Filomat 31:15 (2017), 4833–4844
https://doi.org/10.2298/FIL1715833Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Identities Related to Special Polynomials and Combinatorial Numbers

Eda Yuluklu[a], **Yilmaz Simsek**[b], **Takao Komatsu**[c]

[a]*Department of Mathematics, Faculty of Art and Science University of Usak, Turkey*
[b]*Department of Mathematics, Faculty of Science University of Akdeniz TR-07058 Antalya, Turkey*
[c]*School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China*

Abstract. The aim of this paper is to give some new identities and relations related to the some families of special numbers such as the Bernoulli numbers, the Euler numbers, the Stirling numbers of the first and second kinds, the central factorial numbers and also the numbers $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$ which are given Simsek [31]. Our method is related to the functional equations of the generating functions and the fermionic and bosonic p-adic Volkenborn integral on $\mathbb{Z}_p$. Finally, we give remarks and comments on our results.

1. Introduction

Generating functions (1) and (2) gives us new families of combinatorial numbers, which were defined by Simsek [31]-[32]. These numbers are generalized combinatorial numbers, which were considered in many earlier investigations by (among others) Golombek [12] and Simsek [33].

Definition 1.1. *(See, for details, Simsek [31]). The numbers $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$ are defined by means of the following generating functions, respectively*

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!}\left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}, \tag{1}$$

and

$$F_{y_2}(t,k;\lambda) = \frac{1}{(2k)!}\left(\lambda e^t + \lambda^{-1}e^{-t} + 2\right)^k = \sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!}. \tag{2}$$

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,\ldots\}$ and $\lambda \in \mathbb{C}$, denotes the set of complex numbers.

2010 *Mathematics Subject Classification*. 11B68; 05A15; 05A19; 12D10; 26C05; 30C15.
Keywords. Bernoulli numbers; Euler numbers; Array polynomials; Stirling numbers; Generating functions; Functional equation; Binomial coefficients; Combinatorial sum.
Received: 05 July 2016; Accepted: 08 December 2016
Communicated by Hari M. Srivastava
The present investigation was supported by the Scientific Research Project Administration of Akdeniz University
Email addresses: eda.yuluklu@usak.edu.tr (Eda Yuluklu), ysimsek@akdeniz.edu.tr (Yilmaz Simsek), komatsu@whu.edu.cn (Takao Komatsu)

Furthermore, $0^n = 1$ if $n = 0$, and, $0^n = 0$ if $n \in \mathbb{N}$.

In [31], by using (1) and (2), Simsek defined the numbers $y_1(n, k; \lambda)$ and $y_2(n, k; \lambda)$ as follows:

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^n \lambda^j$$

and

$$y_2(n, k; \lambda) = \frac{1}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \binom{j}{l} (2l - j)^n \lambda^{2l-j}.$$

By using the above formulas, a few values of the numbers $y_1(n, k; \lambda)$ and $y_2(n, k; \lambda)$ are given as follows, respectively:

$$y_1(0, 0; \lambda) = 1, y_1(0, 1; \lambda) = \lambda + 1, y_1(0, 2; \lambda) = \frac{1}{2}\lambda^2 + \lambda + \frac{1}{2},$$

$$y_1(1, 0; \lambda) = 0, y_1(1, 1; \lambda) = \lambda, y_1(1, 2; \lambda) = \lambda^2 + \lambda,$$

$$y_1(2, 0; \lambda) = 0, y_1(2, 1; \lambda) = \lambda, y_1(2, 2; \lambda) = 2\lambda^2 + \lambda$$

and

$$y_2(0, 0; \lambda) = 1, y_2(0, 1; \lambda) = \frac{1}{2\lambda} + \frac{\lambda}{2} + \frac{1}{2}, y_2(0, 2; \lambda) = \frac{\lambda^2 + 4\lambda}{24} + \frac{\lambda}{24\lambda^2} + \frac{1}{4},$$

$$y_2(1, 0; \lambda) = 0, y_2(1, 1; \lambda) = \frac{\lambda}{2} - \frac{1}{2\lambda}, y_2(1, 2; \lambda) = \frac{\lambda^2 + 2\lambda}{12} - \frac{2\lambda + 1}{6\lambda^2},$$

$$y_2(2, 0; \lambda) = 0, y_2(2, 1; \lambda) = \frac{\lambda}{2} + \frac{1}{2\lambda}, y_2(2, 2; \lambda) = \frac{\lambda^2 + \lambda}{6} + \frac{\lambda + 1}{6\lambda^2}.$$

The generating function in (3) gives us with a generalization of the Stirling numbers $S_2(n, v)$ of the second kind, which were considered in many earlier investigations by (among others) Srivastava [36], Luo and Srivastava [22], Srivastava *et al.* [2] and Simsek ([30], [29]).

Luo and Srivastava [22, Definition 5] defined the λ-Stirling numbers $S_2(n, v; \lambda)$ of the second kind by means of the following generating function:

$$F_S(t, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} = \sum_{n=0}^{\infty} S_2(n, v; \lambda) \frac{t^n}{n!}, \tag{3}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. Substituting $\lambda = 1$ into (3), the λ-Stirling numbers $S_2(n, v; \lambda)$ reduce to the Stirling numbers $S_2(n, v)$ of the second kind defined. That is

$$S_2(n, v) = S_2(n, v; 1)$$

(*cf.* [1]-[43]; and the references cited therein).

By using (3), one easily compute the following values for the λ-Stirling numbers $S_2(n, v; \lambda)$:

$$S_2(0, 0; \lambda) = 1, S_2(1, 0; \lambda) = 0, S_2(1, 1; \lambda) = \lambda, S_2(2, 0; \lambda) = 0, S_2(2, 1; \lambda) = \lambda$$

and

$$S_2(0, v; \lambda) = \frac{(\lambda - 1)^v}{v!}$$

(*cf.* [36], [22, Definition 5]).

Remark 1.2. *Replacing λ by $-\lambda$ in (1), then the numbers $y_1(n,k;\lambda)$ reduces to the λ-Stirling numbers $S_2(n,v;\lambda)$.*

Fundamental properties of the numbers $y_1(n,k;\lambda)$ are investigated by Simsek ([31], [32], [33]). The numbers $y_1(n,k;\lambda)$ are related to the following combinatorial sum, which was given by Golombek [12]:

$$y_1(n,k;1) = \sum_{j=0}^{k} \binom{k}{j} j^n = \frac{d^n}{dt^n}\left(e^t+1\right)^k|_{t=0}, \tag{4}$$

where $n = 1,2,\ldots$(*cf.* see, also [31], [33]). In [31], Simsek gave a conjecture with two open questions associated with the numbers $y_1(n,k;1)$.

Srivastava [35] gave not only many useful formulas and identities related to the Bernoulli and Euler polynomials with their generating functions, but also gave finite sums involving the Bernoulli and Euler polynomials with their interpolation functions.

The Bernoulli numbers (or the Bernoulli numbers of the first kind) B_n are defined by means of the following generating functions:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

where $|t| < 2\pi$ (*cf.* [2]-[43]; see also the references cited in each of these earlier works).

By using the above generating functions, few values of the Bernoulli numbers of the first kind are given as follows:

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}$$

and for $n > 1$,

$$B_{2n+1} = 0.$$

Computation of the Bernoulli and Euler numbers of higher order (k or $-k$) were considered in many earlier investigations by (among others) Srivastava and Luo [22], Srivastava *et al.* [36]-[42], Ozden and Simsek [23].

The first kind Apostol-Euler polynomials of order k are defined by means of the following generating function:

$$F_{P1}(t,x;k,\lambda) = \left(\frac{2}{\lambda e^t+1}\right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \tag{5}$$

($|t| < \pi$ when $\lambda = 1$ and $|t| < |\ln(-\lambda)|$ when $\lambda \neq 1$), $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$.

Substituting $x = 0$ into (5), we have the first kind Apostol-Euler numbers of higher order k:

$$E_n^{(k)}(\lambda) = E_n^{(k)}(0,\lambda)$$

Setting $k = \lambda = 1$ into (5), one has the first kind Euler numbers

$$E_n = E_n^{(1)}(1)$$

(*cf.* [2]-[43]; see also the references cited in each of these earlier works).

A few values of the Euler numbers of the first kind are given as follows:

$$E_0 = 1, E_1 = -\frac{1}{2}, E_3 = \frac{1}{4}$$

and for $n > 0$,

$$E_{2n} = 0.$$

The Euler numbers of the second kind are defined by means of the following generating functions:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}$$

(*cf.* [2]-[43]; see also the references cited in each of these earlier works). From this generating function, One can easily compute the following few values for the numbers E_n^*:

$$E_0^* = 1, E_2^* = -1, E_4^* = 5, E_6^* - 61$$

and for $n \geq 0$,

$$E_{2n+1}^* = 0.$$

One can also easily see that

$$E_n^* = 2^n E_n\left(\frac{1}{2}\right)$$

(*cf.* [25], [32], [40], [41], [42]; and the references cited therein).

The Euler numbers of the second kind of negative order. $E_n^{*(-k)}$ are defined by means of the following generating function:

$$F_{E2}(t,k) = \left(\frac{2}{e^t + e^{-t}}\right)^{-k} = \sum_{n=0}^{\infty} E_n^{*(-k)} \frac{t^n}{n!}, \tag{6}$$

where $|t| < \frac{\pi}{2}$ (*cf.* [3]-[43]; and the references cited therein).

In [29], Šimsek defined the λ-array polynomials $S_v^n(x;\lambda)$ by means of the following generating function:

$$F_A(t,x,v;\lambda) = \frac{(\lambda e^t - 1)^v}{v!} e^{tx} = \sum_{n=0}^{\infty} S_v^n(x;\lambda) \frac{t^n}{n!}, \tag{7}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* see also, [2], [5], [4], [29], [30]; and the references cited therein).

The Bernoulli polynomials of the second kind are defined by means of the following generating function:

$$F_{b2}(t,x) = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \tag{8}$$

(*cf.* [25, pp. 113-117]; and the references cited therein).

The Bernoulli numbers of the second kind $b_n(0)$ are defined by means of the following generating function:

$$F_{b2}(t) = \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n(0) \frac{t^n}{n!}$$

(cf. [25], [19], and the references cited therein). These numbers are computed by the following formula:

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} b_k(0) = n!\delta_{n,1},$$

where $\delta_{n,1}$ denotes the Kronecker delta (*cf.* [25, p. 116]).

The Bernoulli polynomials of the second kind are also defined by the following integral representation:

$$b_n(x) = \int_x^{x+1} (u)_n du.$$

Substituting $x = 0$ into the above equation, one has integral representation of the Bernoulli numbers of the second kind:

$$b_n(0) = \int_0^1 (u)_n du.$$

The Bernoulli numbers of the second kind are also so-called the *Cauchy numbers* (*cf.* [19], [24], [25]; and the references cited therein).

By using the above formula for the Bernoulli numbers of the second kind, few of these numbers are computed as follows:

$$b_0(0) = 1, b_1(0) = \frac{1}{2}, b_2(0) = -\frac{1}{12}, b_3(0) = \frac{1}{24}, b_4(0) = -\frac{19}{720}.$$

1.1. p-adic q-integral

In order to give our new identities, we also need the p-adic q-integral, which defined by Kim [17]. Here p is a fixed prime. The q-Haar distribution on $\mathbb{Z}_p$ is given by

$$\mu_q(x + p^N\mathbb{Z}_p) = \frac{q^x}{[p^N]_q},$$

where $q \in \mathbb{C}_p$ with $\mid 1 - q \mid_p < 1$ (*cf.* [17], [27]). Let $\mathbb{Z}_p$ be a set of p-adic integers. Let $\mathbb{K}$ be a field with a complete valuation and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be a set of continuous derivative functions. That is $C^1(\mathbb{Z}_p \to \mathbb{K})$ is contained in the following set

$$\left\{ f : \mathbb{X} \to \mathbb{K} : f(x) \text{ is differentiable and } \frac{d}{dx} f(x) \text{ is continuous} \right\}.$$

Let $UD\left(\mathbb{Z}_p\right)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. The p-adic q-integral of the function $f \in UD\left(\mathbb{Z}_p\right)$ is defined by Kim [17] as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x,$$

where

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, q \neq 1 \\ x, q = 1 \end{cases}$$

Observe that

$$\lim_{q\to 1} [x] = x.$$

The *bosonic* p-adic Volkenborn integral is given by

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{9}$$

where

$$\mu_1\left(x+p^N\mathbb{Z}_p\right)=\frac{1}{p^N}$$

(*cf.* [26], [17]; see also the references cited in each of these earlier works).

Witt formula for the Bernoulli numbers B_n is given by

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x)=B_n \tag{10}$$

(*cf.* [17], [18], [26]; see also the references cited in each of these earlier works).

The *fermionic* p-adic integral on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1}(-1)^x f(x) \tag{11}$$

where $p\neq 2$ and

$$\mu_1\left(x+p^N\mathbb{Z}_p\right)=\frac{(-1)^x}{p^N}$$

(*cf.* [18]). By using (11), Witt formula for the Euler numbers E_n is given by

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x)=E_n, \tag{12}$$

(*cf.* [18], [13]; see also the references cited in each of these earlier works).

Theorem 1.3.

$$\int_{\mathbb{Z}_p}\binom{x}{j}d\mu_1(x)=\frac{(-1)^j}{j+1}. \tag{13}$$

Theorem 1.3 was proved by Schikhof [26].

Theorem 1.4.

$$\int_{\mathbb{Z}_p}\binom{x}{j}d\mu_{-1}(x)=\frac{(-1)^j}{2^j}. \tag{14}$$

Theorem 1.4 was proved by Kim *et al* [15].

We summarize our results as follows.

In the next section, by using p-adic Volkenborn integral and generating functions and functional equation techniques, we derive some identities and relations including the numbers $y_1(n,k;\lambda)$, $y_2(n,k;\lambda)$, the Stirling numbers, the Bernoulli numbers, the Euler numbers and the λ-array polynomials.

2. Identities

In this section we give some relationships between the numbers $y_1(n,k;\lambda)$, $y_2(n,k;\lambda)$, the λ-Stirling numbers, the central factorial numbers, the array polynomials and the Euler numbers. In order to give our results, we are able to give functional equations and using these equations, we obtain our results.

Theorem 2.1.

$$\sum_{l=0}^{k} \frac{S_2(n,l;\lambda)+(-1)^{k-l+1}y_1(n,l;\lambda)}{(k-l)!}=0.$$

Proof. We set

$$g(t,k;\lambda)=\lambda^k e^{tk}.$$

Combining the above equation with (3) and (1), respectively, we get

$$g(t,k;\lambda)=k!\sum_{j=0}^{k}\frac{1}{(k-j)!}F_S(t,j;\lambda) \tag{15}$$

and

$$g(t,k;\lambda)=k!\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k-j)!}F_{y_1}(t,j;\lambda). \tag{16}$$

Therefore

$$\sum_{n=0}^{\infty}\sum_{l=0}^{k}\frac{S_2(n,l;\lambda)}{(k-l)!}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^{k-l}y_1(n,l;\lambda)}{(k-l)!}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.2.

$$\lambda^{2k}(2k)^n=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}^2(j!)^2\sum_{l=0}^{n}\binom{n}{l}S_2(l,j;\lambda)y_1(n-l,j;\lambda).$$

Proof. Multiplying both sides of the equations (15) and (16), we get the following functional equation:

$$\lambda^{2k}e^{2tk}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}^2(j!)^2F_{y_1}(t,j;\lambda)F_S(t,j;\lambda).$$

Combining the above equation with (3) and (1), we obtain

$$\lambda^{2k}\sum_{n=0}^{\infty}(2k)^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}^2(j!)^2\sum_{l=0}^{n}\binom{n}{l}S_2(l,j;\lambda)y_1(n-l,j;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.3.

$$\sum_{m=0}^{n}(-1)^m S_1(n,m)B_m=n!\sum_{m=0}^{n}m!S_2(n,m)b_m(0).$$

Proof. We set

$$h(t,\lambda) = e^{\lambda t} = \sum_{n=0}^{\infty} \lambda^n \frac{t^n}{n!}$$

and

$$h(t,\lambda) = \sum_{m=0}^{\infty} \binom{\lambda}{m} m! \sum_{n=0}^{\infty} S_2(n,m) \frac{t^n}{n!},$$

where

$$\binom{\lambda}{m} = \frac{\lambda(\lambda-1)\cdots(\lambda-m+1)}{m!} = \frac{(\lambda)_m}{m!}.$$

Combining the above equations, since for $m > n$, $S_2(n,m) = 0$, we get

$$\lambda^n = \sum_{m=0}^{n} \binom{\lambda}{m} m! S_2(n,m) \tag{17}$$

(*cf.* [1], [6], [8]). By applying the Riemann integral the above equation from 0 to 1 with respect to λ, we have

$$\frac{1}{n+1} = \sum_{m=0}^{n} m! S_2(n,m) b_l(0). \tag{18}$$

In ([1], [6], [8], [11], [41], [42]), we see that

$$n! \binom{\lambda}{n} = \sum_{m=0}^{n} (-1)^{n-m} S_1(n,m) \lambda^m. \tag{19}$$

By applying p-adic bosonic integral to the above equation with (13), we get

$$\frac{n!(-1)^n}{n+1} = \sum_{m=0}^{n} (-1)^{n-m} S_1(n,m) B_m. \tag{20}$$

Combining (18) with (20), we obtain

$$\sum_{m=0}^{n} (-1)^m S_1(n,m) B_m = n! \sum_{m=0}^{n} m! S_2(n,m) b_l(0).$$

Thus, the proof of the theorem is completed. □

We are ready to express the following comments on the λ-Bernoulli numbers and polynomials and the λ-Euler numbers and polynomials, which have been studied in different sets. That is, on the set of complex numbers, we assume that $\lambda \in \mathbb{C}$ and on set of p-adic numbers or p-adic integrals, we assume that $\lambda \in \mathbb{Z}_p$.

By applying the fermionic p-adic integral on $\mathbb{Z}_p$ to equation (14) and equation (17) with respect to λ, respectively and using (13), obtain

$$E_n = \sum_{m=0}^{n} (-1)^m \frac{m!}{2^m} S_2(n,m) \tag{21}$$

(*cf.* [14], [16]) and

$$n! \frac{(-1)^n}{2^n} = \sum_{m=0}^{n} (-1)^{n-m} S_1(n,m) E_m. \tag{22}$$

Substituting (21) into (22), we get a relationship between the Stirling numbers of the first and the second kind by the following theorem:

Theorem 2.4.

$$\sum_{m=0}^{n}\sum_{j=0}^{m}(-1)^{j-m}2^{n-j}S_1(n,m)S_2(m,j)j! = n!. \tag{23}$$

Remark 2.5. *By using proof method that of (23), one may be prove inverses relations for the Stirling numbers of the first and second kinds:*

$$\sum_{j=0}^{m} S_1(m,j)S_2(j,k) = \delta_{mk}$$

or

$$\sum_{j=0}^{m} S_2(m,j)S_1(j,k) = \delta_{nk},$$

where δ_{nk} is the Kronecker delta (cf. [1], [6], [8], [11], [41], [42]; and the references cited therein).

Theorem 2.6.

$$y_1(n,k,\lambda) = \frac{\lambda^{\frac{1}{2}}}{k!}\sum_{m=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c}k\\ j\end{array}\right)\left(\begin{array}{c}n\\ m\end{array}\right)j!k^{n-m}2^{n+k-j-m}y_2\left(m,j,\lambda^{\frac{1}{2}}\right).$$

Proof. We set the following functional equation:

$$k!F_{y_1}(t,k,\lambda) = \lambda^{\frac{k}{2}}e^{\frac{t}{2}}\sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right)(-2)^{k-j}j!F_{y_2}\left(t,k,\lambda^{\frac{1}{2}}\right).$$

Combining (1) and (2) with the above functional equation, we get

$$k!\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!} = \lambda^{\frac{k}{2}}\sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right)(-2)^{k-j}j!\sum_{n=0}^{\infty}y_2(j,k;\lambda^{\frac{1}{2}})\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{t^n}{n!}.$$

By using the Cauchy product of the above series on the right-hand side, we obtain

$$\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!} = \frac{\lambda^{\frac{1}{2}}}{k!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c}k\\ j\end{array}\right)\left(\begin{array}{c}n\\ m\end{array}\right)j!k^{n-m}2^{n+k-j-m}y_2\left(m,j,\lambda^{\frac{1}{2}}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

A computation formula for the first kind Apostol-Euler numbers of higher order with aid of the numbers $y_1(n,k;\lambda)$ is given by

$$E_n^{(-k)}(\lambda) = \frac{k!}{2^k}y_1(n,k;\lambda)$$

(*cf.* [31]). The first kind Apostol-Euler numbers of higher order have been also computed by Srivastava [36], Lou and Srivastava [22] and Ozden and Simsek [23]. Therefore, the following convolution formula gives us another computation for the first kind Apostol-Euler numbers of higher order and the numbers $y_1(n,k;\lambda)$.

Theorem 2.7.

$$E_n^{(v-d)}(\lambda) = \frac{d!}{2^d}\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) y_1(m,d;\lambda)E_{n-m}^{(v)}(\lambda)$$

Proof. We set the following functional equation

$$\frac{1}{2^{v-d}}F_{P1}(t,0;v-d,\lambda) = \frac{d!}{2^d}F_{P1}(t,0;v,\lambda)F_{y_1}(t,d,\lambda).$$

Substituting (1) and (5) into the above equation, we obtain

$$\frac{1}{2^{v-d}}\sum_{n=0}^{\infty}E_n^{(v-d)}(\lambda)\frac{t^n}{n!} = \frac{d!}{2^d}\sum_{n=0}^{\infty}E_n^{(v)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_1(n,d;\lambda)\frac{t^n}{n!}.$$

By using the Cauchy product of the above series on the right-hand side, we obtain

$$\frac{1}{2^{v-d}}\sum_{n=0}^{\infty}E_n^{(v-d)}(\lambda)\frac{t^n}{n!} = \frac{d!}{2^d}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) y_1(m,d;\lambda)E_{n-m}^{(v)}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.8.

$$y_1(m,d-v;\lambda) = \frac{v!}{2^v}\left(\begin{array}{c} d \\ v \end{array}\right)\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) y_1(m,d;\lambda)E_{n-m}^{(v)}(\lambda).$$

Proof. We set the following functional equation

$$(d-v)!F_{y_1}(t,d-v,\lambda) = \frac{d!}{2^v}F_{P1}(t,0;v,\lambda)F_{y_1}(t,d,\lambda).$$

By combining (1) and (5) with the above equation, we get

$$(d-v)!\sum_{n=0}^{\infty}y_1(n,d-v;\lambda)\frac{t^n}{n!} = \frac{d!}{2^v}\sum_{n=0}^{\infty}E_n^{(v)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_1(n,d;\lambda)\frac{t^n}{n!}.$$

Therefore

$$(d-v)!\sum_{n=0}^{\infty}y_1(n,d-v;\lambda)\frac{t^n}{n!} = \frac{d!}{2^v}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) y_1(m,d;\lambda)E_{n-m}^{(v)}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2.9.

$$y_2(n,k;-\lambda) = \frac{(-1)^k\lambda^{-k}(2k)!}{4^n}\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right) S_{2k}^{m}\left(-2k,\lambda^{\frac{1}{2}}\right)y_1\left(n-m,2k;\lambda^{\frac{1}{2}}\right).$$

Proof. We set the following functional equation:

$$F_{y_2}(t,k;-\lambda) = (-1)^k\lambda^{-k}(2k)!F_A\left(\frac{t}{2},-2k,2k;\lambda^{\frac{1}{2}}\right)F_{y_2}\left(\frac{t}{2},2k;\lambda^{\frac{1}{2}}\right).$$

By combining (7), (1) and (5) with the above equation, we obtain

$$\sum_{n=0}^{\infty} y_2(n,k;-\lambda)\frac{t^n}{n!} = (-1)^k \lambda^{-k}(2k)! \sum_{n=0}^{\infty} \frac{1}{2^n} S_{2k}^n\left(-2k,\lambda^{\frac{1}{2}}\right)\frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{1}{2^n} y_1\left(n,2k;\lambda^{\frac{1}{2}}\right)\frac{t^n}{n!}.$$

By using the Cauchy product of the above series on the right-hand side, we obtain

$$\sum_{n=0}^{\infty} y_2(n,k;-\lambda)\frac{t^n}{n!} = \frac{(-1)^k \lambda^{-k}(2k)!}{4^n} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} S_{2k}^m\left(-2k,\lambda^{\frac{1}{2}}\right) y_1\left(n-m,2k;\lambda^{\frac{1}{2}}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Acknowledgements The authors would like to thank the reviewers for their comments, which improved the previous version of this paper.

References

[1] M. Bona, *Introduction to Enumerative Combinatorics*, The McGraw-Hill Companies, Inc. New York, 2007

[2] A. Bayad, Y. Simsek and H. M. Srivastava, Some array type polynomials associated with special numbers and polynomials, *Appl. Math. Comput.* 244 (2014), 149-157.

[3] P. F. Byrd, New relations between Fibonacci and Bernoulli numbers, *Fibonacci Quart.* 13 (1975), 111-114.

[4] N. P. Cakić and G. V. Milovanović, On generalized Stirling numbers and polynomials, *Math. Balk.* 18 (2004), 241-248.

[5] C.-H. Chang and C.-W. Ha, A multiplication theorem for the Lerch zeta function and explicit representations of the Bernoulli and Euler polynomials, *J. Math. Anal. Appl.* 315 (2006), 758-767.

[6] C. A. Charalambides, *Ennumerative Combinatorics*, Chapman&Hall/Crc, Press Company, London, New York, 2002.

[7] J. Choi, P. J. Anderson, and H. M. Srivastava, Some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n, and the multiple Hurwitz zeta function, *Appl. Math. Comput.* 199 (2008), 723-737.

[8] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Reidel, Dordrecht and Boston, 1974 (Translated from the French by J. W. Nienhuys).

[9] G. Dattoli and H. M. Srivastava, A note on harmonic numbers, umbral calculus and generating functions, *Appl. Math. Lett.* 21 (2008), 686-693.

[10] R. Dere, Y. Simsek, and H. M. Srivastava, A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, *J. Number Theory* 133 (2013), 3245-3263.

[11] G. B. Djordjević and G. V. Milovanović, *Special classes of polynomials*, University of Nis, Faculty of Technology Leskovac, 2014.

[12] R. Golombek, Aufgabe 1088, *El. Math.* 49 (1994) 126-127.

[13] L. C. Jang and T. Kim, A new approach to q-Euler numbers and polynomials, *J. Concr. Appl. Math.* 6 (2008), 159-168.

[14] L. C. Jang, W.-J. Kim and Y. Simsek, A study on the p-adic integral representation on Z_p associated with Bernstein and Bernoulli polynomials, *Adv. Difference Equ.* 2010 (2010), 1-6, doi:10.1155/2010/163217.

[15] D. S. Kim, T. Kim and J. Seo, A note on Changhee numbers and polynomials, *Adv. Stud. Theor. Phys.* 7 (2013), 993-1003.

[16] D. S. Kim and T. Kim, Some identities of degenerate special polynomials, *Open Math.* 13 (2015), 380-389.

[17] T. Kim, q-Volkenborn integration, *Russian J. Math. Phys.* 19 (2002), 288-299..

[18] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integral and basic q-zeta function, *Trend Math. Information Center Math. Sciences* 9 (2006), 7-12.

[19] T. Komatsu, Convolution identities for Cauchy numbers, *Acta Math. Hungar.* 144 (2014), 76-91.

[20] D.-Q. Lu and H. M. Srivastava, Some series identities involving the generalized Apostol type and related polynomials, *Comput. Math. Appl.* 62 (2011), 3591-3602.

[21] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, *J. Math. Anal. Appl.* 308 (2005), 290-302.

[22] Q. M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, *Appl. Math. Comput.* 217 (2011), 5702-5728.

[23] H. Ozden and Y. Simsek, Modification and unification of the Apostol-type numbers and polynomials and their applications, *Appl. Math. Comput.* 235 (2014), 338-351.

[24] F. Qi, Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind, *Filomat* 28(2) (2014), 319-327.

[25] S. Roman, *The Umbral Calculus*, Dover Publ. Inc., New York, 2005.

[26] W. H. Schikhof, *Ultrametric Calculus: An Introduction to p-Adic Analysis*, Cambridge Studies in Advanced Mathematics 4, Cambridge University Press Cambridge, 1984.

[27] Y. Simsek and H. M. Srivastava, A family of p-adic twisted interpolation functions associated with the modified Bernoulli numbers, Appl. Math. Comput. 216 (2010), 2976-2987.

[28] Y. Simsek, Special functions related to Dedekind-type DC-sums and their applications, *Russian J. Math. Phys.* 17 (4) (2010), 495-508.

[29] Y. Simsek, Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their alications, *Fixed Point Theory Appl.* 87 (2013), 343-1355.
[30] Y. Simsek, Special numbers on analytic functions, *Applied Math.* 5 (2014), 1091-1098.
[31] Y. Simsek, New families of special numbers for computing negative order Euler numbers and related numbers and polynomials, to appear *Appl. Anal. Discrete Math.*, arXiv:1604.05601v1.
[32] Y. Simsek, Computation methods for combinatorial sums and Euler type numbers related to new families of numbers, to appear *Math. Meth. Appl. Sci.* (2017), DOI: 10.1002/mma.4143.
[33] Y. Simsek, Combinatorial applications of the special numbers and polynomials, *Permutation Patterns 2016, 14th International Conference on Permutation Patterns*, Howard University, Washington, DC, USA, June 27-July 1, (2016), 116-120.
[34] M. Z. Spivey, Combinatorial Sums and Finite Differences, *Discrete Math.* 307(24) (2007), 3130-3146.
[35] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.* 129 (2000), 77-84.
[36] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inform. Sci.* 5 (2011), 390-444.
[37] H. M. Srivastava, T. Kim and Y. Simsek, q-Bernoulli numbers and polynomials associated with q-zeta functions and basic L-series, *Russian J. Math. Phys.* 12, (2005), 241-268.
[38] H. M. Srivastava, M. A. Özarslan, C. Kaanoglu, Some generalized Lagrange-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, *Russian J. Math. Phys.* 20 (2013), 110-120.
[39] H. M. Srivastava, K. S. Nisar, and M. A. Khan, Some umbral calculus presentations of the Chan-Chyan-Srivastava polynomials and the Erkus-Srivastava polynomials, *Proyecciones J. Math.* 33 (2014), 77-90.
[40] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Limited Publisher, Chichester, 1984.
[41] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.
[42] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers: Amsterdam, London and New York, 2012.
[43] H. M. Srivastava and G.-D. Liu, Some identities and congruences involving a certain family of numbers, *Russian J. Math. Phys.* 16 (2009), 536-542.