



## Some New Inequalities of Simpson's Type for $s$ -convex Functions via Fractional Integrals

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**Abstract.** In this paper, we establish some new inequalities of Simpson's type based on  $s$ -convexity via fractional integrals. Our results generalize the results obtained by Sarikaya et al. [1].

### 1. Introduction and Preliminaries

It is well known that the following inequality, named Simpson's inequality, is one of the best known results in the literature.

$$\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right] \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4,$$

where  $f : [a, b] \rightarrow \mathfrak{R}$  be a four times continuously differentiable mapping on  $(a, b)$  and  $\|f^{(4)}\|_{\infty} = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty$ .

In [2], the class of functions which are  $s$ -convex in the second sense has been introduced by Breckner as the following.

**Definition 1.1.** Let  $s$  be a real number  $s \in (0, 1]$ . A function  $f : [0, \infty) \rightarrow \mathfrak{R}$ , is said to be  $s$ -convex (in the second sense), or  $f$  belongs to the class  $K_s^2$ , if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds for all  $x, y \in [0, \infty)$  and  $\lambda \in [0, 1]$ .

It can be easily seen that for  $s = 1$ ,  $s$ -convexity reduces to ordinary convexity of functions defined on  $[0, \infty)$ .

In [3], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard's inequality which holds for  $s$ -convex functions in the second sense. In [13], the authors proved some new integral inequalities of these classes of functions via  $(h - (\alpha, m))$ -logarithmically convexity.

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**Theorem 1.2.** Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is an  $s$ -convex function in the second sense, where  $s \in (0, 1)$ , and let  $a, b \in [0, \infty)$ ,  $a < b$ . If  $f \in L_1([a, b])$ , then the following inequalities hold:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1}$$

The following Lemma is proved by Sarikaya et al. (see [1]).

**Lemma 1.3.** Let  $f : I \rightarrow \mathfrak{R}$  be an absolutely continuous mapping on  $I^o$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^o$  with  $a < b$ . Then the following equality holds:

$$\begin{aligned} & \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{b-a}{2} \int_0^1 \left[ \left(\frac{t}{2} - \frac{1}{3}\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t}{2}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right] dt. \end{aligned} \tag{2}$$

Using Lemma 1.3, Sarikaya et al. in [1] established the following results which hold for  $s$ -convex functions in the second sense.

**Theorem 1.4.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^o$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^o$  with  $a < b$ . If  $|f'|$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) \frac{(s-4)6^{s+1} + 2 \times 5^{s+2} - 2 \times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)} \left[ |f'(a)| + |f'(b)| \right]. \end{aligned} \tag{3}$$

**Theorem 1.5.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^o$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^o$  with  $a < b$ . If  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{12} \left( \frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left( \frac{|f'(b)|^q + |f'(\frac{a+b}{2})|^q}{s+1} \right)^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + |f'(\frac{a+b}{2})|^q}{s+1} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{4}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Theorem 1.6.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^o$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^o$  with  $a < b$ . If  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{12} \left( \frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left( \frac{(2^{s+1}-1)|f'(b)|^q + |f'(a)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} + \left( \frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{5}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Theorem 1.7.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^o$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^o$  with  $a < b$ . If  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{2} \left( \frac{5}{36} \right)^{1-\frac{1}{q}} \left\{ \left( \frac{2 \times 5^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3 \times 6^{s+1}(s+1)(s+2)} |f'(b)|^q + \frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |f'(a)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |f'(b)|^q + \frac{2 \times 5^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3 \times 6^{s+1}(s+1)(s+2)} |f'(a)|^q \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{6}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [4-5].

**Definition 1.8.** Let  $f \in L_1([a, b])$ . The Riemann-Liouville integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$  is Euler gamma function. Here  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

In the case of  $\alpha = 1$ , the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities, see [6-12]. For more fractional integral applications, please see [14-21]

The aim of this paper is to establish some new inequalities for  $s$ -convex functions in the second sense via Riemann-Liouville fractional integral. Our results generalize the results obtained by Sarikaya [1] and provide new estimates on these types of inequalities for fractional integrals.

## 2. Main Results

In this section, we introduce some inequalities via fractional integrals. First, a new identity is presented as follows:

**Lemma 2.1.** Let  $f : I \rightarrow \mathfrak{R}$  be an absolutely continuous mapping on  $I^0$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^0$  with  $a < b$ . Then the following equality holds:

$$\begin{aligned} & \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \\ &= \frac{b-a}{2} \int_0^1 \left[ \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t^\alpha}{2}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right] dt. \end{aligned} \tag{7}$$

**Proof.** It suffices to note that

$$\begin{aligned} I &= \int_0^1 \left[ \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t^\alpha}{2}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right] dt \\ &= \int_0^1 \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt + \int_0^1 \left(\frac{1}{3} - \frac{t^\alpha}{2}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\ &= I_1 + I_2. \end{aligned} \tag{8}$$

Integrating by parts

$$\begin{aligned}
 I_1 &= \int_0^1 \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) f' \left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt \\
 &= \frac{2}{b-a} \int_0^1 \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) d\left(f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right) \\
 &= \frac{2}{b-a} \left\{ \left[ \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right]_0^1 - \int_0^1 f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) d\left(\frac{t^\alpha}{2} - \frac{1}{3}\right) \right\} \\
 &= \frac{2}{b-a} \left[ \frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right) - \int_0^1 f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) d\left(\frac{t^\alpha}{2} - \frac{1}{3}\right) \right] \\
 &= \frac{2}{b-a} \left[ \frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right) - \frac{\alpha}{2} \int_0^1 t^{\alpha-1} f\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt \right] \tag{9} \\
 &= \frac{2}{b-a} \left[ \frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right) - \frac{\alpha}{2} \int_{\frac{a+b}{2}}^b \left(\frac{x - \frac{a+b}{2}}{\frac{b-a}{2}}\right)^{\alpha-1} f(x) \frac{1}{\frac{b-a}{2}} dx \right] \\
 &= \frac{2}{b-a} \left[ \frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right) - \frac{\alpha}{2} \left(\frac{2}{b-a}\right)^\alpha \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right)^{\alpha-1} f(x) dx \right] \\
 &= \frac{2}{b-a} \left[ \frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2} \left(\frac{2}{b-a}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right)^{\alpha-1} f(x) dx \right] \\
 &= \frac{2}{b-a} \left[ \frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} J_{b-}^\alpha f\left(\frac{a+b}{2}\right) \right],
 \end{aligned}$$

and similarly we have,

$$\begin{aligned}
 I_2 &= \int_0^1 \left(\frac{1}{3} - \frac{t^\alpha}{2}\right) f' \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\
 &= \frac{2}{b-a} \left[ \frac{1}{6}f(a) + \frac{1}{3}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right]. \tag{10}
 \end{aligned}$$

From (8), (9) and (10), we have conclusion (7). This completes the proof.  $\square$

**Remark 2.2.** In Lemma 2.1, if  $\alpha = 1$ , then we obtain Lemma 1.3.

Using this lemma, we can obtain the following fractional integral inequalities which a new result of Simpson’s inequality for  $s$ -convex functions.

**Theorem 2.3.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$ , then the following inequality holds:

$$\begin{aligned}
 &\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\
 &\leq \frac{b-a}{2^{s+1}} \left[ |f'(a)| + |f'(b)| \right] \{ I_3(\alpha, s) + I_4(\alpha, s) \}, \tag{11}
 \end{aligned}$$

where

$$\begin{aligned}
 I_3(\alpha, s) &= \int_0^{\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}} \left(\frac{1}{3} - \frac{t^\alpha}{2}\right) [(1+t)^s + (1-t)^s] dt, \\
 I_4(\alpha, s) &= \int_{\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}}^1 \left(\frac{t^\alpha}{2} - \frac{1}{3}\right) [(1+t)^s + (1-t)^s] dt. \tag{12}
 \end{aligned}$$

**Proof.** From Lemma 2.1 and since  $|f'|$  is  $s$ -convex on  $[a, b]$ , we get

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{2} \int_0^1 \left[ \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right| + \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| \right] dt \\ & \leq \frac{b-a}{2} \int_0^1 \left[ \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f'\left(\frac{1+t}{2}b + \left(1 - \frac{1+t}{2}\right)a \right) \right| + \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left| f'\left(\frac{1+t}{2}a + \left(1 - \frac{1+t}{2}\right)b \right) \right| \right] dt \\ & = \frac{b-a}{2} \int_0^1 \left[ \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f'\left(\frac{1+t}{2}b + \left(1 - \frac{1+t}{2}\right)a \right) \right| \right] dt + \int_0^1 \left[ \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left| f'\left(\frac{1+t}{2}a + \left(1 - \frac{1+t}{2}\right)b \right) \right| \right] dt \\ & \leq \frac{b-a}{2} \int_0^1 \left[ \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left( \frac{1+t}{2} \right)^s |f'(b)| + \left( 1 - \frac{1+t}{2} \right)^s |f'(a)| \right] dt \\ & \quad + \int_0^1 \left[ \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left( \frac{1+t}{2} \right)^s |f'(a)| + \left( 1 - \frac{1+t}{2} \right)^s |f'(b)| \right] dt \\ & \leq \frac{b-a}{2^{s+1}} [|f'(a)| + |f'(b)|] \left\{ \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| [(1+t)^s + (1-t)^s] dt \right\} \\ & \leq \frac{b-a}{2^{s+1}} [|f'(a)| + |f'(b)|] \left\{ \int_0^{(\frac{2}{3})^{\frac{1}{\alpha}}} \left( \frac{1}{3} - \frac{t^\alpha}{2} \right) [(1+t)^s + (1-t)^s] dt + \int_{(\frac{2}{3})^{\frac{1}{\alpha}}}^1 \left( \frac{t^\alpha}{2} - \frac{1}{3} \right) [(1+t)^s + (1-t)^s] dt \right\} \\ & \leq \frac{b-a}{2^{s+1}} [|f'(a)| + |f'(b)|] [I_3(\alpha, s) + I_4(\alpha, s)], \end{aligned}$$

where  $I_3(\alpha, s)$  and  $I_4(\alpha, s)$  are defined in (12). This completes the proof.  $\square$

**Corollary 2.4.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{4} [|f'(a)| + |f'(b)|] [I_3(\alpha, 1) + I_4(\alpha, 1)]. \end{aligned} \tag{13}$$

**Proof.** Setting  $s = 1$  in (11), we get the required result.  $\square$

**Remark 2.5.** In Theorem 2.3, if  $\alpha = 1$ , then we obtain Theorem 1.4.

In the following theorem, we shall propose a new upper bound for the right-hand side of Simpsons inequality for  $s$ -convex mapping with fractional integral type.

**Theorem 2.6.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{(b-a)}{2} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left\{ \left( \frac{|f'(b)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{s+1} \right)^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{s+1} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{14}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** From Lemma 2.1 and Hölder’s inequality, we get

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &= \frac{b-a}{2} \int_0^1 \left[ \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right| + \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| \right] dt \\ &\leq \frac{b-a}{2} \left\{ \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , by using in (1), we get

$$\int_0^1 \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right|^q dt \leq \frac{|f'(b)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{s+1}$$

and

$$\int_0^1 \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right|^q dt \leq \frac{|f'(a)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{s+1}.$$

Hence

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &\leq \frac{(b-a)}{2} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left\{ \left( \frac{|f'(b)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{s+1} \right)^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{s+1} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

This completes the proof.  $\square$

**Corollary 2.7.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is convex on  $[a, b]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &\leq \frac{(b-a)}{2} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left\{ \left( \frac{|f'(b)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{2} \right)^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + |f'\left(\frac{a+b}{2}\right)|^q}{2} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{15}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** Setting  $s = 1$  in (14), we get the required result.  $\square$

**Remark 2.8.** In Theorem 2.6, if  $\alpha = 1$ , then we obtain Theorem 1.5.

Next, we shall give another versions of Simpson’s type inequality for  $s$ -convex functions with fractional integral type as follows:

**Theorem 2.9.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &\leq \frac{(b-a)}{2} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left\{ \left( \frac{(2^{s+1}-1)|f'(b)|^q + |f'(a)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} + \left( \frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{16}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** From Lemma 2.1 and Hölder’s inequality, we get

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &= \frac{b-a}{2} \int_0^1 \left[ \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right| + \left| \frac{1-t^\alpha}{2} - \frac{1}{3} \right| \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| \right] dt \\ &\leq \frac{b-a}{2} \left\{ \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_0^1 \left| \frac{1-t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{17}$$

Since  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , we know that for  $t \in [0, 1]$  and  $s \in (0, 1]$

$$\left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right|^q \leq \left(\frac{t+1}{2}\right)^s |f'(b)|^q + \left(\frac{1-t}{2}\right)^s |f'(a)|^q. \tag{18}$$

From (17) and (18), we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &\leq \frac{b-a}{2} \left\{ \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_0^1 \left| \frac{1-t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right|^q dt \right)^{\frac{1}{q}} \right\} \\ &\leq \frac{b-a}{2} \left\{ \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left[ \left(\frac{t+1}{2}\right)^s |f'(b)|^q + \left(\frac{1-t}{2}\right)^s |f'(a)|^q \right] dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_0^1 \left| \frac{1-t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left[ \left(\frac{t+1}{2}\right)^s |f'(a)|^q + \left(\frac{1-t}{2}\right)^s |f'(b)|^q \right] dt \right)^{\frac{1}{q}} \right\} \\ &= \frac{b-a}{2} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left\{ \left( \int_0^1 \left(\frac{t+1}{2}\right)^s |f'(b)|^q + \left(\frac{1-t}{2}\right)^s |f'(a)|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_0^1 \left(\frac{t+1}{2}\right)^s |f'(a)|^q + \left(\frac{1-t}{2}\right)^s |f'(b)|^q dt \right)^{\frac{1}{q}} \right\} \\ &\leq \frac{(b-a)}{2} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left\{ \left( \frac{(2^{s+1}-1)|f'(b)|^q + |f'(a)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} + \left( \frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

This completes the proof.  $\square$

**Corollary 2.10.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is convex on  $[a, b]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &\leq \frac{(b-a)}{2} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right|^p dt \right)^{\frac{1}{p}} \left\{ \left( \frac{3|f'(b)|^q + |f'(a)|^q}{4} \right)^{\frac{1}{q}} + \left( \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{19}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** Setting  $s = 1$  in (16), we get the required result.  $\square$

**Remark 2.11.** In Theorem 2.9, if  $\alpha = 1$ , then we obtain Theorem 1.6.

**Theorem 2.12.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{2} I_5(\alpha, s) \times \{ I_6(\alpha, s)^{\frac{1}{q}} + I_7(\alpha, s)^{\frac{1}{q}} \}, \end{aligned} \tag{20}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  and

$$\begin{aligned} I_5(\alpha, s) &= \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| dt \right)^{1-\frac{1}{q}} = \left( \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| dt \right)^{1-\frac{1}{q}}, \\ I_6(\alpha, s) &= \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left[ \left( \frac{1+t}{2} \right)^s |f'(b)|^q + \left( \frac{1-t}{2} \right)^s |f'(a)|^q \right] dt, \\ I_7(\alpha, s) &= \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left[ \left( \frac{1+t}{2} \right)^s |f'(a)|^q + \left( \frac{1-t}{2} \right)^s |f'(b)|^q \right] dt. \end{aligned} \tag{21}$$

**Proof.** Using Lemma 2.1, the power mean inequality and Hölder’s inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &= \frac{b-a}{2} \int_0^1 \left[ \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right| + \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| \right] dt \\ &\leq \frac{b-a}{2} \left\{ \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| |f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| |f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)|^q dt \right)^{\frac{1}{q}} \right\} \end{aligned} \tag{22}$$

Since  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , we know that for  $t \in [0, 1]$  and  $s \in (0, 1]$

$$\left| f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) \right|^q \leq \left(\frac{1+t}{2}\right)^s |f'(b)|^q + \left(\frac{1-t}{2}\right)^s |f'(a)|^q. \tag{23}$$

From (22) and (23), we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ &\leq \frac{b-a}{2} \left\{ \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| \frac{t^\alpha}{2} - \frac{1}{3} \right| \left[ \left(\frac{1+t}{2}\right)^s |f'(b)|^q + \left(\frac{1-t}{2}\right)^s |f'(a)|^q \right] dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| \frac{1}{3} - \frac{t^\alpha}{2} \right| \left[ \left(\frac{1+t}{2}\right)^s |f'(a)|^q + \left(\frac{1-t}{2}\right)^s |f'(b)|^q \right] dt \right)^{\frac{1}{q}} \right\} \\ &= \frac{b-a}{2} I_5(\alpha, s) \times \{ I_6(\alpha, s)^{\frac{1}{q}} + I_7(\alpha, s)^{\frac{1}{q}} \} \end{aligned}$$

where  $I_5(\alpha, s)$ ,  $I_6(\alpha, s)$  and  $I_7(\alpha, s)$  are defined in (21). This completes the proof.  $\square$

**Corollary 2.13.** Let  $f : I \subset [0, \infty) \rightarrow \mathfrak{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L_1([a, b])$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^q$  is convex on  $[a, b]$  and  $q > 1$ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{b-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a+}^\alpha f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{2} I_5(\alpha, 1) \times \{ I_6(\alpha, 1)^{\frac{1}{q}} + I_7(\alpha, 1)^{\frac{1}{q}} \}, \end{aligned} \tag{24}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** Setting  $s = 1$  in (20), we get the required result.  $\square$

**Remark 2.14.** In Theorem 2.12, if  $\alpha = 1$ , then we obtain Theorem 1.7.



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