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On Topological Complete Hypergroups

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Abstract. One of the main obstacles before the development of the theory of topological hypergroups is the fact that translation of open sets may not be open in this setting. In this paper, we get rid of such obstacle by introducing the concept of topological complete hypergroups and investigate some of their properties.

1. Introduction and Preliminaries

The year 1934 saw the raise of the concept of *hypergroups* by Marty [12], later it was studied by Corsini [2], Corsini and Leoreanu [3], Davvaz [4], Dresher and Ore [6], Freni [7], Koskas [11], Massouros [13], R. Migliorato [15], Mittas [14], Tallini [17], Vougiouklis [18], and many others. Till now, only a few papers treated the notion of topological hyperstructures, for example see [1, 8–10, 16]. Heidari et al. [8, 9] introduced the concepts of topological hypergroups and topological polygroups, respectively. In this paper we study the concept of topological complete hypergroups, which is a special class of topological hypergroups.

Let us begin with some basic definitions and results that will be used as ready references. For any nonempty set H, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation*, where $\mathcal{P}^*(H)$ is the family of nonempty subsets of H. The ordered pair $(H, \circ)$ is called a *hypergroupoid* and if A, B are two nonempty subsets of it and $x \in H$, then

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if *reproduction axiom* holds, that is if $x \in H \Rightarrow x \circ H = H = H \circ x$. The ordered pair $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup as well as a quasihypergroup. *Subhypergroup* is defined as in general case, that is a nonempty subset K of a hypergroup $(H, \circ)$ is a *subhypergroup* if (1) for all $a, b \in K \Rightarrow a \circ b \subseteq K$ and (2) for all a of K, we have $a \circ K = K = K \circ a$.

A subhypergroup K of a hypergroup $(H, \circ)$ is said to be

(1) *closed on the left* (*on the right*) if for all k_1, k_2 of K and x of H, from $k_1 \in x \circ k_2$ ($k_1 \in k_2 \circ x$, respectively), it follows that $x \in K$. K is said to be *closed* if it is that on the both left and right;

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(2) *invertible on the left* (*on the right*) if for all x, y of H, from $x \in K \circ y$ ($x \in y \circ K$), it follows that $y \in K \circ x$ ($y \in x \circ K$, respectively). K is called *invertible* if it is so on the left as well as right;

(3) *normal* in H if for all $x \in H$, $x \circ K = K \circ x$ [5].

For $n > 1$, β_n defines a relation, which is reflexive as well as symmetric, on a semihypergroup H as follows:

$$a\beta_n b \Leftrightarrow \exists(x_1, x_2, ..., x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

and let $\beta = \bigcup_{i=1}^{\infty} \beta_n$, where $\beta_1 = \{(x, x) : x \in H\}$ is the diagonal relation on H. Koskas [11] introduced a relation β^* which is the transitive closure of β and it is seen that if $(H, \circ)$ is a hypergroup, then $\beta^* = \beta$ [7]. The relation β^* is called the *fundamental relation* on H and H/β^* is called the *fundamental group*. Let $(H, \circ)$ be a semihypergroup and A be a nonempty subset of H. We say that A is a *complete part* of H if for any nonzero natural number n and for all $a_1, a_2, ..., a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \phi \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

Let $(H, \circ)$ be a hypergroup and consider the canonical projection $\phi_H : H \to H/\beta^*$. The *heart* of H is the set $\omega_H = \{x \in H : \phi_H(x) = 1\}$, where 1 is the identity of the group H/β^*. It is seen that ω_H is a complete part as well as a subhypergroup of H [5]. A nonempty subset A of H is a complete part if and only if $\omega_H \circ A = A$. Also, $\omega_H = \bigcap_{K \in CPS(H)} K$, where $CPS(H)$ denotes the class of all complete part subhypergroups of H. Let A be a nonempty subset of H. The intersection of the complete parts of H containing A is called the complete closure of A in H; it is denoted by $C(A)$.

A semihypergroup H is *complete*, if it satisfies one of the following conditions:

(1) $\forall(x, y) \in H^2, \forall a \in x \circ y, C(a) = x \circ y;$

(2) $\forall(x, y) \in H^2, C(x \circ y) = x \circ y;$

(3) $\forall(m, n) \in N^2, 2 \leq m, n, \forall(x_1, x_2, ..., x_n) \in H^n, \forall(y_1, y_2, ..., y_m) \in H^m,$

$$\prod_{i=1}^{n} x_i \cap \prod_{j=1}^{m} y_j \neq \phi \Rightarrow \prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j.$$

A hypergroup is complete if it is a complete semihypergroup. If $(H, \circ)$ is a complete semihypergroup, then either there exist $a, b \in H$ such that $\beta^*(x) = a \circ b$ or $\beta^*(x) = \{x\}$. An element e of a hypergroup $(H, \circ)$ is called an *identity* if $a \in e \circ a \cap a \circ e$ for all $a \in H$. An element x' is called an *inverse* of x in H if there exists an identity e in H such that $e \in x \circ x' \cap x' \circ x$. A hypergroup H is said to be *regular* if it has at least one identity and every element has at least one inverse. A regular hypergroup H is said to be *reversible* if $\forall(a, b, x) \in H^3$ such that $a \in b \circ x$ and $a \in x \circ b \Rightarrow \exists\, x', x'' \in i(x)$ such that $b \in a \circ x'$ and $b \in x'' \circ a$, respectively, where $i(x)$ denote the set of inverses of x [4]. A mapping f from a hypergroup $(H_1, \circ)$ to a hypergroup $(H_2, *)$ is called

(1) a *homomorphism* if for all x, y of H, we have $f(x \circ y) \subseteq f(x) * f(y)$;

(2) a *good homomorphism* if for all x, y of H, we have $f(x \circ y) = f(x) * f(y)$.

Theorem 1.1. ([5]) *If $(H, \circ)$ is a complete hypergroup, then*

(1) *$\omega_H = \{e \in H : \forall x \in H, x \in x \circ e \cap e \circ x\}$, which means that ω_H is the set of two-sided identities of H.*

(2) *H is regular and reversible.*

Lemma 1.2. ([10]) *Let (H, τ) be a topological space, then the family $\mathcal{B}$ consisting of all $S_V = \{U \in \mathcal{P}^*(H) : U \subseteq V\}, V \in \tau$ is a base for a topology on $\mathcal{P}^*(H)$. This topology is denoted by τ^*.*

Definition 1.3. ([8]) Let $(H, \circ)$ be a hypergroup and (H, τ) be a topological space. Then the system $(H, \circ, \tau)$ is called a *topological hypergroup* if with respect to the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$

(1) the mapping $(x, y) \mapsto x \circ y$ from $H \times H \mapsto \mathcal{P}^*(H)$ and

(2) the mapping $(x, y) \mapsto x/y$ from $H \times H \mapsto \mathcal{P}^*(H)$
are continuous, where $x/y := \{z \in H : x \in z \circ y\}$.

For any nonempty subsets A, B of a hypergroup $(H, \circ)$, A/B is defined as $\cup\{a/b : a \in A, b \in B\}$.

Lemma 1.4. ([8]) *Let $(H, \circ)$ be a hypergroup and τ be a topology on H. Then the following assertions hold:*

(1) *the mapping $(x, y) \mapsto x \circ y$ is continuous if and only if for every $x, y \in H$ and $U \in \tau$ such that $x \circ y \subseteq U$, there exist $V, W \in \tau$ such that $x \in V$, $y \in W$ and $V \circ W \subseteq U$;*

(2) *the mapping $(x, y) \mapsto x/y$ is continuous if and only if for every $x, y \in H$ and $U \in \tau$ such that $x/y \subseteq U$, there exist $V, W \in \tau$ such that $x \in V, y \in W$ and $V/W \subseteq U$.*

2. Compactness in Topological Hypergroups with Special Emphasis on Topological Complete Hypergroups

In case of topological groups the translation maps are homeomorphisms, but for topological hypergroups they are continuous in general as shown by the following Lemma 2.1, which will be used in sequel.

Lemma 2.1. *Let $(H, \circ, \tau)$ be a topological hypergroup. Then the following translation maps*

$$L_a : H \to \mathcal{P}^*(H) \text{ by } x \mapsto a \circ x \text{ and } R_a : H \to \mathcal{P}^*(H) \text{ by } x \mapsto x \circ a$$

are continuous for every $a \in H$.

Proof. Let $U \in \tau$ such that $a \circ x \subseteq U$. Then by the continuity of the mapping $(x, y) \mapsto x \circ y$, $\exists\, V, W \in \tau$ such that $a \in V$ and $x \in W$ and $V \circ W \subseteq U$. This implies that $a \circ W \subseteq V \circ W \subseteq U$. This shows that L_a is continuous on H. Continuity of R_a can be shown in a similar way. □

Example 2.2. Consider the translation map L_2 on the topological hypergroup $(\mathbb{R}, \circ, \tau)$, where the hyperoperation $\circ$ is defined as, $x \circ y = \{x, y\}$ for every $x, y \in \mathbb{R}$ and τ is the standard topology on $\mathbb{R}$. Here $L_2((0, 1)) = (0, 1) \cup \{2\}$ which shows that L_2 is not a homeomorphism.

Definition 2.3. Let $(H, \circ, \tau)$ be a topological hypergroup. We say H is a *compact Hausdorff topological hypergroup* if (H, τ) is compact as well as a Hausdorff space.

Example 2.4. As in [8] if (X, τ) is a Hausdorff topological space, then $(X, \circ, \tau)$ is a topological hypergroup with respect to the hyperoperation $\circ$, defined as, for every $x, y \in X$, $x \circ y = \{x, y\}$. So for every compact Hausdorff space (X, τ) one can find a compact Hausdorff topological hypergroup. For instance, let $X = [0, 1]$ and consider the standard topology τ_u on it. Then $(X, \circ, \tau_u)$ is a compact Hausdorff topological hypergroup.

Theorem 2.5. *Let $(H, \circ, \tau)$ be a compact Hausdorff topological hypergroup and K be a subset of H. Then $x \circ \overline{K} = \overline{x \circ K}$, for all $x \in H$.*

Proof. Using Lemma 2.1 we have $x \circ \overline{K} \subseteq \overline{x \circ K}$, $\forall x \in H$. To prove $\overline{x \circ K} \subseteq x \circ \overline{K}$, let $p \in \overline{x \circ K}$. Now, $p \in x \circ K \Rightarrow p \in x \circ \overline{K}$ but if $p \notin x \circ K$, then p is a limit point of $x \circ K$. Let $U \in \tau$ such that $p \in U$, then $x \circ K \cap U \neq \phi \Rightarrow x \circ \overline{K} \cap U \neq \phi \Rightarrow p$ is a limit point of $x \circ \overline{K} \Rightarrow p \in \overline{x \circ \overline{K}}$. Now, we show that $\overline{x \circ \overline{K}} = x \circ \overline{K}$, i.e., $x \circ \overline{K}$ is closed. Here $\overline{K}$ is compact for being a closed subset of the compact space H. Also, $x \circ \overline{K}$ is compact, since translation maps are continuous (by Lemma 2.1). So being a compact subset of a Hausdorff space, $x \circ \overline{K}$ is closed. Hence, $\overline{x \circ K} \subseteq x \circ \overline{K}$. Thus, we conclude that $x \circ \overline{K} = \overline{x \circ K}$. □

Hausdorffness of hypergroup is necessary in Theorem 2.5 as it is illustrated in the following example.

Example 2.6. Let $H = \{1, 2\}$ and a hyperoperation $\circ$ on H is defined as follows

$\circ$	1	2
1	{1}	{2}
2	{2}	{1,2}

Then $(H, \circ)$ is a hypergroup. If $\tau = \{\phi, \{1\}, \{1, 2\}\}$, then $(H, \circ, \tau)$ is a compact topological hypergroup and it is not Hausdorff. Now if $A = \{1\}$, then $\overline{A} = \{1, 2\}$, $2 \circ A = \{2\}$ and $\overline{2 \circ A} = \{2\}$. But we have $2 \circ \overline{A} = \{1, 2\}$.

Necessity of the compactness of hypergroup in Theorem 2.5 is shown by the example below.

Example 2.7. Consider the set of real numbers $\mathbb{R}$. For all $x, y \in \mathbb{R}$, we define a hyperoperation as

$$x \circ y = \begin{cases} (-\infty, x] & \text{if} \quad x = y, \\ \{max\{x, y\}\} & \text{if} \quad x \neq y, \end{cases}$$

then $(\mathbb{R}, \circ)$ is a hypergroup. Now consider the upper limit topology τ_{up} on $\mathbb{R}$, then $(\mathbb{R}, \circ, \tau_{up})$ is a Hausdorff topological hypergroup and it is not compact. Now if $A = (2, 3)$, then $3 \circ \overline{A} = 3 \circ (2, 3] = (-\infty, 3]$. But we have $\overline{3 \circ A} = \overline{3 \circ (2, 3)} = \overline{\{3\}} = \{3\}$.

Proposition 2.8. *Let $(H, \circ, \tau)$ be a topological hypergroup and A, B are compact subsets of H. Then, $A \circ B$ is compact.*

Proof. Since A, B are compact subsets of H, it follows that $A \times B$ is compact subset of $H \times H$ with respect to the product topology induced from the topology τ on H. Now, the continuity of the map $(x, y) \mapsto x \circ y$ implies that $A \circ B$ is compact. □

Example 5 in [8] shows that, unlike in topological groups, translation of open sets may not be open in topological hypergroups. In the later part of this paper we see that how this difference may be avoided by restricting the domain of thoughts into a special class of topological hypergroups which we call topological complete hypergroups. Before that we state a result in the form of a proposition on complete hypergroup which will be used frequently.

Proposition 2.9. *Let A and B be nonempty subsets of a complete hypergroup $(H, \circ)$ such that A is a complete part and $x \in H$. Then,*

(1) $x^{-1} \circ x \circ A = x \circ x^{-1} \circ A = A$, *where* $x^{-1} \in i(x)$;

(2) $x \circ A$ *and* $A \circ x$ *are complete parts;*

(3) $B \subseteq x^{-1} \circ A$ *if and only if* $x \circ B \subseteq A$, *where* $x^{-1} \in i(x)$.

Proof. The proof is omitted. □

Definition 2.10. Let $(H, \circ, \tau)$ be a topological hypergroup. Then, we say H is a *topological complete hypergroup* if H is a complete hypergroup. Also, we say H is a *topological regular hypergroup* if H is a regular hypergroup.

Note that in this section the completeness and regularity of a topological hypergroup are purely algebraic.

Corollary 2.11. *Every topological complete hypergroup is a topological regular hypergroup (by Theorem 1.1).*

Evidently, every topological group is a topological complete hypergroup. Here, we present some other examples.

Example 2.12. The total hypergroup $(H, \circ)$ (i.e., for all $x, y \in H$, $x \circ y = H$) with an arbitrary topology is a topological complete hypergroup.

Example 2.13. Consider the set of integers $\mathbb{Z}$ with the hyperoperation $*$ on it as

$$m * n = \begin{cases} 2\mathbb{Z} & \text{if } m + n \in 2\mathbb{Z} \\ (2\mathbb{Z})^c & \text{otherwise,} \end{cases}$$

then $(\mathbb{Z}, *)$ is a complete hypergroup. Let $\tau = \{\phi, 2\mathbb{Z}, (2\mathbb{Z})^c, \mathbb{Z}\}$. Then τ is a topology on $\mathbb{Z}$, and $(\mathbb{Z}, *, \tau)$ is a topological complete hypergroup.

Example 2.14. Consider the topological group $(\mathbb{Z}, +, \tau)$, where τ is the subspace topology on $\mathbb{Z}$ induced by the standard topology on $\mathbb{R}$. Now for $n \in \mathbb{Z}^+$, let $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \cdots, \overline{n-1}\}$ be the set of all congruence classes of integers modulo n. For all $a, b \in \mathbb{Z}$, we define the hyperoperation $a \circ b = \overline{a+b}$, then $(\mathbb{Z}, \circ, \tau_{\mathbb{Z}})$ is a topological complete hypergroup, where $\tau_{\mathbb{Z}} = \{\bigcup_{x \in U} \bar{x} : U \in \tau\}$.

Note that every open subset of the topological complete hypergroups shown by Example 2.13 and Example 2.14 is a complete part. Now, let's develop a tool which will be used after a while.

Lemma 2.15. *Let U be an open subset of a topological complete hypergroup $(H, \circ, \tau)$ such that U is a complete part. Then, $a \circ U$ and $U \circ a$ are open subsets of H for every $a \in H$.*

Proof. Suppose U be an open subset as well as a complete part of H and $a \in H$. Then, for some $a^{-1} \in i(a)$ we have

$$\begin{aligned} L^{-1}_{a^{-1}}(S_U) &= \{x \in H : L_{a^{-1}}(x) \in S_U\} \\ &= \{x \in H : a^{-1} \circ x \subseteq U\}. \end{aligned}$$

We claim that $\{x \in H : a^{-1} \circ x \subseteq U\} = a \circ U$. For, let $p \in \{x \in H : a^{-1} \circ x \subseteq U\}$, then $a^{-1} \circ p \subseteq U$. Now, there exists $e \in \omega_H$ such that $e \in a \circ a^{-1}$ and this implies that $p \in e \circ p \subseteq a \circ a^{-1} \circ p \subseteq a \circ U$.

For the converse, let

$$\begin{aligned} t \in a \circ U &\Rightarrow t \in a \circ u \text{ for some } u \in U \\ &\Rightarrow u \in a' \circ t \text{ for some } a' \in i(a) \\ &\Rightarrow u \in a' \circ t \subseteq a' \circ a \circ a^{-1} \circ t = \omega_H \circ a^{-1} \circ t = C(a^{-1} \circ t) = a^{-1} \circ t \\ &\Rightarrow a^{-1} \circ t \cap U \neq \phi \\ &\Rightarrow a^{-1} \circ t \subseteq U, \text{ since } U \text{ is a complete part of } H. \\ &\Rightarrow t \in \{x \in H : a^{-1} \circ x \subseteq U\}. \end{aligned}$$

Hence, $L^{-1}_{a^{-1}}(S_U) = a \circ U$. Since the translation maps are continuous, it follows that $a \circ U$ is open in H.

Similarly, $U \circ a$ is open in H. □

Theorem 2.16. *Let H be a topological complete hypergroup and A, B be open subsets of H. If A or B is a complete part of H, $A \circ B$ is open.*

Proof. Suppose that A is a complete part of H, then $A \circ b$ is open(by Lemma 2.15). Now, $A \circ B = \bigcup_{b \in B} A \circ b$, this shows that $A \circ B$ is open. □

Lemma 2.17. *Let H be a topological complete hypergroup such that every open subset of H is a complete part. Let $\mathcal{U}$ be a basis at some identity e. Then, the families $\{x \circ U : x \in H, U \in \mathcal{U}\}$ and $\{U \circ x : x \in H, U \in \mathcal{U}\}$ are basis for H.*

Proof. Let W be an open subset of H and $a \in W$. Then, there exists $a' \in i(a)$ such that $e \in \omega_H = a' \circ a \subseteq a' \circ W$. Since $\mathcal{U}$ is a basis at e, there exists $U \in \mathcal{U}$ such that $e \in U \subseteq a' \circ W$. This implies $a \in a \circ U \subseteq a \circ a' \circ W = W$ (by Proposition 2.9), i.e., $a \in a \circ U \subseteq W$. This shows that $\{x \circ U : x \in H, U \in \mathcal{U}\}$ is a basis for H.

Similarly, $\{U \circ x : x \in H, U \in \mathcal{U}\}$ is also a basis for H. □

Lemma 2.18. *Let H be a topological complete hypergroup such that every open subset of it is a complete part and $\mathcal{U}$ be a basis at some identity e. Then, the following assertions hold:*

(1) *for every $W \in \tau$ with $x \in W$, there exists $V \in \mathcal{U}$ such that $x \circ V \subseteq W$ and $V \circ x \subseteq W$;*

(2) *for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.*

Proof. (1) Suppose that $W \in \tau$ with $x \in W$. Then, there exists $x' \in i(x)$ such that $e \in \omega_H = x' \circ x \subseteq x' \circ W$. Since $\mathcal{U}$ is a basis at e, there exists $V \in \mathcal{U}$ such that $e \in V \subseteq x' \circ W$. This implies $x \circ V \subseteq W$ (by Proposition 2.9).

Similarly, we can show that there exists $V \in \mathcal{U}$ such that $V \circ x \subseteq W$.

(2) Suppose that $U \in \mathcal{U}$, then $e \in U$. Since U is a complete part of H, it follows that $e \circ e \subseteq U$. So by the continuity of the map $(x, y) \mapsto x \circ y$ there exists $V \in \tau$ such that $e \in V$, i.e., $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. □

Theorem 2.19. *Let $(H, \circ, \tau)$ be a topological complete hypergroup such that every open subset of it is a complete part. If A and B are two nonempty subsets of H, then*

(1) $\overline{A} \circ \overline{B} \subseteq \overline{A \circ B}$;

(2) *$IntA \circ IntB \subseteq Int(A \circ B)$, where $Int(A)$ denotes the interior of the subset A.*

Proof. (1) The map $f(x, y) = x \circ y$ is continuous from $H \times H$ to $\mathcal{P}^*(H)$, then $f(\overline{A \times B}) \subseteq \overline{f(A \times B)} \Rightarrow \overline{A} \circ \overline{B} \subseteq \overline{A \circ B}$.

(2) Let $p \in IntA \circ IntB$, then $p \in a \circ b$ for some $a \in IntA$ and $b \in IntB$. Since a and b are interior points of A and B, respectively, then there exist $U, V \in \tau$ such that $a \in U \subseteq A$ and $b \in V \subseteq B \Rightarrow p \in a \circ b \subseteq U \circ V \subseteq A \circ B \Rightarrow p \in Int(A \circ B)$, since $U \circ V$ is open (by Theorem 2.16). Thus, $IntA \circ IntB \subseteq Int(A \circ B)$. □

Theorem 2.20. *Let H be a topological complete hypergroup such that every open subset of it is a complete part. Let F be a compact subset of H and P be a closed subset of H such that $F \cap P = \phi$. Then, there exists an open neighborhood V containing some identity e such that $F \circ V \cap P = \phi$ and $V \circ F \cap P = \phi$.*

Proof. Since P is closed, it follows that for each $x \in F$ there exists an open neighborhood V_x of some identity e in H such that $x \circ V_x \cap P = \phi$. By Lemma 2.18 there exists an open neighborhood W_x of e such that $W_x \circ W_x \subseteq V_x$. Now, $\{x \circ W_x\}_{x \in F}$ is an open cover for the compact set F, so there exist $x_1, x_2, ..., x_n \in F$ such that $F \subseteq \bigcup_{i=1}^{n} x_i \circ W_{x_i}$. Let $V_1 = \bigcap_{i=1}^{n} W_{x_i}$. We claim that $F \circ V_1 \cap P = \phi$. It suffices to verify that $y \circ V_1 \cap P = \phi$ for each $y \in F$. Let $y \in F$, then $y \in x_k \circ W_{x_k}$ for some $k \in \{1, 2, ..., n\}$ and $y \circ V_1 \subseteq (x_k \circ W_{x_k}) \circ V_1 \subseteq x_k \circ (W_{x_k} \circ W_{x_k}) \subseteq x_k \circ V_{x_k} \subseteq H \setminus P$, by our choice of the sets V_x and W_x. This proves that $F \circ V_1$ and P are disjoint.

Similarly, one can find an open neighborhood V_2 of e in H such that $V_2 \circ F \cap P = \phi$. Then, the set $V = V_1 \cap V_2$ is the required open neighborhood of e. □

Theorem 2.21. *Let $(H_1, \circ, \tau)$ and $(H_2, *, \tau')$ be two topological complete hypergroups such that every open subset of them is a complete part. Let f be a homomorphism from H_1 into H_2. Then, f is continuous if and only if it is continuous at some identity of H_1.*

Proof. If f is continuous, then the condition is obvious.

For the converse, let the map f is continuous at some identity e of H_1. Let $x \in H_1$ and W be an open set containing $f(x)$ in H_2. Since W is a complete part, $f(x) \in W \Rightarrow f(x)*f(e) \subseteq W$. By the continuity of translation map there exists an open set V in H_2 such that $f(e) \in V$ and $f(x)*V \subseteq W$. Since f is continuous at e, it follows that there exists an open set U containing e such that $f(U) \subseteq V$. Now, $f(x \circ U) \subseteq f(x) * f(U) \subseteq f(x) * V \subseteq W$. This shows that f is continuous on H_1. □

Theorem 2.22. *Let $(H_1, \circ, \tau)$ and $(H_2, *, \tau')$ be two topological complete hypergroups such that every open subset of them is a complete part. Also let f be a good homomorphism from H_1 into H_2. Then, f is an open map if and only if for every open set V containing some identity e_1 of H_1, $f(V)$ is open in H_2 containing some identity e_2.*

Proof. If f is an open map, then the condition holds as $f(\omega_{H_1}) = \omega_{H_2}$.

For the converse, let the given condition holds. Let U be an open set in H_1. We show $f(U)$ is open in H_2. Let $y \in f(U)$, then $y = f(x)$ for some $x \in U$. Since $x \in U$, there exists an open neighborhood V of some identity e_1 such that $x \in x \circ V \subseteq U$ (By Lemma 2.18). Then, $y = f(x) \in f(x) * f(V) = f(x \circ V) \subseteq f(U)$. Since $f(V)$ is open in H_2 containing e_2, $f(U)$ is open and hence f is an open map. □

Now, let us define a special kind of identity element in a regular hypergroup.

Definition 2.23. Let $(H, \circ)$ be a regular hypergroup. Let e be an identity in H and $g \in H$. We say e is *related* to g if $\exists\, g' \in i(g)$ such that $e \in g \circ g' \cap g' \circ g$.
We say an identity e is related to H or a *related identity* of H if it is related to every element of H, i.e., for every $g \in H$, $\exists\, g' \in i(g)$ such that $e \in g \circ g' \cap g' \circ g$.

Example 2.24. Consider the additive group $(\mathbb{Z}, +)$ of integers, and define the hyperoperation $\circ$ on it as $m \circ n = < m, n >=$ the subgroup generated by m and n. Then, $(\mathbb{Z}, \circ)$ is a regular hypergroup with 0 as a related identity.

Example 2.25. Consider the set of integers $\mathbb{Z}$ with the hyperoperation $*$ on it as

$$m * n = \begin{cases} 2\mathbb{Z} & \text{if } m + n \in 2\mathbb{Z} \\ (2\mathbb{Z})^c & \text{otherwise.} \end{cases}$$

Then, $(\mathbb{Z}, *)$ is a regular hypergroup with 0 as a related identity.

Let us develop some algebraic tools which will be used later in sequel.

Lemma 2.26. *Every subhypergroup of a complete hypergroup is a complete part.*

Proof. Let K be a subhypergroup of a complete hypergroup H. Now, $\omega_H \circ K = \omega_K \circ K = \bigcup_{x \in \omega_K} x \circ K = K$. This shows that K is a complete part of H. □

Corollary 2.27. *Let K be a subhypergroup of a complete hypergroup H. Then, $\{x \circ K\}_{x \in H}$ and $\{K \circ x\}_{x \in H}$ are partitions for H.*

Proof. By Lemma 2.26, K is a complete part subhypergroup of H. Since any complete part subhypergroup is invertible [5], it follows that $\{x \circ K\}_{x \in H}$ and $\{K \circ x\}_{x \in H}$ are partitions for H [5]. □

Theorem 2.28. *Let K be a subhypergroup of a complete hypergroup H. Then, K is normal in H if and only if for every $k \in K$ and for every $x \in H$, $x \circ k \circ x^{-1} \subseteq K$, i.e., $x \circ K \circ x^{-1} \subseteq K$, where $x^{-1} \in i(x)$.*

Proof. Let K be a normal subhypergroup of H. Now, for $x \in H$ and $k \in K$, $x \circ k \subseteq x \circ K = K \circ x$. Then, $x \circ k \circ x^{-1} \subseteq K \circ x \circ x^{-1} = K \circ \omega_H = K$ (by Lemma 2.26).

For the converse, suppose the given condition holds. Let $p \in x \circ K \Rightarrow p \in x \circ k$ for some $k \in K \Rightarrow p \in (x \circ k \circ x^{-1}) \circ x \subseteq K \circ x$. Therefore, we have $x \circ K \subseteq K \circ x$. Now, let $q \in K \circ x \Rightarrow q \in k_1 \circ x$ for some $k_1 \in K$ $\Rightarrow q \in x \circ x^{-1} \circ k_1 \circ x \Rightarrow q \in x \circ (x^{-1} \circ k_1 \circ (x^{-1})^{-1}) \circ x^{-1} \circ x \subseteq x \circ K \circ \omega_H = x \circ K$ (by Lemma 2.26). Therefore, we have $K \circ x \subseteq x \circ K$. Hence, $x \circ K = K \circ x$ for every $x \in H$. This shows that K is normal in H. □

It is observed that if K is a normal subhypergroup of a complete hypergroup H, then for every $x \in H$ with $x^{-1} \in i(x)$, $x \circ K \circ x^{-1} = K \circ x \circ x^{-1} = K \circ \omega_H = K$ (by Lemma 2.26). Hence, the above theorem can be restated as follows:

Corollary 2.29. *Let K be a subhypergroup of a complete hypergroup H. Then, K is normal in H if and only if for every $x \in H$, $x \circ K \circ x^{-1} = K$, where $x^{-1} \in i(x)$.*

Proposition 2.30. *Let $(H, \circ)$ be a complete hypergroup and M, N are two normal subhypergroups of it. Then,*

(1) *$(N \circ a) \circ (N \circ b) = N \circ a \circ b$, for all $a, b \in H$;*

(2) *$N \circ a = N \circ b$ if and only if $b \in N \circ a$;*

(3) *$M \cap N$ is a normal subhypergroup of H.*

Proof. (1) $(N \circ a) \circ (N \circ b) = N \circ (a \circ N) \circ b = N \circ N \circ a \circ b = N \circ a \circ b$.

(2) First we suppose $N \circ a = N \circ b$. Then, $b \in \omega_N \circ b \subseteq N \circ b = N \circ a$.

For the converse, let $b \in N \circ a$. Then, $N \circ b \subseteq N \circ N \circ a = N \circ a$. Since any complete part subhypergroup is invertible [5], it follows that $b \in N \circ a \Rightarrow a \in N \circ b$. So, $N \circ a \subseteq N \circ N \circ b = N \circ b$ and hence $N \circ a = N \circ b$.

(3) Being complete part subhypergroups of H (by Lemma 2.26), M, N are invertible and hence closed [5]. Since $\omega_M = \omega_N = \omega_H$, it follows that the intersection of M, N is nonempty. Therefore, $M \cap N$ is a closed subhypergroup of H [13]. Since M, N are normal in H, it follows that for $x \in H$ with $x^{-1} \in i(x)$ we have $x \circ M \circ x^{-1} \subseteq M$ and $x \circ N \circ x^{-1} \subseteq N$. So, for $x \in H$ with $x^{-1} \in i(x)$ we have $x \circ (M \cap N) \circ x^{-1} \subseteq M \cap N$. This shows that $M \cap N$ is normal in H. □

Now, we show that the component(or connected component) of an element can be obtained from the component of its related identity by using translation map in a topological regular hypergroup. In a topological hypergroup, we use the notation C_g to denote the component of g.

Lemma 2.31. *Let $(H, \circ, \tau)$ be a topological regular hypergroup. Then, for each $g \in H$, $L_g(C_e) = C_g$, where e is an identity related to g.*

Proof. $L_g(C_e)$ is a continuous image of C_e, so it is connected and $g \in L_g(C_e)$, so $L_g(C_e) \subseteq C_g$ as C_g is the maximal connected set containing g. Since e is an identity related to g, there exists $g' \in i(g)$ such that $e \in g' \circ g \cap g \circ g'$. This shows that $L_{g'}(C_g)$ is a connected set containing e, so $L_{g'}(C_g) \subseteq C_e$. This implies $C_g \subseteq g \circ g' \circ C_g = L_g(L_{g'}(C_g)) \subseteq L_g(C_e)$. Hence, $L_g(C_e) = C_g$. □

Theorem 2.32. *Let $(H, \circ, \tau)$ be topological regular hypergroup and e be an identity related to H. Then, C_e is a closed(topologically) subhypergroup. Furthermore, if H is a complete hypergroup, then C_e is a normal subhypergroup of H.*

Proof. Being the component of e, C_e is a closed set. We prove C_e is a subhypergroup. Let $g, h \in C_e$. Then, $g \circ C_e$ is a connected set containing g and $g \circ h$, i.e., $g \circ h \subseteq g \circ C_e$, so $g \circ h \subseteq C_e$.

Let $g \in C_e$, then $g \circ C_e = C_g = C_e$ (By Lemma 2.31). Similarly, $C_e \circ g = C_e$, for all $g \in C_e$. Hence, C_e is a subhypergroup of H.

Now, suppose H be a complete hypergroup. For $g \in H$, $C_e \circ g'$ is connected, where $g' \in i(g)$, so $g \circ C_e \circ g'$ is connected and contains e. Hence, $g \circ C_e \circ g' \subseteq C_e$. This shows that C_e is normal in H (by Theorem 2.28). □

Let us introduce topological subhypergroup.

Definition 2.33. Let H be a topological hypergroup and K be a subhypergroup of H. Let K be endowed with relative topology induced from H. Since the mappings $(x, y) \mapsto x \circ y$ and $(x, y) \mapsto x/y$ of $H \times H$ into $\mathcal{P}^*(H)$ are continuous, so are their restrictions from $K \times K$ into $\mathcal{P}^*(K)$. Thus, K is a topological hypergroup endowed with relative topology. In this case, K is called a *topological subhypergroup*.

Proposition 2.34. *Let $(H, \circ, \tau)$ be a topological hypergroup such that every open subset of it is a complete part and K be a subhypergroup of H. Then, every open subset of K is a complete part.*

Proof. Let U be an open subset of K and for $n \in \mathbb{N}$, $\prod_{i=1}^{n} a_i \cap U \neq \phi$, where $a_i \in K$. Then, there exists $V \in \tau$ such that $U = V \cap K$. Therefore, $\prod_{i=1}^{n} a_i \cap V \neq \phi$ and so, $\prod_{i=1}^{n} a_i \subseteq V$. Also, $\prod_{i=1}^{n} a_i \subseteq K$ and hence $\prod_{i=1}^{n} a_i \subseteq V \cap K = U$. This shows that U is a complete part of K. □

Theorem 2.35. *Let $(H, \circ, \tau)$ be a topological complete hypergroup. Then, every open subhypergroup is closed(topologically).*

Proof. Let K be an open subhypergroup of H, then $x \circ K$ is open for every $x \in H$, since K is a complete part of H. Now, $\{x \circ K\}_{x \in H}$ is a partition for H (by Corollary 2.27). So, we can write $H = \bigcup_{x \in H} x \circ K = (\bigcup_{x \in K} x \circ K) \cup (\bigcup_{x \notin K} x \circ K) = K \cup (\bigcup_{x \notin K} x \circ K)$. This implies $K = H \setminus (\bigcup_{x \notin K} x \circ K)$ and hence K is closed. □

Lemma 2.36. *Every subhypergroup of a complete hypergroup is complete.*

Proof. Let K be a subhypergroup of a complete hypergroup $(H, \circ)$. Now, for $x, y \in K$, $C(x \circ y) = (x \circ y) \circ \omega_K = (x \circ y) \circ \omega_H = x \circ y$. This shows that K is a complete subhypergroup of H. □

Theorem 2.37. *Let $(H, \circ, \tau)$ be a topological complete hypergroup such that every open subset of it is a complete part and K be a subhypergroup of H. Then, K is open if and only if its interior $IntK \neq \phi$.*

Proof. Let $IntK \neq \phi$ and $x \in IntK$. Then, there exists an open set U containing some identity e of H such that $x \in x \circ U \subseteq K$. Now, take any $y \in K$, then $y \circ U \subseteq y \circ x^{-1} \circ x \circ U \subseteq y \circ x^{-1} \circ K = K$, since $x, y \in K$ and K is complete(by Lemma 2.36). This shows that K is open.

For the converse, let K be open, then $IntK \neq \phi$. □

Proposition 2.38. *Let $(H, \circ, \tau)$ be a topological complete hypergroup such that every open subset of it is a complete part and e be a related identity of H. Let $\mathcal{U}$ be the system of all neighborhoods of e, then for any subset A of H,*

$$\overline{A} = \bigcap_{U \in \mathcal{U}} A/U.$$

Proof. Let $x \in \overline{A}$ and $U \in \mathcal{U}$, $x \circ U$ is a neighborhood of x, and hence $x \circ U \cap A \neq \phi$. This implies there exist $a \in A$ and $u \in U$ such that $a \in x \circ u \Rightarrow x \in a/u \subseteq A/U$. Therefore, $\overline{A} \subseteq A/U$ and hence $\overline{A} \subseteq \cap_{U \in \mathcal{U}} A/U$.

For the converse, suppose $y \in A/U$ for every $U \in \mathcal{U}$. Now, for any open neighborhood V of y, there exists $y^{-1} \in i(y)$ such that $y^{-1} \circ V$ contains e and hence $y^{-1} \circ V \in \mathcal{U}$. This implies that $y \in A/(y^{-1} \circ V)$ $\Rightarrow y \in a/w$ for some $a \in A$ and $w \in y^{-1} \circ V \Rightarrow a \in y \circ w \subseteq y \circ y^{-1} \circ V = V$(by Proposition 2.9) $\Rightarrow V \cap A \neq \phi$ and hence $y \in \overline{A}$. This completes the proof. □

Remark 2.39. Let $(H, \circ)$ be a complete hypergroup and ω_H be the heart of H. Then, for every $e \in \omega_H$ we have $e/e = \omega_H$.

For, let $t \in e/e$, then $e \in t \circ e$. Also, $e \in e \circ e \subseteq e \circ t \circ e \subseteq e \circ t \circ \omega_H = C(e \circ t) = e \circ t$. Now, we show that $t \in \omega_H$, i.e., t is a two sided identity of H. Let $x \in H$, then $x \in x \circ e \subseteq x \circ t \circ e \subseteq x \circ t \circ \omega_H = C(x \circ t) = x \circ t$. This shows that t is a right identity of H. Similarly, $x \in e \circ x \subseteq e \circ t \circ x \subseteq \omega_H \circ t \circ x = C(t \circ x) = t \circ x$. This shows that t is a right identity of H and hence $t \in \omega_H$. Also, $\omega_H \subseteq e/e$. Therefore, we obtain $e/e = \omega_H$.

Theorem 2.40. *Let $(H, \circ, \tau)$ be a topological complete hypergroup such that every open subset of it is a complete part and e be a related identity of H. Now, if U is an open neighborhood of e, then there exists an open neighborhood V of e such that $\overline{V} \subseteq U$.*

Proof. Since U is a complete part of H and $e \in U$, it follows that $\omega_H = e \circ e \subseteq U$. Again, $e/e = \omega_H$, this implies that $e/e \subseteq U$. So, by the continuity of the map $(x, y) \mapsto x/y$, there exists an open neighborhood V of e such that $V/V \subseteq U$. Now, by using Proposition 2.38 we have $\overline{V} \subseteq V/V \subseteq U$, i.e., $\overline{V} \subseteq U$. □

Corollary 2.41. *Let $(H, \circ, \tau)$ be a topological complete hypergroup such that every open subset of it is a complete part and e be a related identity of H. Then, H is locally compact if and only if there exists a compact neighborhood of e.*

Proof. Suppose that H is locally compact. Then, by the definition of locally compactness, there exists a compact neighborhood of e.

For the converse, suppose that U be a compact neighborhood of e. Then, by Theorem 2.40, there exists an open neighborhood V of e such that $\overline{V} \subseteq U$. Now, being a closed subset of a compact set, $\overline{V}$ is compact. So, for each $x \in H$, $x \circ \overline{V}$ is a compact neighborhood of x. This completes the proof. □

Let $(H, \circ)$ be a complete hypergroup and K be a normal subhypergroup of H. By H/K we denote the collection of all left(or right) cosets of K in H, i.e., $H/K = \{K \circ x : x \in H\}$.

Proposition 2.42. *Let $(H, \circ)$ be a complete hypergroup and K be a normal subhypergroup of H. Then, H/K forms a hypergroup with respect to the operation $\odot$ defined by $K \circ x \odot K \circ y = \{K \circ z : z \in x \circ y\}$.*

Proof. Let us check for associativity of $\odot$ on H/K. For all $x, y, z \in H$, we have

$$\begin{aligned}(K \circ x \odot K \circ y) \odot K \circ z &= \{K \circ u : u \in x \circ y\} \odot K \circ z \\ &= \{K \circ v : u \in x \circ y, v \in u \circ z\} \\ &= \{K \circ v : v \in (x \circ y) \circ z\},\end{aligned}$$

$$\begin{aligned}K \circ x \odot (K \circ y \odot K \circ z) &= K \circ x \odot \{K \circ u : u \in y \circ z\} \\ &= \{K \circ v : u \in y \circ z, v \in x \circ u\} \\ &= \{K \circ v : v \in x \circ (y \circ z)\}.\end{aligned}$$

Since $(x \circ y) \circ z = x \circ (y \circ z)$, it follows that $(K \circ x \odot K \circ y) \odot K \circ z = K \circ x \odot (K \circ y \odot K \circ z)$.
Now, for reproduction axiom let $K \circ x \in H/K$, then we have

$$\begin{aligned}K \circ x \odot H/K &= \{K \circ v : v \in x \circ y, y \in H\} \\ &= \{K \circ v : v \in x \circ H = H\} \\ &= H/K.\end{aligned}$$

Similarly, we have $H/K \odot K \circ x = H/K$. Therefore, $(H/K, \odot)$ is a hypergroup. □

Let ϕ be the natural mapping $x \mapsto K \circ x$ of H onto H/K. Then, $(H/K, \overline{\tau})$ is a topological space, where $\overline{\tau}$ is the quotient topology induced by ϕ. i.e., for every subset X of H, $\{K \circ x : x \in X\}$ is open in H/K if and only if $\phi^{-1}(\{K \circ x : x \in X\})$ is an open subset of H. We use the notation X/K to denote the set $\{K \circ x : x \in X\}$.

Lemma 2.43. *Let $(H, \circ, \tau)$ be a topological complete hypergroup and K be a normal subhypergroup of it. Let ϕ be the natural mapping $x \mapsto K \circ x$ of H onto H/K. Then,*

(1) *ϕ is continuous;*

(2) *$\phi^{-1}(\{K \circ x : x \in X\}) = K \circ X$ for every subset X of H;*

(3) *If every open subset of H is a complete part, then ϕ is open;*

(4) *ϕ is a good homomorphism;*

(5) *If H is compact, then H/K is compact;*

(6) *If every open subset of H is a complete part, then the quotient topology is the finest topology on H/K with respect to which ϕ is continuous.*

Proof. (1) ϕ is continuous by the definition of quotient topology.

(2) We have $K\circ X \subseteq \phi^{-1}(\{K\circ x : x \in X\})$ for every subset X of H. For the converse, let $y \in \phi^{-1}(\{K\circ x : x \in X\})$. Then, $\phi(y) = K \circ y \in \{K \circ x : x \in X\}$. So, $K \circ x = K \circ y$ for some $x \in X$, then by Proposition 2.30, $y \in K \circ x \subseteq K \circ X$. Thus, the equality holds.

(3) Let U be an open subset of H. We show $\phi(U)$ is open in H/K. Here, $\phi^{-1}(\phi(U)) = K \circ U$. Since U is a complete part of H, it follows that $K \circ U$ is open in H (by Lemma 2.15). Hence, $\phi(U)$ is open in H/K. This shows that ϕ is open.

(4) Let $x, y \in H$. Then, $\phi(x \circ y) = \{K \circ z : z \in x \circ y\} = K \circ x \odot K \circ y$. This shows that ϕ is a good homomorphism.

(5) We have $\phi(H) = H/K$. So, being the continuous image of a compact set, H/K is compact.

(6) Let τ' be any other topology on H/K with respect to which $\phi : H \to H/K$ is continuous. Now, for any open subset O of H/K there exists some open subset V of H such that $O = V/K$. Here $\phi^{-1}(O) = \phi^{-1}(V/K) = K \circ V$ is open in H(by Lemma 2.15). But by the definition of quotient topology, all such O's are open in quotient topology. This shows that the quotient topology $\overline{\tau}$ is finer than τ'. This completes the proof. □

Theorem 2.44. *Let K be a normal subhypergroup of a topological complete hypergroup $(H, \circ, \tau)$ and every open subset of H is a complete part. Then, $(H/K, \odot, \overline{\tau})$ is a topological hypergroup, where $K \circ x \odot K \circ y = \{K \circ z : z \in x \circ y\}$ and $K \circ x/K \circ y = \{K \circ z : z \in x/y\}$.*

Proof. Let us show that the hyperoperation $\odot$ and / are continuous on H/K. Suppose $K \circ x, K \circ y \in H/K$ and $\mathcal{A}$ be an open subset of H/K such that $K \circ x \odot K \circ y \subseteq \mathcal{A}$. Then, $x \circ y \subseteq \phi^{-1}(\mathcal{A})$. Since $\phi^{-1}(\mathcal{A})$ is open in H, by the continuity of the map $(x, y) \mapsto x \circ y$, there exist open subsets V and W containing x and y respectively, such that $V \circ W \subseteq \phi^{-1}(\mathcal{A})$. Now, $\phi(V)$ and $\phi(W)$ are open subsets of H/K containing $K \circ x$ and $K \circ y$ respectively, it follows that $\phi(V) \odot \phi(W) \subseteq \mathcal{A}$. Therefore, the hyperoperation $\odot$ is continuous on H/K.

Now, suppose $\mathcal{B}$ be an open subset of H/K and $K \circ x/K \circ y \subseteq \mathcal{B}$. Then, $x/y \subseteq \phi^{-1}(\mathcal{B})$. Since $\phi^{-1}(\mathcal{B})$ is open in H, by the continuity of the map $(x, y) \mapsto x/y$, there exist open subsets P and Q containing x and y respectively, such that $P/Q \subseteq \phi^{-1}(\mathcal{B})$. Now, $\phi(P)$ and $\phi(Q)$ are open in H/K containing $K \circ x$ and $K \circ y$, respectively, it follows that $\phi(P)/\phi(Q) \subseteq \mathcal{B}$. Therefore, the hyperoperation / is continuous on H/K and hence $(H/K, \odot, \overline{\tau})$ is a topological hypergroup. □

Theorem 2.45. *Let $(H, \circ, \tau)$ be a topological complete hypergroup such that every open subset of H is a complete part and K be a normal subhypergroup of it. Let $\phi : H \to H/K$ be the natural mapping. Then, the family $\{\phi(U\circ x) : U \in \mathcal{U}\}$ is a local base of the space H/K at the point $K \circ x \in H/K$, where $\mathcal{U}$ is a base for H at some identity e.*

Proof. Let $U \in \mathcal{U}$. Then, U is a complete part of H and so, $U \circ x$ is open in H (by Lemma 2.15). Now, for every $k \in K$, $k \circ (U \circ x)$ is open in H. So, $\phi^{-1}(\phi(U \circ x)) = K \circ (U \circ x) = \bigcup_{k\in K} k \circ (U \circ x)$ is an open subset of H. Therefore, by Lemma 2.43, $\phi(U \circ x)$ is open in H/K. Now, suppose V be an open neighborhood of $K \circ x$ in H/K. Let us take $\phi^{-1}(V) = W$, then W is an open subset of H. Since $K \circ x \subseteq V$, it follows that $x \in \phi^{-1}(K \circ x) \subseteq \phi^{-1}(V) = W$. So, there exists $U \in \mathcal{U}$ such that $U \circ x \subseteq W$ (by Lemma 2.18). Therefore, $K \circ x \in \phi(U \circ x) \subseteq \phi(W) = V$. This shows that $\{\phi(U \circ x) : U \in \mathcal{U}\}$ is a local base of the space H/K at the point $K \circ x$. □

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