properties of the submersions from the complex indicatrix, in the case when the holomorphic tangent bundle considered in a fixed point, T'_zM, is a Kähler-Einstein manifold (as in [15]), i.e. the Ricci tensor $Ric(X,Y) = \sum_{i=1}^{2n} R(E_i, X, Y, E_i)$ is proportional to $G_R(X,Y)$, where $\{E_1, E_2, ..., E_n\}$ is a local orthonormal frame on T'_zM and X, Y are vector fields tangent to T'_zM. Using the totally umbilicality condition of I_zM we obtain the several lemmas:

Lemma 3.6. *For any basic vector fields X and Y on I_zM, with respect to the CR-submersion $\phi : I_zM \to M'$, we have*

$$G_R(\tilde{A}_X\xi, \tilde{A}_Y\xi) = G_R(X,Y).$$

Proof. From (8) applied for any basic vector field X and ξ, which are tangent to I_zM, from the umbilicality of I_zM, $h(X,\xi) = -G_R(X,\xi)N$ from Theorem 2.6 (with $F = 1$), and $G_R(X,\xi) = 0$, we get $\nabla_X\xi = \tilde{\nabla}_X\xi$. Then, we have

$$G_R(\nabla_X JN, Y) = G_R(\tilde{\nabla}_X\xi, Y) = G_R(P(\tilde{\nabla}_X\xi), Y) = G_R(\tilde{A}_X\xi, Y).$$

On the other hand, T'_zM is a Kähler manifold, so that ∇ commute with the complex structure J and thus, using the Weingarten equation (8'), (10) and umbilicality relation, we get

$$\begin{aligned} G_R(\nabla_X JN, Y) &= G_R(J\nabla_X N, Y) = -G_R(\nabla_X N, JY) = G_R(A_N X, JY) = G_R(h(X,JY), N) = G_R(-G_R(X,JY)N, N) \\ &= -G_R(X, JY). \end{aligned}$$

Thus, we have

$$G_R(\tilde{A}_X\xi, Y) = -G_R(X, JY). \tag{29}$$

Consequently, by comparing these results, it is obtained

$$G_R(\tilde{A}_X\xi, \tilde{A}_Y\xi) = -G_R(X, J\tilde{A}_Y\xi) = G_R(JX, \tilde{A}_Y\xi) = -G_R(Y, J^2X) = G_R(X,Y).$$

□

If we replace (29) into (21), we obtain $\tilde{R}(X,\xi,Y,\xi) = G_R(X,Y)$.

Lemma 3.7. *For any basic vector fields X, Y, Z and W on I_zM, with respect to the CR-submersion $\phi : I_zM \to M'$, we have*

$$G_R(C(X,Y), C(Z,W)) = G_R(X,JY)G_R(Z,JW).$$

Proof. Since $C(X,Y) = Q(\tilde{\nabla}_X Y)$ is a vertical vector field, it can be written

$$C(X,Y) = G_R(C(X,Y),\xi)\xi.$$

Moreover, considering that ∇ is Kählerian, the Gauss formula (8) with $h(X,Y)$ normal to I_zM and the umbilicality condition of I_zM, we have

$$G_R(C(X,Y),\xi) = G_R(\tilde{\nabla}_X Y, \xi) = G_R(\nabla_X Y, JN) = -G_R(\nabla_X JY, N) = -G_R(h(X,JY), N) = G_R(X,JY).$$

Then, considering that ξ is a unit vector field, we obtain

$$G_R(C(X,Y), C(Z,W)) = G_R(C(X,Y),\xi)G_R(C(Z,W),\xi)G_R(\xi,\xi) = G_R(X,JY)G_R(Z,JW).$$

□

Now, considering relations (11′′), with $L = 1$ on $I_z M$, (20) and the above Lemma, it results

$$\tilde{R}(X,Y,Z,W) = R'(\underline{X}',\underline{Y}',\underline{Z}',\underline{W}') - [G_R(X,JZ)G_R(Y,JW) - G_R(Y,JZ)G_R(X,JW)) + 2G_R(X,JY)G_R(Z,JW) - G_R(X,W)G_R(Y,Z) + G_R(Y,W)G_R(X,Z))],$$

for any basic vector fields X, Y, Z and W. If we consider $\{E_1,...,E_p, JE_1,..., JE_p\}$ a local J-orthonormal frame of basic vector fields for the horizontal distribution $\mathcal{D}$ of the complex indicatrix, then we get $\{\underline{E}'_1,...,\underline{E}'_p, J\underline{E}'_1,...,J\underline{E}'_p\}$ a local J-orthonormal frame, by taking $\phi_* E_i = \underline{E}'_i$ on the Kähler manifold M'. Using the above lemmas, we can state

Theorem 3.8. *Let $I_z M$ the indicatrix of a complex Finsler space M and we consider the holomorphic tangent bundle in a fixed point $T'_z M$ to be an Einstein manifold. If $\phi : I_z M \to M'$ is a CR-submersion of $I_z M$ on an almost Hermitian manifold M', then M' is a Kähler-Einstein manifold.*

Submersion on the complex projective bundle $P_z M$

As an application we make an approach for the submersion from complex indicatrix to the complex projective bundle $P_z M$, considered in a fixed point $z \in M$. Thus, it provides an isometry between the holomorphic distribution of the indicatrix and the tangent bundle of $P_z M$, and, using this, can be obtained a link between the volume of the projective tangent bundle and the volume of any almost Hermitian manifold of $n-1$ dimension submersed from the complex indicatrix (as it can be seen in [21]).

Let M be an n dimensional complex manifold, TM the real tangent bundle and $T_{\mathbb{C}}M$ its complexified, $T_{\mathbb{C}}M = T'M \oplus T''M$. For each $z \in M$, we take $\mathcal{G} = g_{i\bar{j}} d\eta^i \otimes d\eta^{\bar{k}}$ the Hermitian metric on $T'_z M$. Since $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ acts on $\widetilde{T'M}$ by scalar multiplication, we get the projective holomorphic tangent bundle PM of M, defined by

$$PM = T'M/_{\mathbb{C}^*},$$

where (z,η) is identified with (z',η') iff $z = z'$ and $\eta = \lambda\eta'$, for some $\lambda \in \mathbb{C}^*$. For each $(z,\eta) \in \widetilde{T'M}$, we denote its equivalence class under the $\mathbb{C}^*$ action by $(z,[\eta]) \in PM$ and regard the fibre coordinates $(\eta^1,...,\eta^n)$ of η as the homogeneous coordinates of $[\eta]$. Geometrically, a point $(z,[\eta])$ in PM represents a complex line through η in the holomorphic tangent space $T'_z M$ of M in $z \in M$.

The natural projection $p : PM \to M$ is given by $p(z,[\eta]) = z$ and it pulls back the bundle $T'M$ to an n-dimensional vector bundle $p^{-1}T'M$ over the complex manifold PM, $\dim_C PM = 2n-1$. The elements of $p^{-1}T'M$ will be denoted by $([\eta], v)$, where $v = v^i \frac{\partial}{\partial z^i} \in T'_z M$. In other words, to each complex "ray" $[\eta]$ we assign the vector $v \in p^{-1}T'M$, which is invariant under $\eta \mapsto \lambda\eta$ rescaling. Keeping the notation $\{\frac{\partial}{\partial z^i}\}$, $\{dz^i\}$ for the local bases for $p^{-1}T'M$ and its dual $p^{-1}T^*M$ and considering that the $g_{i\bar{j}}$ is 0-homogeneous, we produce an inner product on $p^{-1}T'M$ by $g_{i\bar{j}}(z,[\eta]) dz^i \otimes d\bar{z}^j$ on the fibre over the point $[\eta] \in PM$, where $\eta \in T'_z M$.

Considering that $T'(T'M)$ splits into the direct sum between the horizontal distribution $H(T'M) = span\{\delta_k\}$ and the vertical subbundle $V(\widetilde{T'M}) = span\{\dot{\partial}_k\}$, the dual adapted base is spanned by $\{dz^k, \delta\eta^k\}$ (as in Section 1). Then, we define a metric on $\widetilde{T'M}$ by

$$h = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j.$$

This metric descends to the metric

$$h = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + (\log L)_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j$$

on the total space PM. Note that the first factor defines an Hermitian inner product on the horizontal subspaces, while the second term defines a Kähler metric on the projectivization of the fibre $T'_z M$, [23].

So, we can state that the Hermitian metric on T'_zM given by $\mathcal{G}$ descends to the metric $G = (\log L)_{i\bar{j}} d\eta^i \otimes d\bar{\eta}^j$ on PM, which is equivalent to

$$G = (\frac{1}{L} g_{i\bar{j}} - \frac{1}{L^2}\eta_i \eta_{\bar{j}}) d\eta^i \otimes d\bar{\eta}^j,$$

and we establish the following result:

Theorem 3.9. *The canonical map $\phi : I_zM \to P_zM$, given by $\phi(\eta/F) := [\eta]$, is a submersion.*

Proof. In order to show that the canonical map $\phi : I_zM \to P_zM$ is a submersion, we consider a tangential map $\phi_* : T_R(I_zM) \to T_R(P_zM)$ such that $\phi_*(\xi) = 0$ and $\phi_* : \mathcal{D}_\eta \to T_{[\eta]}(P_zM)$ is a complex isometry for every $\eta \in I_zM$, i.e. $G(\phi_*(X), \phi_*(Y)) = G_R(X, Y)$, $\forall X, Y \in \mathcal{D}$.

Following these ideas, we take

$$v_\alpha = \frac{1}{F}(v^i_\alpha \dot{\partial}_i + v^{\bar{i}}_\alpha \dot{\partial}_{\bar{i}})$$

with v^i_α and $v^{\bar{i}}_\alpha$ 1-homogeneous functions in η variables, $\alpha \in \{1, .., 2n-1\}$, such that $T_R(I_zM) = span\{v_\alpha\}$. Then, by direct calculus, we obtain that $G_R(v_\alpha, v_\beta) = \frac{1}{L}\mathrm{Re}\{v^i_\alpha v^{\bar{j}}_\beta g_{i\bar{j}}\}$. Further, we take $\xi = v_{2n-1}$ and so $v^j_{2n-1} = \mathrm{i}\eta^j$ and $v^{\bar{j}}_{2n-1} = \mathrm{i}\eta^{\bar{j}}$. Since $\mathcal{D} = span\{v_a\}$, $a \in \{1, .., 2n-2\}$, $\mathcal{D}^\perp = span\{\xi\}$, we have $G_R(v_a, \xi) = G_R(v_a, N) = 0$ and thus $v^i_\alpha \eta_i = 0$ and $v^{\bar{i}}\eta_{\bar{i}} = 0$.

Then, we consider the tangential map between the tangential vector bundles of I_zM and P_zM will be

$$\begin{aligned} \phi_* : T_R(I_zM) &\to T_R(P_zM) \\ v_\alpha &\mapsto e_\alpha := \mathrm{Re}\{(v^j_\alpha - \tfrac{\mathrm{i}\sqrt{2}}{2}\eta^j)\dot{\partial}_j\}. \end{aligned}$$

The vector fields e_α are invariant under complex rescaling $\eta \mapsto \lambda\eta$ so they are well-defined objects living on P_zM. Moreover, by direct calculus we can easily verify that $\phi_*(\xi) = 0$ and $G(e_a, e_b) = \frac{1}{L}\mathrm{Re}\{v^i_\alpha v^{\bar{j}}_\beta g_{i\bar{j}}\}$, so that $G(\phi_*(v_a), \phi_*(v_b)) = G_R(v_a, v_b)$, $\forall a, b \in \{1, .., 2n-2\}$, i.e. ϕ_* is an isometry. □

References

[1] Abate, M., Patrizio, G., *Finsler metrics—a global approach*, Springer-Verlag, Berlin, 1994.
[2] Aldea, N., *On Chern complex linear connection*, Bull. Math. Soc. Sc. Math. Roumanie, vol. 45 (93), no. 3-4 (2002), 119-131.
[3] Anastasiei, M., Gîrţu, M., *Indicatrix of a Finsler vector bundle*, Sci. Stud. Res. Ser. Math. Inform. **20** (2010), no. 2, 21–28.
[4] Bao, D., Chern, S.-S., Shen, Z., *An introduction to Riemann-Finsler geometry*, Springer-Verlag, New York, 2000.
[5] Bejancu, A., Faran, H. R., *The geometry of pseudo-Finsler submanifolds*, Kluwer Acad. Publ., 2000.
[6] Bejancu, A., CR *submanifolds of a Kaehler manifold. I*, Proc. Amer. Math. Soc., **69** (1978), no. 1, 135–142.
[7] Bejancu, A., *CR submanifolds of a Kaehler manifold. II*, Trans. Amer. Math. Soc., **250** (1979), 333–345.
[8] Bejancu, A., *Geometria CR-subvarietăţilor (in romanian)*, Mem. Secţ. Ştiinţ. Acad. Repub. Soc. România Ser. IV, **5** (1982), 53–101.
[9] Chen, B. Y., *Geometry of submanifolds and its applications*, Science University of Tokyo, Tokyo, 1981.
[10] Chen, B. Y., *CR-submanifolds of a Kaehler manifold. I, II*, J. Differential Geom., **16** (1981), 305-322, 493–509.
[11] Crâşmăreanu, M., *The Gaussian curvature for the indicatrix of a generalized Lagrange space.* Finsler and Lagrange geometries (Iaşi, 2001), 83–89, Kluwer Acad. Publ., Dordrecht, 2003.
[12] Dragomir, S., *Cauchy-Riemann submanifolds of Kaehlerian Finsler spaces*, Collect. Math., **40** (1989), 225–240.
[13] Dragomir, S., Grimaldi, R., *On Rund's connection*, Note Mat., **15** (1995), 85–98.
[14] Kobayashi, S., *Submersions of CR-submanifolds*, Tohoku Math. J., **39** (1987), 95–100.
[15] Mangione, V., *Some submersions of CR-hypersurfaces of Kaehler-Einstein manifold*, Int. J. Math. Math. Sci., **18** (2003), 1137–1144.
[16] Matsumoto, M., *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Shigaken, 1986.
[17] Munteanu, G., *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Kluwer Academic Publishers, Dordrecht, 2004.
[18] Nomizu, K., Yano, K., *On circles and spheres in Riemannian geometry*, Math. Ann., **210** (1974), 163-170.
[19] O'Neill, B., *The fundamental equations of a submersion*, Mihigan Mth. J., **13**(1966), 459-469.
[20] Popovici, E., *The CR geometry of the complex indicatrix*, Novi Sad J. Math., 46, 1 (2016), 191-205.
[21] Popovici, E., *On the volume of the indicatrix of a complex Finsler space*, Turk J Math, DOI: 10.3906/mat-1510-66.
[22] Rund, H., *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959.
[23] Yan, R., *On the volume of the projectivized tangent bundle in a complex Finsler manifold*, Arch. Math. (Basel), 86, 5 (2006), 458-463.