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Semi-invariant Submanifolds of a Normal Almost Paracontact Manifold

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Abstract. In this paper, we introduce the notion of semi-invariant submanifolds of a normal almost paracontact manifold. We study their fundamental properties and the particular cases. The necessary and sufficient conditions are given for a submanifold to be invariant or anti-invariant. Also, we give some results for semi-invariant submanifolds of a normal almost paracontact manifold with constant c and we construct an example.

1. Introduction

On the analogy of almost contact Riemannian manifolds, in [20], Sato introduced the notion of almost paracontact Riemannian manifolds. An almost contact manifolds is always odd-dimensional whereas an almost paracontact manifolds could be even or odd-dimensional. Some important classes of such manifolds are almost complex, almost product, almost contact and normal almost paracontact manifolds. The geometry of submanifolds of these manifolds is very rich and interesting subject.

CR-submanifolds in complex manifolds are corresponding semi-invariant submanifolds in paracontact (or Riemannian product) manifolds. But their properties are all different from each other. For example, the invariant submanifold of a Kaehler is always minimal, but it is not true in paracontact metric manifolds.

Nowadays, the study of submanifold theory is growing rapidly. Invariant submanifolds play a crucial role in many applied branches of mathematics. For instance, the method of invariant submanifold is used in the study of non-linear autonomoust systems.

In 1978, A. Bejancu [2, 6] introduced the notion of CR-submanifolds. Later, B.-Y. Chen studied these submanifolds in a Kaehler manifold [9, 10]. He obtained several fundamental results for CR-submanifolds. Since then the geometry of CR-submanifolds is an active field of research. Many articles and books have been published on CR-submanifolds (see [3], [12], [22], [24]). In this sense, A. Bejancu and N. Papaghiuc studied semi-invariant submanifolds of a Sasakian manifold or a Sasakian space form [4, 5] and A. Cabras, P. Matzeu and C. I. Bejan studied them for cosymplectic manifolds [1, 7]. In [19], semi-invariant submanifolds in a locally product manifold were studied by B. Şahin and M. Atçeken. Also, the semi-invariant submanifolds of an almost paracontact Riemannian manifolds were investigated in [13].

Inspired by the studies mentioned above, in the this paper, we study semi invariant submanifolds of an almost paracontact manifold which have not been attempted so far. We characterize the induced

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structures on a submanifold. Also, we obtain some necessary and sufficient conditions that a semi-invariant submanifold to be invariant, anti-invariant and semi-invariant product and we investigate flat, curvature-invariant cases in an almost paracontact manifold with constant c. We give an example to illustrate our results.

2. Preliminaries

A Riemannian manifold $(\tilde{M}, g)$ is called almost paracontact metric manifold if it is endowed with the structure (φ, ξ, η, g), where φ is a $(1,1)$ tensor, ξ and η are vector field and 1-form on $\tilde{M}$, respectively, satisfying

$$\varphi^2 X = X - \eta(X)\xi, \ \ \varphi\xi = 0, \ \ \eta o \varphi = 0, \ \ \eta(\xi) = 1 \tag{1}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ \eta(X) = g(X, \xi), \tag{2}$$

for any $X, Y \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$ denotes the set of all smooth vector fields on $\tilde{M}$ [20, 21].

An almost paracontact metric manifold $\tilde{M}$ is said to be normal if the covariant derivative of φ satisfies

$$(\tilde{\nabla}_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{3}$$

This implies that

$$\tilde{\nabla}_X \xi = \varphi X, \tag{4}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$.

A normal paracontact metric manifold $\tilde{M}$ is said to have a constant c if and only if its Riemannian curvature tensor $\tilde{R}$ is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= \left(\frac{c+3}{4}\right)\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \left(\frac{c-1}{4}\right)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &+ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\} \end{aligned} \tag{5}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$ [18].

Now, let M be an isometrically immersed submanifold of a normal almost paracontact metric manifold $\tilde{M}$ and we also denote the Riemannian metric tensor by g for the induced metric on M. On the other hand, if ∇ denotes the induced Levi-Civita connection on M by $\tilde{\nabla}$, then the Gauss and Weingarten formulas for M in $\tilde{M}$ respectively, given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{6}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{7}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form of M, A_V is the Weingarten operator with respect to V and $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ of M. It is well known that the Weingarten operator A_V and second fundamental form h are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{8}$$

A submanifold M of $\tilde{M}$ is said to be a totally geodesic submanifold if h vanishes identically.

For any submanifold M of a Riemannian manifold $\tilde{M}$, the Gauss equation is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \tag{9}$$

for any $X,Y,Z \in \Gamma(TM)$, where $\tilde{R}$ and R are the Riemannian curvature tensors of M and $\tilde{M}$, respectively. The covariant derivative $\tilde{\nabla}h$ of h is defined by

$$(\tilde{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(\nabla_X Z,Y) \tag{10}$$

and the covariant derivative ∇A of A is defined by

$$(\nabla_X A)_V Y = \nabla_X A_V Y - A_{\nabla^{\perp}_X V}Y - A_V(\nabla_X Y) \tag{11}$$

for any $X,Y,Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The normal component of (9) is said to be Codazzi equation and is given by

$$(\tilde{R}(X,Y)Z)^{\perp} = (\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z). \tag{12}$$

If $(\tilde{R}(X,Y)Z)^{\perp}$ vanishes identically, then the submanifold M is called curvature-invariant submanifold.

The Ricci equation is given by

$$g(\tilde{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V) + g([A_V,A_U]X,Y) \tag{13}$$

for $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$, where $R^{\perp}$ denotes the Riemannian curvature tensor of the normal vector bundle $T^{\perp}M$. If $R^{\perp}$ vanishes identically, then the normal connection of M is called flat [23].

From now on, let us consider a submanifold M of a normal almost paracontact metric manifold $(\tilde{M},\varphi,\xi,\eta,g)$ and M is tangent to the structure vector field ξ. We put

$$\varphi X = TX + \omega X, \tag{14}$$

where TX (resp. ωX) denotes the tangential (resp. normal) component of φX. In the same way, for any $V \in \Gamma(T^{\perp}M)$, we can write

$$\varphi V = BV + CV, \tag{15}$$

where BV (resp. CV) denotes the tangential (resp. normal) component of φV.

In (14), if ω (resp. T) vanishes identically, then submanifold M is said to be invariant (resp. anti-invariant) as special cases. Here, we can define the covariant derivatives of T,ω,B and C, respectively, by

$$(\nabla_Y T)X = \nabla_Y TX - T\nabla_Y X \tag{16}$$

$$(\nabla_Y \omega)X = \nabla^{\perp}_Y \omega X - \omega\nabla_Y X \tag{17}$$

$$(\nabla_X B)V = \nabla_X BV - B\nabla^{\perp}_X V \tag{18}$$

and

$$(\nabla_X C)V = \nabla^{\perp}_X CV - C\nabla_X V \tag{19}$$

for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. On the other hand, we can easily to see that

$$g(TX,Y) = g(X,TY)$$

and

$$g(CU, V) = g(U, CV)$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, that is, T and C are also symmetric tensors. Moreover, by using (14) and (15)

$$g(\omega X, V) = g(\varphi X, V) = g(X, \varphi V) = g(X, BV) \tag{20}$$

which gives the relation between ω and B.

By using (5), (9) and (14), the Riemannian curvature tensor R of a submanifold M in a normal paracontact metric manifold $\tilde{M}(c)$ with constant c is given by

$$\begin{aligned} R(X, Y)Z = &\left(\frac{c+3}{4}\right)\{g(Y, Z)X - g(X, Z)Y\} + \left(\frac{c-1}{4}\right)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(TY, Z)TX \\ &- g(TX, Z)TY - 2g(TX, Y)TZ\} - A_{h(X,Z)}Y + A_{h(Y,Z)}X \end{aligned} \tag{21}$$

and the Codazzi equation becomes

$$\begin{aligned} (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z) = &\left(\frac{c-1}{4}\right)\{g(\varphi Y, Z)\omega X - g(\varphi X, Z)\omega Y \\ &- 2g(\varphi X, Y)\omega Z\}. \end{aligned} \tag{22}$$

On the other hand, for a submanifold M of the normal paracontact metric manifold $\tilde{M}(c)$ with constant c, the Ricci equation reduces to

$$\begin{aligned} g(R^{\perp}(X, Y)U, V) = &\left(\frac{c-1}{4}\right)\{g(Y, \varphi U)g(X, \varphi V) - g(\varphi U, X)g(\varphi V, Y) \\ &- 2g(\varphi X, Y)g(\varphi U, V)\} + g([A_U, A_V]X, Y) \end{aligned} \tag{23}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

3. Semi-invariant Submanifolds of Almost Paracontact Metric Manifolds

In this section, we study semi-invariant submanifolds of a normal paracontact metric manifold. First, we define these submanifolds as follows:

Definition 3.1. *Let M be a Riemannian manifold isometrically immersed in a normal paracontact metric manifold $\tilde{M}$ such that ξ is tangent to M. Then M is called a semi-invariant submanifold of $\tilde{M}$ if there exists a differentiable distribution D: $x \longrightarrow D_x \subset T_x(M)$ on M satisfying the following conditions;*

(i) D is invariant with respect to φ, i.e., $\varphi D_x \subset D_x$, for each $x \in M$.
(ii) The orthogonal complementary distribution $D^{\perp}$: $x \longrightarrow D_x^{\perp} \subset T_x(M)$ is anti-invariant with respect to φ, i.e., $\varphi D_x^{\perp} \subset T_x^{\perp}(M)$ for each $x \in M$.

If we put $\dim \tilde{M} = m$, $\dim M = n$, $\dim D = p$, $\dim D^{\perp} = q$, then $\operatorname{co}\dim M = m - n$. If $q = 0$ (resp. $p = 0$), then the semi-invariant submanifold is invariant (resp. anti-invariant).

On the other hand, if $m - n = q$, then the submanifold M is called a generic submanifold of $\tilde{M}$. For $\xi \in D$, if $p > 1$ and $q > 0$, then M is called a proper (non-trivial) semi-invariant submanifold. So, invariant and anti-invariant submanifolds are special classes of semi-invariant submanifolds.

If M is an invariant submanifold of $\tilde{M}$, then M is also a paracontact metric manifold with respect to the induced structure. So, if M is an invariant submanifold of paracontact metric manifold $\tilde{M}$, then from (14) and (15) we have $\omega = 0$ and $B = 0$ and $\varphi X = TX$ and $\varphi V = CV$ for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. We have the following Lemmas.

Lemma 3.2. *Let M be a semi-invariant submanifold of a normal almost paracontact metric manifold $\tilde{M}$. Then, we have*

$$h(X,\xi) = \omega X \ \text{ and } \ A_V\xi = BV \tag{24}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$

Proof. By using (4) and (6), we have

$$\varphi X = \nabla_X\xi + h(X,\xi),$$

from which $h(X,\xi) = \omega X$ and $\nabla_X\xi = TX$. On the other hand, making use of (8) and (14), we obtain

$$g(A_V\xi, X) = g(h(X,\xi), V) = g(\omega X, V) = g(BV, X)$$

that is,

$$A_V\xi = BV,$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, which proves the lemma. □

Lemma 3.3. *Let M be a submanifold of an almost paracontact metric manifold $\tilde{M}$. Then we have*

$$T^2 + B\omega = I - \eta\otimes\xi, \ \ \omega T + C\omega = 0 \tag{25}$$

and

$$\omega B + C^2 = I, \ \ TB + BC = 0. \tag{26}$$

Proof. Applying φ to (14) and (15) and comparing the tangent and normal components, we obtain (25) and(26), respectively. □

Lemma 3.4. *Let M be a submanifold of a normal almost paracontact metric manifold $\tilde{M}$. Then we have*

$$(\nabla_X T)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi + A_{\omega Y}X + Bh(X,Y) \tag{27}$$

and

$$(\nabla_X\omega)Y = Ch(X,Y) - h(X,TY) \tag{28}$$

for any $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, by using (3), (6), (7), (16) and (17), we have

$$\begin{aligned} (\tilde{\nabla}_X\varphi)Y &= \tilde{\nabla}_X\varphi Y - \varphi\tilde{\nabla}_X Y \\ -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi &= \nabla_X TY + h(X,TY) - A_{\omega Y}X + \nabla^{\perp}_X\omega Y \\ &\quad - T\nabla_X Y - \omega\nabla_X Y - Bh(X,Y) - Ch(X,Y). \end{aligned}$$

From the tangent and normal components of the last equality, respectively, we infer

$$(\tilde{\nabla}_X T)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi + A_{\omega Y}X + Bh(X,Y)$$

and

$$(\tilde{\nabla}_X\omega)Y = Ch(X,Y) - h(X,TY).$$

This proves our assertion. □

Lemma 3.5. *Let M be a submanifold of a normal almost paracontact metric manifold $\tilde{M}$. Then, we have*

$$(\nabla_X B)V = A_{CV}X - TA_V X \tag{29}$$

and

$$(\nabla_X C)V = -h(X, BV) - \omega A_V X \tag{30}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Proof. For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, making use of (3), (6), (7) and (15), we have

$$\begin{aligned}(\tilde{\nabla}_X \varphi)V &= \tilde{\nabla}_X \varphi V - \varphi \tilde{\nabla}_X V\\ -g(X, V)\xi - \eta(V)X + 2\eta(X)\eta(V)\xi &= \nabla_X BV + h(X, BV) - A_{CV}X + \nabla^\perp_X CV\\ &\quad - B\nabla^\perp_X V - C\nabla^\perp_X V + TA_V X + \omega A_V X.\end{aligned}$$

By corresponding the tangent and normal components of the last equality, (29) and (30) are respectively, obtained. □

Now, let M be a semi-invariant submanifold of an almost paracontact metric manifold $\tilde{M}$. Taking into account the Definition 3.1 we derive that the tangent bundle and normal bundle of a semi-invariant submanifold M has the orthogonal decompositions;

$$TM = D \oplus D^\perp \oplus \langle\xi\rangle, \;\; T^\perp M = \varphi(D^\perp) \oplus \upsilon, \;\; \varphi(\upsilon) = \upsilon, \tag{31}$$

where υ denotes the orthogonal complementary subbundle of $\varphi(D^\perp)$ in $T^\perp M$ and $\langle\xi\rangle$ is a 1-dimensional distribution which is spanned by ξ. If we denote by P and Q the projection morphisms of TM on D and $D^\perp$, respectively. Then we have

$$\begin{aligned}&X = PX + QX, \;\; \varphi X = TPX + \omega QX, \;\; \omega PX = 0, \;\; TQX = 0,\\ &TX = \varphi PX, \;\; \omega X = \varphi QX, \;\; i.e., \;\; T = \varphi \circ P \;\; and \;\; \omega = \varphi \circ Q\end{aligned} \tag{32}$$

for any $X \in \Gamma(TM)$. Then, we obtain

$$TPQ = 0, \;\; TP = T = PT$$

and by using (25), from $\omega TP + C\omega P = 0$, we arrive at

$$\omega TP = \omega T = 0 \tag{33}$$

which is equivalent to

$$C\omega = 0. \tag{34}$$

Conversely, let M be a submanifold of an almost paracontact metric manifold $\tilde{M}$ and the condition (34) is satisfied. For $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we have

$$g(X, \varphi^2 V) = g(\varphi^2 X, V)$$

or,

$$g(X, \varphi(BV + CV)) = g(\varphi(TX + \omega X), V)$$

i.e.,

$$g(X, TBV) = g(C\omega X, V) = 0,$$

which implies that $TB = 0$. From (25) we have $T^3 - T = 0$. This tells us that T defines an almost product structure on M. If we put $P = T^2 + \eta \otimes \xi$ and $Q = I - T^2 - \eta \otimes \xi$, then we can easily verify that

$$P^2 = P, \ \ Q^2 = Q \ \ P + Q = I, \ \ PQ = QP = 0. \tag{35}$$

From (35), we can infer that P and Q are orthogonal projections and they define orthogonal distributions such as D and $D^{\perp}$, respectively.

By virtue of $P = T^2 + \eta \otimes \xi$ and $T^3 - T = 0$, we have $TP = T$ and $TQ = 0$. Taking into account of T and P being symmetric, we have

$$g(QTX, Y) = g(TX, QY) = g(X, TQY) = 0,$$

for any $X, Y \in \Gamma(TM)$, that is,

$$QT = 0,$$

which implies that

$$QTP = 0.$$

By virtue of $P = T^2 + \eta \otimes \xi$ and $T\xi = \omega\xi = 0$ and from (33), it is obvious that

$$\omega T = 0. \tag{36}$$

The relations (35) and (36) tell us that D and $D^{\perp}$ are invariant and anti-invariant distributions, respectively. From the definition of D and $D^{\perp}$, it can be verified that $\xi \in D$.

Thus we have the following theorem.

Theorem 3.6. *Let M be a submanifold of an almost paracontact metric manifold $\tilde{M}$. Then M is a semi-invariant submanifold if and only if $\omega T = 0$.*

Proposition 3.7. *Let M be a submanifold of an almost paracontact metric manifold $\tilde{M}$. Then M is a semi-invariant submanifold if and only if $T^3 - T = 0$.*

Proof. If M is a semi-invariant submanifold, then by Theorem 6, we know that $T^3 - T = 0$.

Conversely, if $T^3 - T$ vanishes identically. Again, from Theorem 6, we get $\omega T = 0$. This proves our assertion. □

Proposition 3.8. *Let M be a submanifold of an almost paracontact metric manifold $\tilde{M}$. Then M is a semi-invriant submanifold if only if $C^3 - C = 0$.*

Proof. If $C^3 - C = 0$, then from (26) and (34), we get $\omega T = 0$, which means that M is semi-invariant.

Conversely, if M is a semi-invariant submanifold, then taking into account that (34) and (26) we conclude $C^3 = C$. Hence, the proof is complete. □

For the sake of similarity in results, we notice that the above proposition has been proved in [13] by using different technique.

Proposition 3.9. *Let M be a submanifold of a normal almost paracontact metric manifold $\tilde{M}$. If ω is parallel, then M is semi-invariant.*

Proof. If ω is parallel, from (28), we have

$$Ch(X, Y) - h(X, TY) = 0$$

for any $X, Y \in \Gamma(TM)$. For $X = \xi$, from Lemma 3.2, we have

$$\begin{aligned} Ch(\xi, Y) - h(\xi, TY) &= 0 \\ C\omega Y - \omega TY &= 0. \end{aligned}$$

On the other hand, from (25), we obtain

$$C\omega Y + \omega TY = 0,$$

that is, $C\omega = 0$. This proves our assertion. □

Proposition 3.10. *Let M be a submanifold of a normal almost paracontact metric manifold $\tilde{M}$. The endomorphism T is parallel if and only if M is anti-invariant.*

Proof. If M is an anti-invariant submanifold of $\tilde{M}$, then $T = 0$ and so $\nabla T = 0$.
Conversely, if $\nabla T = 0$, then from (27), we have

$$-g(X, Y) + \eta(X)\eta(Y) + g(h(X, \xi), \omega Y) = 0,$$

for any $X, Y \in \Gamma(TM)$. By using (2),(14) and (24), we obtain

$$-g(\varphi X, \varphi Y) + g(\omega X, \omega Y) = -g(TX, TY) = 0,$$

which implies that $T = 0$, i.e., M is anti-invariant. This ends the proof. □

Theorem 3.11. *Let M be a semi-invariant submanifold of a normal almost paracontact metric manifold $\tilde{M}$. Then B is parallel if and only if ω is parallel.*

Proof. For $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, from (28) and (29), we obtain

$$\begin{aligned} g((\tilde{\nabla}_X \omega)Y, V) &= g(Ch(X, Y), V) - g(h(X, TY), V) \\ &= g(h(X, Y), CV) - g(h(X, TY), V) \\ &= g(A_{CV}X - TA_V X, Y) \\ &= g((\nabla_X B)V, Y). \end{aligned}$$

Thus, the proof follows from the above relation. □

For a semi-invariant submanifold M of $\tilde{M}$, if the invariant distribution D and anti-invariant distribution $D^{\perp}$ are totally geodesics in M, then M is called a semi-invariant product.

Theorem 3.12. *Let M be a semi-invariant submanifold of a normal paracontact metric manifold $\tilde{M}$. Then M is a semi-invariant product if and only if the shape operator A of M satisfies*

$$A_{\varphi W}X = \eta(X)W \tag{37}$$

for any $X \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Proof. For $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$, by using (3) and (6), we have

$$\begin{aligned} g(A_{\varphi W}X - \eta(X)W, Y) &= g(h(X, Y), \varphi W) \\ &= g(\tilde{\nabla}_Y X, \varphi W) \\ &= g(\tilde{\nabla}_Y \varphi X - (\tilde{\nabla}_Y \varphi)X, W) \\ &= g(\nabla_Y TX, W) - g(-g(X, Y)\xi - \eta(X)Y \\ &\quad + 2\eta(Y)\eta(X)\xi, W) \\ &= g(\nabla_Y TX, W) \end{aligned}$$

and

$$\begin{aligned} g(A_{\varphi W}X - \eta(X)W, Z) &= g(h(X,Z), \varphi W) - \eta(X)g(Z,W) \\ &= g(\tilde{\nabla}_Z X, \varphi W) - \eta(X)g(Z,W) \\ &= -g(\tilde{\nabla}_Z \varphi W, X) - \eta(X)g(Z,W) \\ &= -g((\nabla_Z \varphi)W + \varphi\tilde{\nabla}_Z W, X) - \eta(X)g(Z,W) \\ &= -g(-g(Z,W)\xi - \eta(W)Z + 2\eta(W)\eta(Z)\xi, X) \\ &+ g(\nabla_Z W, \varphi X) - \eta(X)\eta(Z), W) \\ &= g(\nabla_Z W, TX). \end{aligned}$$

Therefore, $\nabla_Y X \in \Gamma(D)$ and $\nabla_Z W \in \Gamma(D^{\perp})$ if and only if (37) holds. Hence, the theorem is proved completely. □

4. Submanifolds of a Normal Almost Paracontact Metric Manifold with C onstant *c*

In this section, we present some new results for semi-invariant submanifolds in a normal almost paracontact metric manifold $\tilde{M}$ with constant c and is denoted by $\tilde{M}(c)$.

Theorem 4.1. *Let M be a submanifold of a normal almost paracontact metric manifold $\tilde{M}(c)$ with constant c. If M is a curvature-invariant submanifold such that $c \neq 1$, then M is either invariant or anti-invariant.*

Proof. Let us suppose that M is a curvature-invariant submanifold of $\tilde{M}(c)$ such that $c \neq 1$. Then from (22), we have

$$g(\varphi Y, Z)\omega X - g(\varphi X, Z)\omega Y - 2g(\varphi X, Y)\omega Z = 0$$

for any $X, Y, Z \in \Gamma(TM)$. Here, choosing $X = Y$, we conclude

$$-2g(TX, X)\omega Z = 0.$$

It follows from above relation that either $T = 0$, i.e., M is anti-invariant or $\omega = 0$, i.e., M is invariant because T is an almost product structure. □

Theorem 4.2. *Let M be a submanifold of a normal almost paracontact metric manifold $\tilde{M}(c)$ with constant c ($c \neq 1$). If the normal connection of M is flat and $TA_U = A_U T$, then M is either an anti-invariant submanifold or a generic submanifold of $\tilde{M}(c)$.*

Proof. If the normal connection of M is flat, then from (13) and (23), we obtain

$$\begin{aligned} g([A_U, A_V]X, Y) = -\left(\frac{c-1}{4}\right)\{&g(\varphi Y, U)g(\varphi X, V) - g(\varphi Y, V)g(\varphi X, U) \\ &- 2g(\varphi X, Y)g(\varphi V, U)\}. \end{aligned}$$

Here, if in particular, put $Y = TX$, then this equality reduces to

$$g(A_U A_V X - A_V A_U X, TX) = -\left(\frac{c-1}{2}\right)g(TX, TX)g(\varphi V, U).$$

Also, choosing $V = CU$, we conclude

$$g(A_{CU}X, A_U TX) - g(A_U X, A_{CU}TX) = -\left(\frac{c-1}{2}\right)g(TX, TX)g(CU, CU).$$

Since $TA_U = A_U T$, we reach

$$\left(\frac{c-1}{2}\right)g(TX, TX)g(CU, CU) = 0,$$

which implies that either $T = 0$, i.e., M is anti invariant or $C = 0$, which means that M is a generic submanifold. Hence, the proof is complete. □

Theorem 4.3. *Let M be a proper semi-invariant submanifold of a normal almost paracontact metric manifold $\tilde{M}(c)$ with constant c. If the invariant distribution D is integrable, then $c = 1$.*

Proof. If the invariant distribution D is integrable, it is well known that

$$h(TX, Y) = h(X, TY),$$

which is equivalent

$$TA_U Y = A_U TY, \tag{38}$$

for $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$. From (38), we have

$$g(A_U TY, BV) = 0, \tag{39}$$

for $V \in \Gamma(T^{\perp}M)$. By differentiating covariant derivative of (39) in the direction X, we obtain

$$g(\tilde{\nabla}_X A_U TY, BV) + g(A_U TY, \tilde{\nabla}_X BV) = 0.$$

Using (11) and (18), we obtain

$$g((\nabla_X A)_U TY + A_{\nabla_X^{\perp} U} TY + A_U(\nabla_X TY), BV) + g((\nabla_X B)V + B\nabla_X^{\perp} V, A_U TY) = 0.$$

Taking into account of M being a semi-invariant submanifold with (27) and (29), we reach

$$\begin{aligned} -g((\nabla_X A)_U TY, BV) &= g(A_U\{(\nabla_X T)Y + T\nabla_X Y\}, BV) \\ &+ g(A_{CV}X - TA_V X, A_U TY) \\ &= g(A_U\{-g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi + A_{\omega Y}X \\ &+ Bh(X, Y)\}, BV) + g(A_{CV}X, A_U TY) - g(A_V TX, A_U TY) \\ &= g(-g(X, Y)A_U\xi - \eta(Y)A_U X + 2\eta(X)\eta(Y)A_U\xi \\ &+ A_U A_{\omega Y}X + A_U Bh(X, Y), BV) + g(A_{CV}X, A_U TY) \\ &- g(A_V TX, A_U TY) \\ &= -g(X, Y)g(A_U\xi, BV) - \eta(Y)g(A_U X, BV) \\ &+ 2\eta(X)\eta(Y)g(A_U\xi, BV) + g(A_U A_{\omega Y}X, BV) \\ &+ g(A_U Bh(X, Y), BV) + g(A_{CV}X, A_U TY) \\ &- g(A_V TX, A_U TY) \end{aligned}$$

Here, considering (10), (11) and Lemma 3.2, we have

$$\begin{aligned} -g((\nabla_X A)_U TY, BV) &= -g(X, Y)g(BU, BV) - \eta(Y)g(A_U BV, X) \\ &+ 2\eta(X)\eta(Y)g(BU, BV) + g(A_U BV, A_{\omega Y}X) \\ &+ g(A_U BV, Bh(X, Y)) + g(A_{CV}X, A_U TY) \\ &- g(A_V TX, A_U TY). \end{aligned}$$

Interchanging X by TX in the last equality, we derive

$$\begin{aligned} -g((\nabla_{TX} h)(TY, BV), U) &= -g(TX, Y)g(BU, BV) - \eta(Y)g(A_U TX, BV) \\ &+ 2\eta(TX)\eta(Y)g(BU, BV) + g(A_U BV, A_{\omega Y}TX) \\ &+ g(A_U BV, Bh(TX, Y)) + g(A_{CV}TX, A_U TY) \\ &- g(A_V T^2 X, A_U TY), \end{aligned}$$

from which, we find

$$-g((\nabla_{TX}h)(TY,BV),U) = -g(TX,Y)g(BU,BV) + g(A_UBV,Bh(TX,Y)) + g(A_{CV}TX,A_UTY) - g(A_VTX,A_UY).$$

Thus, we obtain

$$g((\tilde{\nabla}_{TY}h)(TX,BV) - (\tilde{\nabla}_{TX}h)(TY,BV),U) = g(A_{CV}TX,A_UTY) + g(A_VTY,A_UX) - g(A_{CV}TY,A_UTX) - g(A_VTX,A_UY) \tag{40}$$

and from (22), we arrive at

$$g((\tilde{\nabla}_{TY}h)(TX,BV) - (\tilde{\nabla}_{TX}h)(TY,BV),U) = \left(\frac{c-1}{2}\right)g(TX,Y)g(BV,BU). \tag{41}$$

By taking into account of (40) and (41), we compute

$$g(A_{CV}TX,A_UTY) - g(A_{CV}TY,A_UTX) + g(A_VTY,A_UX) - g(A_VTX,A_UY) = \left(\frac{c-1}{2}\right)g(TX,Y)g(BV,BU).$$

Here, taking $U=V$ and TX instead of Y, making use of $T^3 - T = 0$ and (38), we have

$$\begin{aligned}\left(\frac{c-1}{2}\right)g(TX,TX)g(BU,BU) &= g(A_{CU}TX,A_UT^2X) - g(A_{CU}T^2X,A_UTX)\\ &= g(A_{CU}X,A_UX) - g(A_{CU}X,A_UX)\\ &= 0.\end{aligned}$$

Since M is a proper semi-invariant submanifold, then TX and BU are non-zero vectors, it follows from above relation that $c=1$, which proves the theorem completely. □

Now, we present an example of a semi-invariant submanifold of an almost paracontact manifold.

Example 4.4. On a 7-dimensional Euclidean space

$$\mathbb{R}^7 = \{(x_1,x_2,x_3,x_4,x_5,x_6,x_7) | x_i \in \mathbb{R}, 1 \leq i \leq 7\},$$

we define the almost paracontact metric structure (φ, g, ξ, η) as follows;

$$\varphi(\frac{\partial}{\partial x_i}) = \varepsilon_i \frac{\partial}{\partial x_i}, \ \ \xi = \frac{\partial}{\partial x_7}, \ \ \eta = dx_7, \ \ g = \sum_{i=1}^{7} dx_i^2,$$

where

$$\varepsilon_i = \begin{cases} 1 & , \text{ if } i = 1,2,3, \\ -1 & , \text{ if } i = 4,5,6, \\ 0 & , \text{ if } i = 7, \end{cases}$$

and g denotes the standard metric tensor of $\mathbb{R}^7$, where $\left\{\frac{\partial}{\partial x_i}\right\}, 1 \leq i \leq 7$, are the usual basis vectors of $\mathbb{R}^7$. Let Z be an arbitrary vector in $\mathbb{R}^7$, then it can be written as

$$Z = \sum_{i=1}^{7} \lambda_i \frac{\partial}{\partial x_i}.$$

Then, we have

$$g(Z,Z)=\sum_{i=1}^{7}\lambda_i^2.$$

On the other hand, we can easily to see that

$$g(Z,\xi)=\eta(Z)=\lambda_7,\ \ g(\varphi Z,\varphi Z)=g(Z,Z)-\eta^2(Z),$$

and

$$\varphi\xi=0,\ \ \eta(\xi)=1,$$

that is, (R^7,φ,g,ξ,η) becomes an almost paracontact metric manifold. Now, let us consider an immersed submanifold M in $\mathbb{R}^7$ given by the equations

$$x_1^2+x_2^2=x_5^2+x_6^2,\ \ x_3+x_4=0.$$

By direct computation, it is easy to check that the tangent bundle of M is spanned by the vectors

$$Z_1=\cos\theta\frac{\partial}{\partial x_1}+\sin\theta\frac{\partial}{\partial x_2}+\cos\beta\frac{\partial}{\partial x_5}+\sin\beta\frac{\partial}{\partial x_6},\ \ Z_2=-u\sin\theta\frac{\partial}{\partial x_1}+u\cos\theta\frac{\partial}{\partial x_2}$$
$$Z_3=\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_4},\ \ Z_4=-u\sin\beta\frac{\partial}{\partial x_5}+u\cos\beta\frac{\partial}{\partial x_6},\ \ Z_5=\frac{\partial}{\partial x_7}$$

where θ,β and u denote arbitrary parameters, from the definition of the almost paracontact structure φ, we can derive

$$\varphi Z_1=\cos\theta\frac{\partial}{\partial x_1}+\sin\theta\frac{\partial}{\partial x_2}-\cos\beta\frac{\partial}{\partial x_5}-\sin\beta\frac{\partial}{\partial x_6}$$
$$\varphi Z_2=Z_2,\ \ \varphi Z_3=\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4},\ \ \varphi Z_4=-Z_4,\ \ \varphi Z_5=0.$$

Since φZ_1 and φZ_3 are orthogonal to TM and $\varphi Z_2,\varphi Z_4$ are tangent to TM. Hence, we find that $D=\mathrm{Span}\{Z_2,Z_4,Z_5\}$ is an invariant distribution and $D^{\perp}=\mathrm{Span}\{Z_1,Z_3\}$ is an ati-invariant distribution of M.Thus M is a 5-dimensional semi-invariant submanifold of $\mathbb{R}^7$ with its usual almost paracontact metric structure (φ,g,ξ,η).

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