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Completion of Locally Convex Cones

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Abstract. We define the concept of completion for locally convex cones. We show that how a locally convex cone with (SP) can be embedded as an upper dense subcone in an upper complete locally convex cone with (SP). We prove that every upper complete locally convex cone with (SP) is also symmetric complete.

1. Introduction

A cone is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.

The theory of locally convex cones as developed in [5] and [9] uses an order theoretical concept or convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the later. For recent researches see [1–4].

Let $\mathcal{P}$ be a cone. A collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on $\mathcal{P}$, if the following properties hold:
$(U1)$ $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ($\Delta = \{(a, a) : a \in \mathcal{P}\}$);
$(U2)$ for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
$(U3)$ $\lambda U o \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
$(U4)$ $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

Here, for $U, V \subseteq \mathcal{P}^2$, by UoV we mean the set of all $(a, b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a, c) \in U$ and $(c, b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone if

$(U5)$ for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0, a) \in \rho U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b, a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in P : (a, b) \in U\}, \quad U \in \mathfrak{U}.$$

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The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ if for every $W \in \mathcal{W}$ there is $U \in \mathfrak{U}$ such that $U \subseteq W$.

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \mathbb{R}$, $\alpha.(+\infty) = +\infty$ for all $\alpha > 0$ and $0.(+\infty) = 0$. We set $\mathcal{V} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \{(a, b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}.$$

Then $\tilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the intire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a)+T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $T \times T(U) \subseteq W$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. We denote the set of all linear functional on $\mathcal{P}$ by $L(\mathcal{P})$ (the algebraic dual of $\mathcal{P}$). For a subset F of $\mathcal{P}^2$ we define *polar* F° as below

$$F^\circ = \{\mu \in L(\mathcal{P}) : \mu(a) \leq \mu(b) + 1, \ \forall (a, b) \in F\}.$$

Clearly $(\{(0, 0)\})^\circ = L(\mathcal{P})$. A linear functional μ on $(\mathcal{P}, \mathfrak{U})$ is (uniformly) continuous if there is $U \in \mathfrak{U}$ such that $\mu \in U^\circ$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars U° of neighborhoods $U \in \mathfrak{U}$.

We shall say that a locally convex cone $(\mathcal{P}, \mathfrak{U})$ has the strict separation property if the following holds:

(SP) For all $a, b \in \mathcal{P}$ and $U \in \mathfrak{U}$ such that $(a, b) \notin \rho U$ for some $\rho > 1$, there is a linear functional $\mu \in U^\circ$ such that $\mu(a) > \mu(b) + 1$ ([5], II, 2.12).

2. Completion of a Locally Convex Cone

Definition 2.1. *Suppose that $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone. We shall say that the subset A of $\mathcal{P}$ is bounded below (above) whenever for every $U \in \mathfrak{U}$ there is $\lambda > 0$ such that $\{0\} \times A \subseteq \lambda U$ ($A \times \{0\} \subseteq \lambda U$). The subset A of $\mathcal{P}$ is bounded whenever it is bounded below and above.*

In [5], a dual pair is defined as follows.

Definition 2.2. *A dual pair* $(\mathcal{P}, \mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping

$$(a, x) \to < a, x >: \mathcal{P} \times \mathcal{Q} \to \overline{\mathbb{R}}.$$

If $(\mathcal{P}, \mathcal{Q})$ is a dual pair, then every $x \in \mathcal{Q}$ is a linear mapping on $\mathcal{P}$. We denote the coarsest convex quasiuniform structure on $\mathcal{P}$ that makes all $x \in \mathcal{Q}$ continuous by $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$. In fact $(\mathcal{P}, \mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q}))$ is the projective limit of $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ by $x \in \mathcal{Q}$ as linear mappings on $\mathcal{P}$ (projective limits of locally convex cones were defined in [7]).

Let $\mathcal{P}$ be a cone. Then $(\mathcal{P}, L(\mathcal{P}))$ is a dual pair endowed with the bilinear mapping $< a, \mu >= \mu(a)$, for $a \in \mathcal{P}$ and $\mu \in L(\mathcal{P})$.

A Cauchy net in locally convex cones is defined in [6] as follows:

Definition 2.3. *A net $(x_i)_{i\in\mathcal{I}}$ in $(\mathcal{P},\mathfrak{U})$ is called lower (upper) Cauchy if for every $U\in\mathfrak{U}$ there is some $\gamma_U\in\mathcal{I}$ such that $(x_\beta,x_\alpha)\in U$ (respectively, $(x_\alpha,x_\beta)\in U$) for all $\alpha,\beta\in\mathcal{I}$ with $\beta\geq\alpha\geq\gamma_U$. Also $(x_i)_{i\in\mathcal{I}}$ is called symmetric Cauchy if it is lower and upper Cauchy, i.e., if for each $U\in\mathfrak{U}$ there is some $\gamma_U\in\mathcal{I}$ such that $(x_\beta,x_\alpha)\in U$ for all $\alpha,\beta\in\mathcal{I}$ with $\beta,\alpha\geq\gamma_U$.*

The locally convex cone $(\mathcal{P},\mathfrak{U})$ is called lower (upper and symmetric) complete if every lower (respectively, upper and symmetric) Cauchy net converges in lower (respectively, upper and symmetric) topology. It is proved in [6], that $(\mathbb{R},\tilde{\mathcal{V}})$ is upper and symmetric complete but not lower complete. Indeed the net $(-n)_{n\in\mathbb{N}}$ is lower Cauchy but it is not convergent with respect to the lower topology.

Theorem 2.4. *Let $\mathcal{P}$ be a cone. The locally convex cone $(L(\mathcal{P}),\mathfrak{U}_\sigma(L(\mathcal{P}),\mathcal{P}))$ is complete with respect to the upper and symmetric topologies.*

Proof. *If $(\mu_i)_{i\in\mathcal{I}}$ is a Cauchy net in the upper topology of $(L(\mathcal{P}),\mathfrak{U}_\sigma(L(\mathcal{P}),\mathcal{P}))$, then, for every $a\in\mathcal{P}$, $(\mu_i(a))_{i\in\mathcal{I}}$ is a Cauchy net in the upper topology of $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$. Now we consider usual order on $\overline{\mathbb{R}}$ and we set $\mu(a)=\sup_{i\in\mathcal{I}}\mu_i(a)$. Then μ is a linear functional on $\mathcal{P}$ by Lemma 5.5 from [9], and $\mu_i\to\mu$ with respect to the upper topology, since for every $a\in\mathcal{P}$, $\mu_i(a)\leq\mu(a)$ for all $i\in\mathcal{I}$.*

Let $(\mu_i)_{i\in\mathcal{I}}$ be a Cauchy net in the symmetric topology of $(L(\mathcal{P}),\mathfrak{U}_\sigma(L(\mathcal{P}),\mathcal{P}))$. Then, for every $a\in\mathcal{P}$, $(\mu_i(a))_{i\in\mathcal{I}}$ is a Cauchy net in the symmetric topology of $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$. Since $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$ is complete with respect to the symmetric topology, there is $\mu(a)\in\overline{\mathbb{R}}$ such that $\mu_i(a)\to\mu(a)$ with respect to the symmetric topology. We prove that $\mu\in L(\mathcal{P})$ and $\mu_i\to\mu$ with respect to the symmetric topology of $(L(\mathcal{P}),\mathfrak{U}_\sigma(L(\mathcal{P}),\mathcal{P}))$. It is easy to see that $\mu(a+b)=\mu(a)+\mu(b)$ for $a,b\in\mathcal{P}$ (the addition is continuous with respect to the symmetric topology by Proposition 1.1 from [9]). Let $a\in\mathcal{P}$. Since $(\mu_i(a))_{i\in\mathcal{I}}$ is a Cauchy net and $\{+\infty\}$ is an open subset of $\overline{\mathbb{R}}$, there is $i_0\in\mathcal{I}$ such that $\mu_i(a)<+\infty$ for all $i\geq i_0$ or $\mu_i(a)=+\infty$ for all $i\geq i_0$. If $\mu_i(a)<+\infty$ for all $i\geq i_0$, then $\alpha\mu_i(a)\to\alpha\mu(a)$, by the continuity of the scaler multiplication on bounded elements(see [9], Proposition 1.1). On the other hand $\mu_i(\alpha a)\to\mu(\alpha a)$ and $\mu_i(\alpha a)=\alpha\mu_i(a)$. This yields that $\mu(\alpha a)=\alpha\mu(a)$, since the symmetric topology of $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$ is Hausdorff. If $\mu_i(a)=+\infty$ for all $i\geq i_0$, then $\mu(a)=\infty$. This shows that $\alpha\mu_i(a)=+\infty\to+\infty=\alpha\mu(a)$ for $\alpha>0$ and $0.\mu_i(a)\to 0=0.\mu(a)$. Thus $\alpha\mu_i(a)\to\alpha\mu(a)$ for all $\alpha\geq 0$. On the other hand $\mu_i(\alpha a)\to\mu(\alpha a)$ and $\mu_i(\alpha a)=\alpha\mu_i(a)$. Then $\mu(\alpha a)=\alpha\mu(a)$.

We prove that $(\mu_i)_{i\in\mathcal{I}}$ is convergent to μ with respect to the symmetric topology of $(L(\mathcal{P}),\mathfrak{U}_\sigma(L(\mathcal{P}),\mathcal{P}))$. Let $U_\sigma\in\mathfrak{U}_\sigma(L(\mathcal{P}),\mathcal{P})$. For $a\in\mathcal{P}$, we define the linear functional $\varphi_a:L(\mathcal{P})\to\overline{\mathbb{R}}$ as below

$$\varphi_a(\mu)=\mu(a)\qquad \mu\in L(\mathcal{P}).$$

For $a\in\mathcal{P}$, we set $\Phi_a=\varphi_a\times\varphi_a$. There are $n\in\mathbb{N}$ and $a_1,\dots,a_n\in\mathcal{P}$ such that $\bigcap_{j=1}^n\Phi_{a_j}^{-1}(\tilde{1})\subseteq U_\sigma$. Since $\mu_i(a_j)\to\mu(a_j)$ with respect to the symmetric topology of $(\overline{\mathbb{R}},\mathcal{V})$, for each $j=1,2,\dots,n$, there is $i_j\in\mathcal{I}$ such that $(\mu_i(a_j),\mu(a_j))\in\tilde{1}$ and $(\mu(a_j),\mu_i(a_j))\in\tilde{1}$ for all $i\geq i_j$. This shows that $(\mu_i,\mu)\in\Phi_{a_j}^{-1}(\tilde{1})$ and $(\mu,\mu_i)\in\Phi_{a_j}^{-1}(\tilde{1})$ for all $i\geq i_j$. We set $i_0=\max\{i_1,\dots,i_n\}$, then we have $(\mu_i,\mu)\in\bigcap_{j=1}^n\Phi_{a_j}^{-1}(\tilde{1})\subseteq U_\sigma$ and $(\mu,\mu_i)\in\bigcap_{j=1}^n\Phi_{a_j}^{-1}(\tilde{1})\subseteq U_\sigma$ for all $i\geq i_0$. This shows that $\mu_i\in U_\sigma(\mu)U_\sigma$ for all $i\geq i_0$. Thus $\mu_i\to\mu$ with respect to the symmetric topology of $(L(\mathcal{P}),\mathfrak{U}_\sigma(L(\mathcal{P}),\mathcal{P})$. □

Let $(\mathcal{P},\mathcal{Q})$ be a dual pair. We shall say that a subset B of $\mathcal{P}$ is $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{Q})$-bounded below whenever it is bounded below in locally convex cone $(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{Q}))$. Let $\mathfrak{B}$ be a collection of $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{Q})$-bounded below subsets of $\mathcal{P}$ such that

(a) $\alpha B\in\mathfrak{B}$ for all $B\in\mathfrak{B}$ and $\alpha>0$,

(b) For all $X,Y\in\mathfrak{B}$ there is $Z\in\mathfrak{B}$ such that $X\cup Y\subset Z$.

(c) $\mathcal{P}$ is spanned by $\bigcup_{B\in\mathfrak{B}}B$.

For $B\in\mathfrak{B}$ we set

$$U_B=\{(x,y)\in\mathcal{Q}^2 :< b,x>\leq< b,y>+1,\ \textit{for all } b\in B\}\text{ and }\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P})=\{U_B:B\in\mathfrak{B}\}.$$

It is proved in [5], page 37, that $\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P})$ is a convex quasiuniform structure on $\mathcal{Q}$ and $(\mathcal{Q},\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P}))$ is a locally convex cone. If $b\in B$ for $B\in\mathfrak{B}$, then $b\in B\subseteq U_B^\circ$. Now $\mathcal{P}\subset(\mathcal{Q},\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P}))^*$ by (c). This shows that $\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P})$ is finer than $\mathfrak{U}_\sigma(\mathcal{Q},\mathcal{P})$.

If $\mathfrak{B}$ is the collection of all finite subsets of $\mathcal{P}$, then $\mathfrak{U}_{\mathfrak{B}}(\mathcal{Q},\mathcal{P})=\mathfrak{U}_\sigma(\mathcal{P},\mathcal{Q})$.

Lemma 2.5. *Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair and let $\mathfrak{B}$ be a collection of $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$-bounded below subsets of $\mathcal{P}$ that has (a), (b) and (c). Then the upper neighborhoods of $(\mathcal{Q}, \mathfrak{U}_\mathfrak{B}(\mathcal{Q}, \mathcal{P}))$ are closed with respect to the lower topology of $(\mathcal{Q}, \mathfrak{U}_\sigma(\mathcal{Q}, \mathcal{P}))$ and the lower neighborhoods of $(\mathcal{Q}, \mathfrak{U}_\mathfrak{B}(\mathcal{Q}, \mathcal{P}))$ are closed with respect the upper topology of $(\mathcal{Q}, \mathfrak{U}_\sigma(\mathcal{Q}, \mathcal{P}))$.*

Proof. Let $B \in \mathfrak{B}$, $\nu \in \mathcal{Q}$ and $\mu \in \overline{U_B(\nu)}^{l\sigma}$ (the closure of $U_B(\nu)$ with respect to the lower topology of $(\mathcal{Q}, \mathfrak{U}_\sigma(\mathcal{Q}, \mathcal{P}))$). Then there is a net $(\mu_i)_{i \in \mathcal{I}}$ in $U_B(\nu)$ such that $\mu_i \to \mu$ with respect to the lower topology of $(\mathcal{Q}, \mathfrak{U}_\sigma(\mathcal{Q}, \mathcal{P}))$. Let $b \in B$ and $\varepsilon > 0$. $\mu_i(b) \to \mu(b)$ with respect to the lower topology of $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$. Then there is $i_{b,\varepsilon}$ such that $\mu(b) \le \mu_i(b) + \varepsilon \le \nu(b) + 1 + \varepsilon$ for all $i \ge i_{b,\varepsilon}$. Then $\mu(b) \le \nu(b) + 1$, since ε is arbitrary. Thus $\mu \in U_B(\nu)$. The other claim can be proved in a similar way. □

Corollary 2.6. *Under the assumptions of Lemma 2.5, the upper and lower neighborhoods of $(\mathcal{Q}, \mathfrak{U}_\mathfrak{B}(\mathcal{Q}, \mathcal{P}))$ are closed with respect to the symmetric topology of $(\mathcal{Q}, \mathfrak{U}_\sigma(\mathcal{Q}, \mathcal{P}))$.*

Definition 2.7. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone with (SP). We shall say that the locally convex cone $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ is completion of $(\mathcal{P}, \mathfrak{U})$ whenever*

(a) $\hat{\mathfrak{U}}$ induces $\mathfrak{U}$ on $\mathcal{P}^2$,
(b) $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ is complete under the upper topology,
(c) $\mathcal{P}$ is dense in $\hat{\mathcal{P}}$ under the upper topology.

Theorem 2.8. *Let $(\mathcal{P}, \mathcal{Q})$ be a dual pair and let $\mathfrak{B}$ be a collection of $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{Q})$-bounded below subsets of $\mathcal{P}$ that has (a), (b) and (c). Then the completion of $(\mathcal{Q}, \mathfrak{U}_\mathfrak{B}(\mathcal{Q}, \mathcal{P}))$ is the subcone*

$$\hat{\mathcal{Q}} = \bigcap_{B \in \mathfrak{B}} (Q + (\{0\} \times B)^\circ)$$

of $L(\mathcal{P})$, under the convex quasiuniform structure $\mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P})$, the polar $(\{0\} \times B)^\circ$ being taken in $L(\mathcal{P})$.

Proof. If $\hat{z} \in \hat{\mathcal{Q}}$ and $B \in \mathfrak{B}$, then $\hat{z} \in \mathcal{Q} + (\{0\} \times B)^\circ$. Therefore there is $x \in \mathcal{Q}$ such that $\hat{z} \in x + (\{0\} \times B)^\circ$. There is $\lambda > 0$ such that $x \in \lambda(\{0\} \times B)^\circ$, since B is $\mathfrak{U}_\sigma$-bounded below. Then $\hat{z} \in (\lambda + 1)(\{0\} \times B)^\circ$. This shows that each $B \in \mathfrak{B}$ is $\mathfrak{U}_\sigma(\mathcal{P}, \hat{\mathcal{Q}})$-bounded below and we can define $\mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P})$. Also $\mathcal{Q}$ is dense in $\hat{\mathcal{Q}}$ under the upper topology, indeed for every $B \in \mathfrak{B}$ and $\hat{z} \in \hat{\mathcal{Q}}$ there is $x \in \mathcal{Q}$ and $\mu \in (\{0\} \times B)^\circ$ such that $\hat{z} = x + \mu$. For $b \in B$ we have

$$< b, \hat{z} >=< b, x > + < b, \mu >\ge< b, x > -1.$$

This shows that $x \in \mathcal{Q} \cap U_B(\hat{z})$.

Clearly, $\mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P})$ induces $\mathfrak{U}_\mathfrak{B}(\mathcal{Q}, \mathcal{P})$ on $\mathcal{Q}$.

Now, we prove that $(\hat{\mathcal{Q}}, \mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P}))$ is complete under whose upper topology. Let $(\hat{z}_i)_{i \in \mathcal{I}}$ be a Cauchy net with respect to the upper topology of $(\hat{\mathcal{Q}}, \mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P}))$. Then $(\hat{z}_i)_{i \in \mathcal{I}}$ is a Cauchy net with respect to the upper topology of $(L(\mathcal{P}), \mathfrak{U}_\sigma(L(\mathcal{P}), \mathcal{P}))$. Since, $(L(\mathcal{P}), \mathfrak{U}_\sigma(L(\mathcal{P}^*, \mathcal{P}))$ is complete with respect to the upper topology, there is $\mu \in L(\mathcal{P})$ such that $\hat{z}_i \to \mu$ under the upper topology. For $B \in \mathfrak{B}$ there is $i_0 \in \mathcal{I}$ such that $\hat{z}_\alpha \in U_B(\hat{z}_\beta)$ for all $\beta \ge \alpha \ge i_0$. Then $\hat{z}_\beta \in (\hat{z}_\alpha)U_B$ for all $\beta \ge \alpha \ge i_0$. This implies $\mu \in (\hat{z}_\alpha)U_B$ for $\alpha \ge i_0$, since $(\hat{z}_\alpha)U_B$ is closed with respect to the upper topology of $(L(\mathcal{P}), \mathfrak{U}_\sigma(L(\mathcal{P}), \mathcal{P}))$ by Lemma 2.5. Thus $\hat{z}_\alpha \in U_B(\mu)$ for all $\alpha \ge i_0$. This shows that $\hat{z}_\alpha \to \mu$ under the upper topology of $(\hat{\mathcal{Q}}, \mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P}))$. We show $\mu \in \hat{\mathcal{Q}}$. For $\gamma > i_0$ we have $< b, \hat{z}_\gamma >\le< b, \mu > +1$ for all $b \in B$. Since B is $\mathfrak{U}_\sigma(\hat{\mathcal{Q}}, \mathcal{P})$-bounded below and $\hat{z}_\gamma \in \hat{\mathcal{Q}}$, there is $\lambda > 0$ such that $-\lambda \le< b, \hat{z}_\gamma >$ for all $b \in B$. Then $0 \le< b, \mu > +(\lambda + 1)$ for all $b \in B$. Then $\mu \in (\lambda + 1)(\{0\} \times B)^\circ$. This shows that $\mu \in \hat{\mathcal{Q}}$. □

Theorem 2.9. *The locally convex cone $(\hat{\mathcal{Q}}, \mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P}))$ from Theorem 2.8 is complete with respect to the symmetric topology.*

Proof. Let $(\hat{z}_i)_{i \in \mathcal{I}}$ be a Cauchy net in $(\hat{\mathcal{Q}}, \mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P})$ with respect to the symmetric topology. Since $\mathfrak{U}_\mathfrak{B}(\hat{\mathcal{Q}}, \mathcal{P})$ is finer than $\mathfrak{U}_\sigma(\hat{\mathcal{Q}}, \mathcal{P})$ induced on $\hat{\mathcal{Q}}$ by $\mathfrak{U}_\sigma(L(\mathcal{P}), \mathcal{P})$, $(\hat{z}_i)_{i \in \mathcal{I}}$ is a Cauchy net under the symmetric topology of $(L(\mathcal{P}), \mathfrak{U}_\sigma(L(\mathcal{P}), \mathcal{P}))$. Now Theorem 2.4 yields that there is $\mu \in L(\mathcal{P})$ such that $\hat{z}_i \to \mu$ under the symmetric topology of $(L(\mathcal{P}), \mathfrak{U}_\sigma(L(\mathcal{P}), \mathcal{P}))$.

Since $(\hat{z}_i)_{i\in\mathcal{I}}$ is a Cauchy net with respect to the symmetric topology of $(\hat{\mathcal{Q}}, \mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{Q}}, \mathcal{P}))$, for every $B \in \mathfrak{B}$ there is $i_0 \in \mathcal{I}$ such that $(\hat{z}_i, \hat{z}_j) \in U_B$ for all $i, j \geq i_0$. For a fix $j \geq i_0$ we have $\hat{z}_i \in U_B(\hat{z}_j)$ for all $i \geq i_0$. This shows that $\mu \in U_B(\hat{z}_j)$, since $U_B(\hat{z}_j)$ is closed in the symmetric topology of $(\hat{\mathcal{Q}}, \mathfrak{U}_\sigma(\hat{\mathcal{Q}}, \mathcal{P}))$ by Corollary 2.6. Then $(\mu, \hat{z}_j) \in U_B$ for $j \geq i_0$. Similarly, $(\hat{z}_i, \mu) \in U_B$ for $i \geq i_0$. Thus $\hat{z}_i \to \mu$ with respect to the symmetric topology of $(\hat{\mathcal{Q}}, \mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{Q}}, \mathcal{P}))$. □

Now, we are ready to obtain the completion of a locally convex cone with (SP). Since the convex quasiuniform structure of a locally convex cone with (SP) can be represented as a polar-convex quasiuniform structure, we can use Theorem 2.8 for this aim.

Theorem 2.10. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone with (SP). The completion $\hat{\mathcal{P}}$ of $\mathcal{P}$ is the subcone $\bigcap_{U\in\mathfrak{U}}(\mathcal{P} + (\{0\} \times U^\circ)^\circ)$ of $L(\mathcal{P}^*)$ endowed with the convex convex quasiuniform structure $\mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{P}}, \mathcal{P}^*)$, where $\mathfrak{B} = \{U^\circ : U \in \mathfrak{U}\}$.*

Proof. *We consider the dual pair $(\mathcal{P}^*, \mathcal{P})$ and the collection $\mathfrak{B} = \{U^\circ : U \in \mathfrak{U}\}$ of subsets $\mathcal{P}^*$. It is proved in [5], II, that the convex quasiuniform structures $\mathfrak{U}$ and $\mathfrak{U}_{\mathfrak{B}}(\mathcal{P}, \mathcal{P}^*)$ are equivalent. Now Theorem 2.8 yields that $\hat{\mathcal{P}} = \bigcap_{U\in\mathfrak{U}}(\mathcal{P} + (\{0\} \times U^\circ)^\circ)$ and $\hat{\mathfrak{U}} = \mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{P}}, \mathcal{P}^*)$.* □

Corollary 2.11. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone with (SP). Then the locally convex cone $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ is complete with respect to the upper and symmetric topologies by Theorems 2.8 and 2.9.*

Proposition 2.12. *Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones with (SP) and $t : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{Q}, \mathcal{W})$ be a continuous linear mapping. Then t has an extension $\hat{t}$ which is a continuous linear mapping of $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ into $(\hat{\mathcal{Q}}, \hat{\mathcal{W}})$.*

Proof. *Let t' be the transpose of t, mapping $\mathcal{Q}^*$ into $\mathcal{P}^*$, and let t'^* be the transpose of t', mapping $L(\mathcal{P}^*)$ into $L(\mathcal{Q}^*)$. Suppose that $\hat{t}$ be the restriction of t'^* to $\hat{\mathcal{P}}$. Now, we show that $\hat{t}$ is continuous. Let $U_{W^\circ} \in \mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{P}}, \mathcal{P}^*)$, where $\mathfrak{B} = \{U^\circ : U \in \mathfrak{U}\}$ and $W \in \mathcal{W}$. There is $U \in \mathfrak{U}$ such that $t \times t(U) \subset W$, since t is continuous. We claim that $\hat{t} \times \hat{t}(U_{U^\circ}) \subset U_{W^\circ}$. Indeed, if $(\hat{a}, \hat{b}) \in U_{U^\circ}$, then $\mu(\hat{a}) \leq \mu(\hat{b}) + 1$ for all $\mu \in U^\circ$. If $\Lambda \in W^\circ$ then $\Lambda \in (t \times t(U))^\circ$, since $W^\circ \subset (t \times t(U))^\circ$. This shows that $t'(\Lambda) \in U^\circ$. Then $t'(\Lambda)(\hat{a}) \leq t'(\Lambda)(\hat{b}) + 1$. This yields that $(t'^*(\hat{a}))(\Lambda) \leq (t'^*(\hat{b}))(\Lambda) + 1$. Thus $(\hat{t}(\hat{a}), \hat{t}(\hat{b})) \in U_{W^\circ}$.*

Now, we prove that for $\hat{a} \in \hat{\mathcal{P}}$, $\hat{t}(\hat{a}) \in \hat{\mathcal{Q}}$. If $\hat{a} \in \hat{\mathcal{P}}$, then there is a net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ such that $a_i \to \hat{a}$ with respect to the upper topology of $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$. Since $\hat{t}$ is continuous with respect to the upper topologies of $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ and $(\hat{\mathcal{Q}}, \hat{\mathcal{W}})$, then $t(a_i) = \hat{t}(a_i) \to \hat{t}(\hat{a})$. This shows that $\hat{t}(\hat{a}) \in \hat{\mathcal{Q}}$. □

Proposition 2.13. *If, in Proposition 2.12, t is an isomorphism, then $\hat{t}$ is one to one.*

Proof. *Firstly, we prove that t' is onto. If $\mu \in \mathcal{P}^*$, then $\mu o t^{-1} \in \mathcal{Q}^*$, since t is an isomorphism. Now, we have $t'^*(\mu o t^{-1}) = \mu$.*

Let $\hat{t}(\hat{a}) = \hat{t}(\hat{b})$. This implies $< \Lambda, t'^*(\hat{a}) >=< \Lambda, t'^*(\hat{b}) >$ for all $\Lambda \in \mathcal{Q}^*$. Then $< t'(\Lambda), \hat{a} >=< t'(\Lambda), \hat{b} >$ for all $\Lambda \in \mathcal{Q}^*$. This yields $\hat{a} = \hat{b}$, since t' is onto. □

Proposition 2.14. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone with (SP). The completion $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ of $(\mathcal{P}, \mathfrak{U})$ is unique.*

Proof. *Let $(\tilde{\mathcal{P}}, \tilde{\mathfrak{U}})$ be another completion of the locally convex cone $(\mathcal{P}, \mathfrak{U})$. The identity mapping $i : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{P}, \mathfrak{U})$ is an isomorphism. Then the extensions mappings $\hat{i}_1 : (\hat{\mathcal{P}}, \hat{\mathfrak{U}}) \to (\tilde{\mathcal{P}}, \tilde{\mathfrak{U}})$ and $\hat{i}_2 : (\tilde{\mathcal{P}}, \tilde{\mathfrak{U}}) \to (\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ are one to one. This shows that $(\hat{\mathcal{P}}, \hat{\mathfrak{U}})$ and $(\tilde{\mathcal{P}}, \tilde{\mathfrak{U}})$ are isomorphic.* □

Corollary 2.15. *If $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone with (SP) and it is complete with respect to the upper topology, then $(\hat{\mathcal{P}}, \hat{\mathfrak{U}}) = (\mathcal{P}, \mathfrak{U})$.*

Theorem 2.16. *Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone with (SP) which is complete with respect to the upper topology. Then it is complete with respect to the symmetric topology.*

Proof. *We have $(\hat{\mathcal{P}}, \hat{\mathfrak{U}}) = (\mathcal{P}, \mathfrak{U})$, by Corollary 2.15. Then $(\mathcal{P}, \mathfrak{U})$ is complete with respect to the symmetric topology by Corollary 2.11.* □

The locally convex cone $(\overline{\mathbb{R}}, \mathcal{V})$ is complete under whose upper topology. Then $(\hat{\overline{\mathbb{R}}}, \hat{\mathcal{V}}) = (\overline{\mathbb{R}}, \mathcal{V})$ by corollary 2.15. If $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone with (SP) and $\mu : (\mathcal{P}, \mathfrak{U}) \to (\overline{\mathbb{R}}, \mathcal{V})$ be a linear functional, then there is a continuous extension $\hat{\mu} : (\hat{\mathcal{P}}, \hat{\mathfrak{U}}) \to (\hat{\overline{\mathbb{R}}}, \hat{\mathcal{V}}) = (\overline{\mathbb{R}}, \mathcal{V})$ by Proposition 2.12. This yields that $\mathcal{P}^* \subset \hat{\mathcal{P}}^*$.

Let E be a real locally convex Hausdorff topological vector space, and let $\mathcal{V}$ be a neighborhood base of convex, balanced and closed subsets of E. For every $V \in \mathcal{V}$ we set $\tilde{V} = \{(a, b) : a - b \in V\}$. Then $\tilde{\mathcal{V}} = \{\tilde{V} : V \in \mathcal{V}\}$ is a convex quasiuniform structure on E and $(E, \tilde{\mathcal{V}})$ is a locally convex cone. We have $V^\circ = \tilde{V}^\circ$ (V° is taken in the dual space of E as a locally convex space and $\tilde{V}^\circ$ is taken in the dual cone of E as a locally convex cone). This shows that the dual space of E as a locally convex space is equal to the dual cone of $(E, \tilde{\mathcal{V}})$. We denote it by E^*. The locally convex cone $(E, \tilde{\mathcal{V}})$ has (SP). Indeed, let $(a, b) \notin \rho\tilde{V}$ for some $\rho > 1$ and $V \in \mathcal{V}$. Then $a - b \notin \rho V$. Since E is Hausdorff, V is closed, and then there is $\mu \in E^*$ such that $\mu(a - b) > \rho > 1$ by the Hahn-Banach theorem. Then $\mu(a) > \mu(b) + 1$. Also the algebraic dual of E^* as a space and a cone are equal. It is proved in [8], VI, Theorem 3 that the completion of E as a locally convex space is $\hat{E} = \bigcap_{V \in \mathcal{V}}(E + V^{\circ\circ})$. Now we prove that the completion of $(E, \tilde{\mathcal{V}})$ as a locally convex cone is equal to $\hat{E}$. For this we prove that $(\{0\} \times \tilde{V}^\circ)^\circ = V^{\circ\circ}$. Indeed, let $\varphi \in V^{\circ\circ}$. If $\mu \in \tilde{V}^\circ$, then $\mu \in V^\circ$. This shows that $|\varphi(\mu)| \leq 1$. Then $-1 \leq \varphi(\mu)$ and $\varphi(0) = 0 \leq \varphi(\mu) + 1$. Thus $\varphi \in (\{0\} \times \tilde{V}^\circ)^\circ$. Similarly, we can prove that $(\{0\} \times \tilde{V}^\circ)^\circ \subseteq V^{\circ\circ}$.

Example 2.17. *The real number system $\mathbb{R}$ endowed with the convex quasiuniform structure $\mathcal{V} = \{\tilde{\varepsilon} : \varepsilon > 0\}$ is a locally convex cone. We have $\mathbb{R}^* = (\mathbb{R}, \mathcal{V})^* = [0, +\infty)$ and $L(\mathbb{R}^*) = L([0, +\infty)) = \overline{\mathbb{R}} = \mathbb{R} \cup \{\overline{+\infty}\}$, where $\overline{+\infty}$ is a linear functional on $[0, +\infty)$ acting as*

$$\overline{+\infty}(a) = \begin{cases} 0 & a=0 \\ +\infty & else. \end{cases}$$

Now, we have $\hat{\mathbb{R}} = \overline{\mathbb{R}}$, since for every $\varepsilon > 0$ we have $\overline{+\infty} \in (\{0\} \times \tilde{\varepsilon}^\circ)^\circ$.

Example 2.18. *Let $(E, \|.\|)$ be a normed space with unit ball B. Let $\mathcal{Q}$ be the collection of all sets $a + \rho B$, $a \in E$, $\rho \geq 0$. Then Q is a cone endowed with the usual addition and scaler multiplication. Its neutral element is $\{0\}$. We set*

$$\tilde{B} = \{(A, C) \in \mathcal{Q}^2 : A - C \subseteq B\}.$$

Then $\mathfrak{U} = \{\alpha\tilde{B} : \alpha > 0\}$ is a convex quasiuniform on $\mathcal{Q}$ and $(\mathcal{Q}, \mathfrak{U})$ is a locally convex cone. We have $\mathcal{Q}^* = \{\mu \oplus r : r \geq 0, \mu \in E^*, \|\mu\| \leq r\}$, where

$$\mu \oplus r(a + \rho B) = \mu(a) + r\rho,$$

for $a \in E$ (see [5], II, Example 2.17). If E is the real number system $\mathbb{R}$ endowed with the usual topology, then $\hat{Q}$ is the set $\{a + \rho B : a \in E, \rho \geq 0\} \cup \{\{\infty\}\}$, where $\{\infty\}$ is a linear functional on $\mathcal{Q}^*$ acting as

$$\{\infty\}(\mu \oplus r) = \begin{cases} 0 & r=0 \\ +\infty & else. \end{cases}$$

The locally convex cone $(\mathcal{Q}, \mathfrak{U})$ has (SP) but it is not complete with respect to the upper topology. In fact the net $(nB)_{n\in\mathbb{N}}$ is a Cauchy net in the upper topology but is not convergent. Since $\{+\infty\} \in (\{0\} \times (\alpha\tilde{B})^\circ)^\circ$ for all $\alpha > 0$, we have

$$\hat{\mathcal{Q}} = \mathcal{Q} \cup \{\{+\infty\}\}.$$

Then the completion of $(\mathcal{Q}, \mathfrak{U})$ is the locally convex cone $(\hat{\mathcal{Q}}, \mathfrak{U}_{\mathfrak{B}}(\hat{\mathcal{Q}}, \mathcal{Q}^*))$, where $\mathfrak{B} = \{(\alpha\tilde{B})^\circ : \alpha > 0\}$.

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