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# Characterization of Linear Preservers of Generalized Majorization on $c_0$

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**Abstract.** In this work we investigate a natural preorder on $c_0$, the Banach space of all real sequences tend to zero with the supremum norm, which is said to be "convex majorization". Some interesting properties of all bounded linear operators $T : c_0 \to c_0$, preserving the convex majorization, are given and we characterize such operators.

## 1. Introduction and Preliminaries

For any two vectors $x, y \in \mathbb{R}^n$, we say $x$ is majorized by $y$, denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad (\text{for } k = 1, \dots, n-1)$$

and

$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}.$$

Here $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ is the decreasing order of components of a vector $x$. There are several equivalent conditions of vector majorization. Hardy, Littlewood, and Polya in [4] proved that $x = (x_1, \dots, x_n) \prec y = (y_1, \dots, y_n)$ is equivalent to

$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i),$$

for all continuous convex function $\phi : \mathbb{R} \to \mathbb{R}$. In fact, the previous characterization shows that if $x \prec y$, then the set of the components of $x$, lies in the convex hull spanned by the components of $y$, i.e.,

$$\mathrm{co}(x) \subseteq \mathrm{co}(y). \tag{1}$$

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The topic of linear preservers is of interest to a large group of matrix theorists. For some references on this subject we refer the reader to [1–3, 5–8]. On the basis of (1), Khalooei et al. [5, 6], introduced the concept of left matrix majorization and determined all linear operator preserving left matrix majorization on $\mathbb{R}^n$.

Throughout this work, $c_0$ is the Banach space of all convergent real sequences tend to zero with the supremum norm. An element $f \in c_0$ can be represented by $\sum_{i\in\mathbb{N}} f(i)e_i$, where $e_i : \mathbb{N} \to \mathbb{R}$ is defined by $e_i(j) = \delta_{ij}$, the Kronecker delta. Let $T : c_0 \to c_0$, be a bounded linear operator. Then an easy computation shows that, $T$ is represented by a matrix $(t_{ij})_{i,j\in\mathbb{N}}$ in the sense that

$$(Tf)(i) = \sum_{j\in\mathbb{N}} t_{ij} f(j), \quad \text{for } f \in c_0 \text{ and } i \in \mathbb{N},$$

where $t_{ij} = (Te_j)(i)$. To simplify notation, we can incorporate $T$ to its matrix form $(t_{ij})_{i,j\in\mathbb{N}}$.

In the following of this paper, by using (1), the notion of the left matrix majorization is extended to $c_0$. Then all of the bounded linear operators, preserving such a majorization, together with some important properties of them, are obtained and determined. We also investigate the linear operators $T : c_0 \to c_0$, which satisfy $\text{co}(Tf) = \text{co}(f)$, for all $f \in c_0$. Then we prove that any row sum of them belongs to $[0, 1]$.

## 2. Main Results

First, we define a preorder on $c_0$, as the following.

**Definition 2.1.** *Let $f, g \in c_0$. We say that $f$ is convex majorized by $g$, and denoted by $f \prec_c g$, if $\text{co}(f) \subseteq \text{co}(g)$. Also, $f$ is said to be convex equivalent to $g$, denoted by $f \sim_c g$, whenever $f \prec_c g \prec_c f$, i.e., $\text{co}(f) = \text{co}(g)$, where $\text{co}(f)$ means convex hull spanned by the components of $f$.*

**Remark 2.2.** *For $f, g \in c_0$, some consequences of the previous definition are as follows.*

- *If $f \prec_c g$, then $\|f\| \leq \|g\|$.*
- *$f \prec_c g$, iff $\lambda f \prec_c \lambda g$, for all $\lambda \in \mathbb{R}$, iff $f \prec_c ng$, for all $n \in \mathbb{N}$.*
- *If $nf \prec_c g$, for each $n \in \mathbb{N}$, then $f = 0$.*

**Definition 2.3.** *A bounded linear operator $T : c_0 \to c_0$ is said to be order preserving, if $T$ preserves $\prec_c$, that is, for $f, g \in c_0$, the relation $f \prec_c g$ implies $Tf \prec_c Tg$. The set of all such operators is denoted by $\mathcal{P}_{cm}$.*

One of the concepts, appears in the study of order preserver operators, is the generalization of the concept of convex combination, which appears in [2].

**Definition 2.4.** *Let $(X, \|.\|)$ be a normed linear space and $A \subseteq X$. The countable convex hull of $A$, denoted by $\text{cco}(A)$, is defined to be the set*

$$\left\{ \sum_{i=1}^{\infty} \lambda_i x_i;\ x_i \in A, \lambda_i \geq 0, \sum_{i=1}^{\infty} \lambda_i = 1, \sum_{i=1}^{\infty} \lambda_i x_i \textit{ converges} \right\}.$$

The following assertions come from [2].

- $\text{co}(A) \subseteq \text{cco}(A) \subseteq \overline{\text{co}(A)}$, so $\text{cco}(A)$ is a convex set.
- If $X$ is a Banach space and $A \subseteq X$ is bounded, then in the definition of $\text{cco}(A)$, $\sum_{i=1}^{\infty} \lambda_i x_i$ is always a convergent series.
- If $A \subseteq \mathbb{R}$, then $\text{cco}(A) = \text{co}(A)$.

It can be proved that, for $f \in c_0$ if $0 \in \text{co}(f)$, then $\text{co}(f) = [a, b]$, for some $a, b \in \mathbb{R}$ with $a \leq 0 \leq b$, and if $0 \notin \text{co}(f)$, then $\text{co}(f)$ is equal to either an interval $[a, 0)$, for some $a < 0$, or $(0, b]$, for some $b > 0$.

In this section, we characterize all linear operators $T : c_0 \to c_0$ which preserve $\prec_c$ .

**Some elementary properties of $\mathcal{P}_{cm}$**

- $0, \mathrm{id} \in \mathcal{P}_{cm}$.
- If $T_1, T_2 \in \mathcal{P}_{cm}$, then $T_1 \circ T_2 \in \mathcal{P}_{cm}$. In particular, $\lambda T \in \mathcal{P}_{cm}$ for $\lambda \in \mathbb{R}$ and $T \in \mathcal{P}_{cm}$.
- Any constant coefficient of a permutation lies in $\mathcal{P}_{cm}$.

**Example 2.5.** *Let $a, b \in \mathbb{R}$ and $S : c_0 \to c_0$ be defined by*

$$Sf = (af_1, bf_1, af_2, bf_2, \dots),$$

*for $f = (f_1, f_2, \dots) \in c_0$. It is obvious that $S \in \mathcal{P}_{cm}$.*
*In general case, let $(n_k)$ be a sequence of natural numbers. Then the bounded linear operator $T : c_0 \to c_0$, defined by*

$$Tf = (\underbrace{af_1, \dots, af_1}_{n_1}, \underbrace{bf_1, \dots, bf_1}_{n_2}, \underbrace{af_2, \dots, af_2}_{n_3}, \underbrace{bf_2, \dots, bf_2}_{n_4}, \underbrace{af_3, \dots, af_3}_{n_5}, \underbrace{bf_3, \dots, bf_3}_{n_6}, \dots),$$

*for $f = (f_1, f_2, \dots) \in c_0$, belongs to $\mathcal{P}_{cm}$.*

**Lemma 2.6.** *Let $f \in c_0$, $\lambda_i \geq 0$ and $0 < \sum_{i=1}^{\infty} \lambda_i \leq 1$. Then $\sum_{i=1}^{\infty} \lambda_i f(i) \in \mathrm{co}(f)$.*

*Proof.* Put $\lambda = \sum_{i=1}^{\infty} \lambda_i$. We consider two cases. If $0 \in \mathrm{co}(f)$, then

$$\sum_{i=1}^{\infty} \lambda_i f(i) = \sum_{i=1}^{\infty} \lambda_i f(i) + (1 - \lambda)0 \in \mathrm{cco}(f) = \mathrm{co}(f).$$

But if $0 \notin \mathrm{co}(f)$, then $\mathrm{co}(f)$ has one of the forms $[a, 0)$ or $(0, b]$, where $a < 0 < b$. If $\mathrm{co}(f) = (0, b]$, then for all $i \in \mathbb{N}$, we have $0 < f(i) \leq b$. This implies $0 < \sum_{i=1}^{\infty} \lambda_i f(i) \leq \sum_{i=1}^{\infty} \lambda_i b \leq b$, i.e., $\sum_{i=1}^{\infty} \lambda_i f(i) \in \mathrm{co}(f)$. Similarly, the result follows for the case $\mathrm{co}(f) = [a, 0)$. □

In Lemma 2.6, if all the $\lambda_i$ are equal to zero, then $\sum_{i=1}^{\infty} \lambda_i f(i) = 0$, but it may be $0 \notin \mathrm{co}(f)$. For example suppose that $f = (1, \frac{1}{2}, \frac{1}{3}, \dots)$.

The previous lemma gives a different example of order preserver operators.

**Example 2.7.** *Let $(\lambda_i)$ be a sequence of nonnegative real numbers such that $\sum_{i=1}^{\infty} \lambda_i \leq 1$. Then the bounded linear operator $T : c_0 \to c_0$ defined by*

$$Tx = \left( \sum_{i=1}^{\infty} \lambda_i x_i, x_1, x_2, x_3, \dots \right), \qquad \text{for } x = (x_i) \in c_0$$

*belongs to $\mathcal{P}_{cm}$. To see this, suppose that $f \prec_c g$, for some $f, g \in c_0$. Now in the case $\sum_{i=1}^{\infty} \lambda_i = 0$, that is, for all $i \in \mathbb{N}$, $\lambda_i = 0$, then*

$$\mathrm{co}(Tf) = \mathrm{co}\{0, f(1), f(2), f(3), \dots\},$$

*which leads to*

$$\mathrm{co}(Tf) = \mathrm{co}\{0, f(1), f(2), \dots\} \subseteq \mathrm{co}\{0, g(1), g(2), \dots\} = \mathrm{co}(Tg).$$

*But whenever* $\sum_{i=1}^{\infty} \lambda_i > 0$, *then by Lemma 2.6,* $\sum_{i=1}^{\infty} \lambda_i f(i) \in \text{co}(f)$, *and so*

$$\text{co}(Tf) = \text{co}\left\{\sum_{i=1}^{\infty} \lambda_i f(i), f(1), f(2), f(3), \dots\right\} = \text{co}(f).$$

*This implies* $\text{co}(Tf) = \text{co}(f) \subseteq \text{co}(g) = \text{co}(Tg)$.

Now, in the next theorem, we obtain an important property of order preserving linear operators on $c_0$, that is, their rows belong to $\ell^1$.

**Theorem 2.8.** *For $T \in \mathcal{P}_{cm}$, all rows of $T$ lie in $\ell^1$. Moreover for any fixed $i \in \mathbb{N}$, we have* $\sum_{j\in\mathbb{N}} |Te_j(i)| \leq \|T\|$.

*Proof.* Let $i \in \mathbb{N}$ be fixed. For any $j, n \in \mathbb{N}$, we set $\delta_j = \text{sgn}(Te_j)(i)$ and $x_n = \sum_{j=1}^{n} \delta_j e_j \in c_0$. Then $Tx_n = \sum_{j=1}^{n} \delta_j Te_j$, which implies

$$(Tx_n)(i) = \sum_{j=1}^{n} \delta_j (Te_j)(i) = \sum_{j=1}^{n} |(Te_j)(i)|.$$

Since $\|x_n\| \leq 1$, we have $\sum_{j=1}^{n} |(Te_j)(i)| = (Tx_n)(i) = |(Tx_n)(i)| \leq \|T\|$. Letting $n \to \infty$, this completes the proof. □

**Corollary 2.9.** *Let the bounded linear operator $T : c_0 \to c_0$ be in $\mathcal{P}_{cm}$. Then for $j_1, j_2 \in \mathbb{N}$, where $j_1 \neq j_2$, we have $\|Te_{j_1} - Te_{j_2}\| = \|T\|$, independent of chosen $j_1, j_2$.*

*Proof.* Let $x \in c_0$, such that $\|x\| \leq 1$. Then $x \prec_c e_1 - e_2$, and since $T \in \mathcal{P}_{cm}$, we have $Tx \prec_c Te_1 - Te_2$. Remark 2.2 implies that $\|Tx\| \leq \|Te_1 - Te_2\|$, which follows that

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| \leq \|Te_1 - Te_2\|.$$

On the other hand, $\|Te_1 - Te_2\| = \|T(e_1 - e_2)\| \leq \|T\|$. This completes the proof. □

**Remark 2.10.** *Note that for $T \in \mathcal{P}_{cm}$ and $j_1, j_2 \in \mathbb{N}$, as $\text{co}(Te_{j_1}) = \text{co}(Te_{j_2})$, the following equalities hold*

$$\inf_{i\in\mathbb{N}}\{Te_{j_1}(i)\} = \inf_{i\in\mathbb{N}}\{Te_{j_2}(i)\}, \ \sup_{i\in\mathbb{N}}\{Te_{j_1}(i)\} = \sup_{i\in\mathbb{N}}\{Te_{j_2}(i)\}.$$

*Hence both the values of* $\inf_{i\in\mathbb{N}}\{Te_j(i)\}$ *and* $\sup_{i\in\mathbb{N}}\{Te_j(i)\}$ *are independent of the choice of $j \in \mathbb{N}$. In what follows, for brevity we denote them by $a$ and $b$, respectively. That is, for $T \in \mathcal{P}_{cm}$, there is a bounded real interval $I$, such that*

$$\text{co}(Te_j) = I,$$

*for all $j \in \mathbb{N}$. Thus $a = \inf I$, and $b = \sup I$, for each $T \in \mathcal{P}_{cm}$.*

**Lemma 2.11.** *Assume that $T \in \mathcal{P}_{cm}$ and $i \in \mathbb{N}$. Then*

$$a \leq \sum_{j\in I^-} Te_j(i) \leq 0 \leq \sum_{j\in I^+} Te_j(i) \leq b, \tag{2}$$

*where*

$$I^+ = \{j \in \mathbb{N};\ Te_j(i) > 0\},\ I^- = \{j \in \mathbb{N};\ Te_j(i) < 0\}. \tag{3}$$

*Proof.* Let $i \in \mathbb{N}$, and $F$ be a nonempty finite subset of $I^+$. As $\mathrm{co}(\sum_{j\in F} Te_j) = \mathrm{co}(Te_{j_0})$, for $j_0 \in \mathbb{N}$, we have

$$0 \leq \sum_{j\in F} Te_j(i) \in \mathrm{Im}(\sum_{j\in F} Te_j) \subseteq \mathrm{co}(\sum_{j\in F} Te_j).$$

Thus

$$0 \leq \sum_{j\in F} Te_j(i) \leq \sup_{i\in\mathbb{N}} Te_{j_0}(i) = b.$$

Since the last inequality holds for all finite subset $F \subseteq I^+$, we conclude that

$$0 \leq \sum_{j\in I^+} Te_j(i) \leq b.$$

The rest of the proof runs as before. □

In what follows, we assume that $I^+$ and $I^-$, is defined as in (3).

**Corollary 2.12.** *Let $T \in \mathcal{P}_{cm}$. Then each row sums of $T$, lies in $[a, b]$.*

*Proof.* By adding both inequalities in (2), the assertion follows. □

**Theorem 2.13.** *Let $T \in \mathcal{P}_{cm}$. Then $||T|| = ||Te_j||$, for all $j \in \mathbb{N}$.*

*Proof.* It follows from Corollary 2.9, that for distinct $j, j' \in \mathbb{N}$, we have

$$||T|| = ||Te_j - Te_{j'}||.$$

Also, as $e_j \sim_c e_{j'}$, we have $Te_j \sim_c Te_{j'}$, which follows that $||Te_j|| = ||Te_{j'}||$, (Remark 2.2). Assume that $\alpha = ||Te_j||$, for $j \in \mathbb{N}$, and $\varepsilon > 0$. As $Te_j \in c_0$, we have $\lim_{i\to\infty}(Te_j)(i) = 0$, so there is $k \in \mathbb{N}$, such that for $i > k$, we have

$$|(Te_j)(i)| < \varepsilon. \tag{4}$$

On the other hand, Theorem 2.8 implies that the rows of $T$ belong to $\ell^1$, and so all the following sequences

$$\Big((Te_j)(1)\Big)_{j\in\mathbb{N}}, \ldots, \Big((Te_j)(k)\Big)_{j\in\mathbb{N}},$$

converge to zero, and so there is $k' \in \mathbb{N}$, such that for $j' > k'$, we have

$$|(Te_{j'})(1)| < \varepsilon, \ldots, |(Te_{j'})(k)| < \varepsilon. \tag{5}$$

The relations (4) and (5) imply that, for $j' > k'$, we have

$$\begin{aligned} |(Te_j)(i) - (Te_{j'})(i)| &\leq |(Te_j)(i)| + |(Te_{j'})(i)| \\ &\leq \begin{cases} |Te_j)(i)| + \varepsilon & \text{if } 1 \leq i \leq k, \\ \varepsilon + |(Te_{j'})(i)| & \text{if } i > k, \end{cases} \\ &\leq \alpha + \varepsilon, \end{aligned}$$

for any $i \in \mathbb{N}$. Therefore for $\varepsilon > 0$, it follows that

$$||T|| = ||Te_j - Te_{j'}|| = \sup_{i\in\mathbb{N}} |(Te_j)(i) - (Te_{j'})(i)| \leq \alpha + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we have

$$||T|| = ||Te_j - Te_{j'}|| \leq \alpha.$$

Also, $\alpha = ||Te_j|| \leq ||T||$. Therefore $||T|| = \alpha = ||Te_j||$, for all $j \in \mathbb{N}$. □

**Lemma 2.14.** *If $T \in \mathcal{P}_{cm}$, and $j_0 \in \mathbb{N}$, then $0 \in \mathrm{Im}(Te_{j_0})$.*

*Proof.* Assume that $j_0, j_1 \in \mathbb{N}$ with $j_0 \neq j_1$. If $a = b = 0$, then $Te_{j_0} = 0$ and we are done. Otherwise, if $a < 0$ or $b > 0$, then $\|Te_{j_1}\| = \max\{b, -a\} > 0$. Now if $\|Te_{j_1}\| = b > 0$, then there is $i_0 \in \mathbb{N}$ such that $Te_{j_1}(i_0) = b$. Applying Theorems 2.8 and 2.13, we can assert that $b = |Te_{j_1}(i_0)| \leq \sum_{j\in\mathbb{N}} |Te_j(i_0)| \leq \|T\| = \|Te_{j_1}\| = b$. The latter relation yields $|Te_j(i_0)| = 0$ for all $j \neq j_1$. Thus $Te_{j_0}(i_0) = 0$, which follows $0 \in \mathrm{Im}(Te_{j_0})$. In case $\|Te_{j_1}\| = -a > 0$, the result follows by a similar argument. □

**Lemma 2.15.** *Let $T \in \mathcal{P}_{cm}$ and $j \in \mathbb{N}$. Then $a, b \in \mathrm{Im}(Te_j)$ and $\mathrm{co}(Te_j) = [a, b]$.*

*Proof.* Remark 2.10 yields that $\mathrm{co}(Te_j) = I$, where $I$ is a bounded real interval and $a = \inf I$ and $b = \sup I$. The zero at most can be a limit point of $\mathrm{Im}(Te_j)$ and $a \leq 0 \leq b$. If $a < 0$, then $a$ will not be a limit point of $\mathrm{Im}(Te_j)$. Since $a = \inf_{i\in\mathbb{N}}\{Te_j(i)\}$, we see that $a \in \mathrm{Im}(Te_j)$. But if $a = 0$, then Lemma 2.14, yields $a = 0 \in \mathrm{Im}(Te_j)$. For $b$, we can use a similar argument. □

**Lemma 2.16.** *If $T \in \mathcal{P}_{cm}$ and $a < 0 < b$, then for any $j_1, j_2 \in \mathbb{N}$, where $j_1 \neq j_2$, we have*

$$\max_{i\in\mathbb{N}} \left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = 1.$$

*Proof.* If $j_1, j_2 \in \mathbb{N}$ $(j_1 \neq j_2)$, then obviously

$$\max_{i\in\mathbb{N}} \left\{\frac{1}{a}Te_{j_1}(i)\right\} = \max_{i\in\mathbb{N}} \left\{\frac{1}{b}Te_{j_2}(i)\right\} = 1. \tag{6}$$

Now, assume that $0 < \varepsilon < 1$ is arbitrary and we choose finite subset $F \subseteq \mathbb{N}$ such that

$$\forall i \in \mathbb{N} \setminus F, \qquad \left|\frac{1}{a}Te_{j_1}(i)\right| < \varepsilon, \tag{7}$$

such an $F$ exists, because $Te_{j_1} \in c_0$.

Theorem 2.8 implies that for all $i \in F$, $\sum\limits_{j\in\mathbb{N}} |Te_j(i)| < \infty$. So there is a finite set $G \subseteq \mathbb{N}$ such that

$$\forall i \in F, \quad \forall j \in \mathbb{N} \setminus G, \qquad \left|\frac{1}{b}Te_j(i)\right| < \varepsilon. \tag{8}$$

Let $j^* \in \mathbb{N} \setminus G$ and $j^* \neq j_1$. Then for all $i \in \mathbb{N}$; if $i \in F$, then

$$\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j^*}(i) \leq 1 + \varepsilon, \tag{9}$$

and if $i \in \mathbb{N} \setminus F$, then

$$\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j^*}(i) \leq \varepsilon + 1, \tag{10}$$

Since $\frac{1}{a}Te_{j_1} + \frac{1}{b}Te_{j_2} \sim_c \frac{1}{a}Te_{j_1} + \frac{1}{b}Te_{j^*}$, the relations (9) and (10) follow that for all $\varepsilon > 0$ we have

$$\sup_{i\in\mathbb{N}} \left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = \sup_{i\in\mathbb{N}} \left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j^*}(i)\right\} \leq \varepsilon + 1,$$

since $\varepsilon > 0$ is arbitrary, we have

$$\sup_{i\in\mathbb{N}} \left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = \sup_{i\in\mathbb{N}} \left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j^*}(i)\right\} \leq 1. \tag{11}$$

On the other hand, (6) shows that there is $i^* \in \mathbb{N}$ such that $\frac{1}{a}Te_{j_1}(i^*) = 1$. But $\varepsilon < 1$, thus (7) implies $i^* \in F$ and by (8) we deduce that

$$\frac{1}{a}Te_{j_1}(i^*) + \frac{1}{b}Te_{j^*}(i^*) \geq 1 - \varepsilon,$$

thus for all $\varepsilon > 0$,

$$\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = \sup_{i\in\mathbb{N}}\left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j^*}(i)\right\} \geq 1 - \varepsilon,$$

and hence

$$\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = \sup_{i\in\mathbb{N}}\left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j^*}(i)\right\} \geq 1,$$

the last inequality and (11) follow that $\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = 1$. But one is not a limit point of $\mathrm{Im}\left\{\frac{1}{a}Te_{j_1} + \frac{1}{b}Te_{j_2}\right\}$, so $1 \in \mathrm{Im}\left\{\frac{1}{a}Te_{j_1} + \frac{1}{b}Te_{j_2}\right\}$, that is $\max_{i\in\mathbb{N}}\left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = 1$. □

**Theorem 2.17.** *If $T \in \mathcal{P}_{cm}$, and $a < 0 < b$, then for any $i \in \mathbb{N}$, we have*

$$\frac{1}{a}\sum_{j\in I^-} Te_j(i) + \frac{1}{b}\sum_{j\in I^+} Te_j(i) \leq 1.$$

*Proof.* Let $j_1, j_2 \in \mathbb{N}$ ($j_1 \neq j_2$). If $I^+ = \emptyset$, then by Lemma 2.11, we have

$$a \leq \sum_{j\in I^-} Te_j(i) \leq 0.$$

By multiplying $\frac{1}{a}$ to the latter inequalities, the assertion follows. Similar arguments apply to the case $I^- = \emptyset$.

Now, suppose that $I^+$ and $I^-$ are both nonempty. Then Theorem 2.8 yields

$$\sum_{j\in I^+} |Te_j(i)| + \sum_{j\in I^-} |Te_j(i)| = \sum_{j\in\mathbb{N}} |Te_j(i)| < \infty,$$

that implies $I^+$ and $I^-$ are countable. For $F \subseteq I^-$ and $G \subseteq I^+$, where $F$ and $G$ are nonempty finite sets, since $\frac{1}{a}\sum_{j\in F} e_j + \frac{1}{b}\sum_{j\in G} e_j \sim_c \frac{1}{a}e_{j_1} + \frac{1}{b}e_{j_2}$, it follows that

$$\frac{1}{a}\sum_{j\in F} Te_j + \frac{1}{b}\sum_{j\in G} Te_j \sim_c \frac{1}{a}Te_{j_1} + \frac{1}{b}Te_{j_2}.$$

According to Lemma 2.16 and the latter relation, we have

$$\frac{1}{a}\sum_{j\in F} Te_j(i) + \frac{1}{b}\sum_{j\in G} Te_j(i) \leq \max_{i\in\mathbb{N}}\left\{\frac{1}{a}Te_{j_1}(i) + \frac{1}{b}Te_{j_2}(i)\right\} = 1.$$

Since the latter inequality holds for any finite subsets $F \subseteq I^-$ and $G \subseteq I^+$, we have $\frac{1}{a}\sum_{j\in I^-} Te_j(i) + \frac{1}{b}\sum_{j\in I^+} Te_j(i) \leq 1$. □

**Corollary 2.18.** *Let $T \in \mathcal{P}_{cm}$ and consider the matrix form of $T$. Then the following conditions hold.*

(i) *If $a < 0$, then in any row which appears $a$, the other entries are equal to zero.*

(ii) *If $b > 0$, then in any row which appears $b$, the other entries are equal to zero.*

*Proof.* (i) In the matrix form of $T$, suppose that $a < 0$ and it appears in the row $i \in I$. If $b = 0$, then $I^+ = \emptyset$ and $I^- \neq \emptyset$. According to Lemma 2.11, $a \leq \sum_{j \in I^-} Te_j(i)$. Since $Te_j(i) \leq 0$, for each $j \in I^-$, and one of them equals $a$,we have $\sum_{j \in I^-} Te_j(i) = a$. Now, let $j_0 \in I^-$ be such that $Te_{j_0}(i) = a$. Thus

$$a = \sum_{\substack{j \in I^- \\ j \neq j_0}} Te_j(i) + a,$$

and so $Te_j(i) = 0$, for all $j \in I^-$ with $j \neq j_0$.
Now if $b > 0$, then by Theorem 2.17, we have $\sum\limits_{j \in I^-} \frac{Te_j(i)}{a} + \sum\limits_{j \in I^+} \frac{Te_j(i)}{b} \leq 1$. Since the elements of both series are nonnegative and there is $j_0 \in I^-$ such that $Te_{j_0}(i) = a$, that is, $\frac{Te_{j_0}(i)}{a} = 1$, we conclude that for all $j \in \mathbb{N}$, where $j \neq j_0$, $Te_j(i) = 0$.

The assertion (ii) follows by a similar argument. □

**Theorem 2.19.** *(Characterization theorem) Let $T : c_0 \to c_0$ be a linear operator. Then $T \in \mathcal{P}_{cm}$ if and only if*

(i) *For any $j \in \mathbb{N}$, the value of $\min\limits_{i \in \mathbb{N}} Te_j(i)$ exists and independent of $j$ is equal to $a$.*

(ii) *For any $j \in \mathbb{N}$, the value of $\max\limits_{i \in \mathbb{N}} Te_j(i)$ exists and independent of $j$ is equal to $b$.*

(iii) *If $a < 0 < b$, we have $\frac{1}{a} \sum\limits_{j \in I^-} Te_j(i) + \frac{1}{b} \sum\limits_{j \in I^+} Te_j(i) \leq 1$; if $a < 0 = b$, then we have $\sum\limits_{j \in \mathbb{N}} Te_j(i) \geq a$, and if $a = 0 < b$, then it implies $\sum\limits_{j \in \mathbb{N}} Te_j(i) \leq b$,*

*where $(Te_j(i))_{j \in \mathbb{N}}$ is an arbitrary row of the matrix representation of $T$.*

*Proof.* If $T \in \mathcal{P}_{cm}$, obviously the conditions (i)-(iii) are satisfied. So suppose that the conditions (i)-(iii) are satisfied and $a < 0 < b$. Then (i), (ii) follow that in any column, the values $a$ and $b$ are appeared. So for $j \in I$, there are $i_1, i_2 \in I$, such that $Te_j(i_1) = a$, $Te_j(i_2) = b$, and according to Corollary 2.18, (iii) implies that all of the other entries of the rows $i_1, i_2$ are zero. That is, for all $s \in I$, where $s \neq j$, we have $Te_s(i_1) = 0$, $Te_s(i_2) = 0$. Thus for, $f \in c_0$ we have

$$Tf(i_1) = \sum_{s \in \mathbb{N}} Te_s(i_1) f(s) = \sum_{\substack{s \in \mathbb{N} \\ s \neq j}} Te_s(i_1) f(s) + Te_j(i_1) f(j) = af(j).$$

Similarly, $Tf(i_2) = bf(j)$. Thus for $j \in \mathbb{N}$, we have $af(j), bf(j) \in \mathrm{Im}(Tf)$, which implies $\mathrm{co}\{af,\ bf\} \subseteq \mathrm{co}(Tf)$. For $i \in \mathbb{N}$, we have

$$\begin{aligned}(Tf)(i) &= \sum_{j \in \mathbb{N}} Te_j(i) f(j) = \sum_{j \in I^-} Te_j(i) f(j) + \sum_{j \in I^+} Te_j(i) f(j) \\ &= \sum_{j \in I^-} \frac{Te_j(i)}{a} af(j) + \sum_{j \in I^+} \frac{Te_j(i)}{b} bf(j) \\ &\in \mathrm{cco}\{af,\ bf\} = \mathrm{co}\{af,\ bf\}.\end{aligned}$$

Hence $i \in \mathbb{N}$ deduce that $(Tf)(i) \in \mathrm{co}\{af,\ bf\}$ and $\mathrm{co}(Tf) \subseteq \mathrm{co}\{af,\ bf\}$. We thus prove that (i)-(iii) imply that for all $f \in c_0$, $\mathrm{co}(Tf) = \mathrm{co}\{af,\ bf\}$. Now let $f, g \in c_0$ and $f \prec_c g$. Thus

$$\mathrm{co}(Tf) = \mathrm{co}\{af,\ bf\} \subseteq \mathrm{co}\{ag,\ bg\} = \mathrm{co}(Tg),$$

that is $Tf \prec_c Tg$. If $a < 0 = b$, then we need only consider the following two cases:

(i) The operator $T$ has a zero row, and so $\text{co}(Tf) = \text{co}\{af,\ 0\}$.

(ii) The operator $T$ has no zero row, and so $\text{co}(Tf) = \text{co}(af)$.

But (i),(ii) follow that $T \in \mathcal{P}_{cm}$. By a similar argument, the case $a = 0 < b$ implies the assertion. □

Now we investigate the operators $T : c_0 \to c_0$ which for all $f \in c_0$ satisfy the condition $\text{co}(Tf) = \text{co}(f)$. Let $\mathcal{P}_{ecm}$ be the set of such operators.

**Some Properties of $\mathcal{P}_{ecm}$**

- $\mathcal{P}_{ecm} \subseteq \mathcal{P}_{cm}$.
- Any permutation lies in $\mathcal{P}_{ecm}$.
- If $T_1, T_2 \in \mathcal{P}_{ecm}$, then $T_1 \circ T_2 \in \mathcal{P}_{ecm}$.
- If $T \in \mathcal{P}_{ecm}$, then for any constant $\lambda \neq 1$, $\lambda T \notin \mathcal{P}_{ecm}$.
  **Proof**. Let $T \in \mathcal{P}_{ecm}$ and $\lambda \in \mathbb{R}$ such that $\lambda T \in \mathcal{P}_{ecm}$. Since

  $$\lambda[0,1] = \{\lambda x\ ;\ x \in [0,1]\} = \lambda\text{co}(e_i) = \lambda\text{co}(Te_i) = \text{co}(\lambda Te_i) = \text{co}(e_i) = [0,1],$$

  we have $\lambda = 1$.□
- If $T \in \mathcal{P}_{ecm}$, then $T$ is a positive operator (i.e. $Tf \geq 0$, for each $f \geq 0$).
  **Proof**. For any $i, j \in \mathbb{N}$, we have $Te_j(i) \in \text{Im}(Te_j) \subseteq \text{co}(Te_j) = \text{co}(e_j) = [0,1]$. Thus $0 \leq Te_j(i) \leq 1$. Now suppose that $f \in c_0$ and $f \geq 0$. As for all $i, j \in \mathbb{N}$, $0 \leq Te_j(i)$ and $f(j) \geq 0$, we have $0 \leq (Tf)(i) = \sum_{j \in \mathbb{N}} Te_j(i) f(j)$. Since $i \in \mathbb{N}$ is arbitrary, it follows that $Tf \geq 0$.□

**Theorem 2.20.** *If $T \in \mathcal{P}_{ecm}$, then*

(i) *for all $j \in \mathbb{N}$, $\max_{i \in \mathbb{N}}\{Te_j(i)\} = 1$, $\min_{i \in \mathbb{N}}\{Te_j(i)\} = 0$.*

(ii) *if $(Te_j(i))_{j \in \mathbb{N}}$ is the ith row of the matrix form of T, then $\sum_{j \in \mathbb{N}} Te_j(i) \leq 1$.*

*Proof.* The definition of $\mathcal{P}_{ecm}$ follows (i) and since $T \in \mathcal{P}_{ecm} \subseteq \mathcal{P}_{cm}$, Theorem 2.19 and (i) imply (ii). □

In the following example, we show that the conditions (i) and (ii) in Theorem 2.20 do not follow $T \in \mathcal{P}_{ecm}$.

**Example 2.21.** *Let $T : c_0 \to c_0$ be a bounded linear operator defined by*

$$Tx = (0, x_1, x_2, x_3, \ldots), \qquad \text{for } x = (x_i) \in c_0.$$

*Then for $f = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots) \in c_0$, we have $Tf = (0, 1, \frac{1}{2}, \frac{1}{4}, \ldots)$. So, $\text{co}(Tf) = [0,1] \neq \text{co}(f) = (0,1]$, which leads to $T \notin \mathcal{P}_{ecm}$. However, by Theorem 2.19, $T \in \mathcal{P}_{cm}$.*

**Theorem 2.22.** *If $T \in \mathcal{P}_{ecm}$, then the matrix form of T has no zero row.*

*Proof.* On the contrary, suppose that the matrix form of the operator $T$ has a zero row. Thus, for all $f \in c_0$, $0 \in \text{Im}(Tf) \subseteq \text{co}(Tf)$. On the other hand, there is $f \in c_0$, such that $f > 0$, and $0 \notin \text{co}(f)$. Hence $\text{co}(Tf) \neq \text{co}(f)$ which contradicts our assumption. □

**Remark 2.23.** *Example 2.21 shows that although $\mathcal{P}_{ecm} \subseteq \mathcal{P}_{cm}$, but $\mathcal{P}_{cm} \nsubseteq \mathcal{P}_{ecm}$. Also, Theorem 2.20 implies that any row sum of the elements of $\mathcal{P}_{ecm}$ belongs to $[0,1]$.*

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