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On the g-Hypergroupoids Associated with g-Hypergraphs

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Abstract. In this paper, we associate a partial g-hypergroupoid with a given g-hypergraph and analyze the properties of this hyperstructure. We prove that a g-hypergroupoid may be a commutative hypergroup without being a join space. Next, we define diagonal direct product of g-hypergroupoids. Further, we construct a sequence of g-hypergroupoids and investigate some relationships between it's terms. Also, we study the quotient of a g-hypergroupoid by defining a regular relation. Finally, we describe fundamental relation of an H_v-semigroup as a g-hypergroupoid.

1. Introduction and Preliminaries

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, let $P(X)$ be the set of all subsets of a given set X. A *partial hypergroupoid* is a pair $(X, *)$, where X is a non-empty set and $*$ is a *partial hyperoperation*, i.e.,

$$* : X \times X \longrightarrow P(X), \quad (x, y) \mapsto x * y.$$

Every map from $X \times X$ to $P^*(X)$ is called a *hyperoperation*, where $P^*(X) = P(X) - \{\emptyset\}$. If $A, B \in P^*(X)$, then we define $A * B = \bigcup \left\{a * b \mid a \in A, b \in B\right\}$, $x * B = \{x\} * B$ and $A * y = A * \{y\}$. If $A = \emptyset$ or $B = \emptyset$ we define $A * B = \emptyset$. A partial hypergroupoid $(X, *)$ is called a *hypergroupoid* if $*$ is a hyperoperation. A hypergroupoid $(X, *)$ is called a *semihypergroup* if the associative axiom is valid, i.e., $x * (y * z) = (x * y) * z$, for all $x, y, z \in X$ and it is called *reproductive* if $x * X = X * x = X$, for all $x \in X$. A *hypergroup* is a reproductive semihypergroup. A commutative hypergroup $(X, *)$ (i.e., $x * y = y * x$ for all $x, y \in X$) is called a *join space* if the following implication holds for all elements a, b, c, d of X:

$$a/b \bigcap c/d \neq \emptyset \Rightarrow a * d \bigcap b * c \neq \emptyset,$$

where $a/b = \{x \mid a \in x * b\}$.

H_v-structures which satisfy the corresponding structure-like axioms are the largest class of algebraic hyperstructures. The notion of H_v-structures has been introduced by Vougiouklis [13] as a generalization

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of well-known algebraic hyperstructures (semihypergroups, hypergroups, hyperrings and so on) which satisfy the weak axioms where the non-empty intersection replaces the equality. A comprehensive review of the theory of H_v-structures appears in [1, 5, 6]. A hypergroupoid $(X, *)$ is called an *H_v-semigroup* if the weak associative axiom is valid, i.e.,

$$x * (y * z) \bigcap (x * y) * z \neq \emptyset \ , \text{ for all } x, y, z \in X$$

and it is called an *H_v-group* if it is a reproductive H_v-semigroup.

Let X be a non-empty set. By an *h-relation* R on X we mean a subset of $X \times P^*(X)$. The domain of R is the set $Dom(R) = \Big\{x \in X \mid (x, A) \in R \text{ for some } A \in P(X)\Big\}$ and codomain of R is the set $Cod(R) = \Big\{A \in P(X) \mid (x, A) \in R \text{ for some } x \in X\Big\}$. Also, for any $x \in X$, we define $x_R = \{A \mid (x, A) \in R\}$.

The notion of hypergraph has been introduced around 1960 as a generalization of graph and one of the initial concerns was to extend some classical results of graph theory. In [2], there is a very good presentation of graph and hypergraph theory. Connections between hypergraphs and hyperstructures are studied by many authors, for example, see [4, 8, 10, 11]. A *hypergraph* is a pair $\Gamma = (X, A)$, where X is a finite set of vertices and $A = \{A_1, \ldots, A_m\}$ is a set of hyperedges which are non-empty subsets of X. Figure 1 is an example of a hypergraph with 2 hyperedges $A_1 = \{1, 2, 3\}$ and $A_2 = \{2, 3, 4\}$.

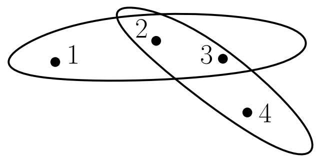


Figure 1: An example of hypergraph with 2 hyperedges.

2. Partial g-Hypergroupoids

In this section we generalize the notion of hypergraphs to generalized hypergraphs and then we associate a partial hypergroupoid to each generalized hypergraph.

Definition 2.1. *[12] A generalized hypergraph or, in short, a g-hypergraph is an ordered pair $\mathcal{G} = (X, R)$, where X is a non-empty set and R is an h-relation on X. The elements of X are called the vertices and the sets in $\mathcal{E} = Cod(R)$ are called the hyperedges of the g-hypergraph.*

It is worth mentioning that in this paper we deal only with g-hypergraphs $\mathcal{G} = (X, R)$ in which X is a finite set. A g-hypergraph $\mathcal{G} = (X, R)$ is called *v-linked* if $x_R \neq \emptyset$, for all $x \in X$ and it is called *plenary* if $\bigcup_{A \in Cod(R)} A = X$.

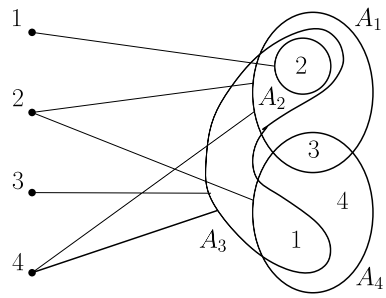


Figure 2: An example of a g-hypergraph.

Let $\mathcal{G} = (X, R)$ be a g-hypergraph. The partial hypergroupoid $X_{\mathcal{G}} = (X, \circ)$ where the partial hyperoperation $\circ$ is defined by

$$x \circ y = \mathcal{N}(x) \bigcup \mathcal{N}(y), \text{ for all } (x, y) \in X^2,$$

is called the *partial g-hypergroupoid* associated with $\mathcal{G}$, where $\mathcal{N}(x) = \bigcup\limits_{(x,A)\in R} A$. In the case that $\circ$ is a hyperoperation, $X_{\mathcal{G}}$ is called a *g-hypergroupoid*.

Lemma 2.2. *$X_{\mathcal{G}}$ is a g-hypergroupoid if and only if $\mathcal{G}$ is v-linked.*

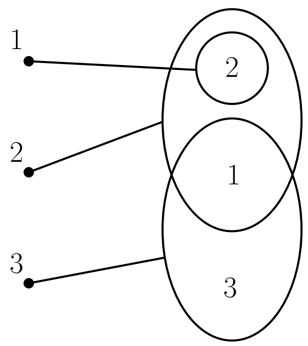
Proof. It is obvious. □

Remark 2.3. *In [4], Corsini associated to a given hypergraph $\Gamma = (H, \{A_i\}_i)$ an h.g. hypergroupoid $H_\Gamma = (H, \circ)$ where the hyperoperation $\circ$ has defined as follows:*

$$x \circ y = E(x) \bigcup E(y), \text{ for all } x, y \in H^2,$$

where $E(x) = \bigcup\limits_{x\in A_i} A_i$. Let $\Gamma = (H, \{A_i\}_i)$ be a hypergraph. If we define the h-relation $R = \left\{(x, A_i) \mid x \in A_i\right\}$ on H, then Γ becomes a v-linked and plenary g-hypergraph. Thus, every hypergraph can be considered as a g-hypergraph and there is no difference between h.g. hypergroupoids and g-hypergroupoids when we deal with hypergraphs. In other words, each h.g. hypergroupoid can be considered as a g-hypergroupoid. As we will see, h.g. hypergroupoids does not coincide with g-hypergroupoids. For example, by Theorem 3 of [4], each h.g. hypergroupoid is a join space whereas there are g-hypergroupoids which are not join spaces (see Example 2.4).

Example 2.4. *Consider the following g-hypergraph and the table of it's associated g-hypergroupoid:*

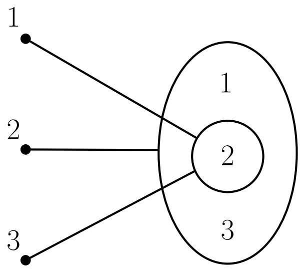


$\circ$	1	2	3
1	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
2	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
3	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,3\}$

It is not difficult to see that $(X = \{1, 2, 3\}, \circ)$ is a hypergroup. We have $1/3 = \{1, 2, 3\}$ and $3/1 = \{3\}$. It implies that $1/3 \bigcap 3/1 \neq \emptyset$, but $1 \circ 1 \bigcap 3 \circ 3 = \emptyset$. Hence $(X, \circ)$ is not a join space.

Here, we give an example of a g-hypergraph such that it's associated g-hypergroup is a join space.

Example 2.5. *In the following, we have drawing a g-hypergraph $\mathcal{G}$ and the table of the g-hypergroupoid associated with $\mathcal{G}$:*



$\circ$	1	2	3
1	$\{2\}$	$\{1,2,3\}$	$\{2\}$
2	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$
3	$\{2\}$	$\{1,2,3\}$	$\{2\}$

One can check that $(X,\circ)$ is a hypergroup. On the other hand, we have $x\circ y\bigcap z\circ w\neq\emptyset$, for all $x,y,z,w\in X$. This implies that $(X,\circ)$ is a join space.

Definition 2.6. *A partial hypergroupoid $(X,\circ)$ is called separable if the following property holds:*

$$x\circ y = x\circ x\bigcup y\circ y, \text{ for all } x,y\in X.$$

Remark 2.7. *Let $(X,\circ)$ be a separable hypergroupoid. Define $R=\Big\{(x,x\circ x)\mid x\in X\Big\}$. Then, $(X,\circ)$ is the g-hypergroupoid associated with the v-linked g-hypergraph $\mathcal{G}=(X,R)$. Therefore, every separable hypergroupoid can be considered as a g-hypergroupoid.*

The next lemma can be proved easily by using the previous notions.

Lemma 2.8. *Let $(X,\circ)$ be a partial g-hypergroupoid. Then, for all $x,y\in X$ and $A\subseteq X$ we have*

(1) $x\circ y = y\circ x$,

(2) $(x\circ x)\circ(x\circ x)=\bigcup\limits_{t\in x\circ x} t\circ t$,

(3) $(A\circ A)\circ(A\circ A)=\bigcup\limits_{t\in A\circ A} t\circ t$.

Lemma 2.9. *Let $(X,\circ)$ be a separable hypergroupoid. Then*

(1) for each $x,y,z\in X$ we have
$(x\circ y)\circ z=\Big[(x\circ x)\circ(x\circ x)\Big]\bigcup z\circ z\bigcup\Big[(y\circ y)\circ(y\circ y)\Big]$,
$x\circ(y\circ z)=\Big[(y\circ y)\circ(y\circ y)\Big]\bigcup x\circ x\bigcup\Big[(z\circ z)\circ(z\circ z)\Big]$.

(2) $(X,\circ)$ is an H_v-semigroup.

Proof. (1) For each $x,y,z\in X$ we have

$$(x\circ y)\circ z=(x\circ x\bigcup y\circ y)\circ z=(x\circ x)\circ z\bigcup(y\circ y)\circ z,$$

and

$$x\circ(y\circ z)=(y\circ z)\circ x=(y\circ y\bigcup z\circ z)\circ x=(y\circ y)\circ x\bigcup(z\circ z)\circ x.$$

Moreover,

$$(x\circ x)\circ z=\bigcup_{t\in x\circ x} t\circ z=\Big(\bigcup_{t\in x\circ x} t\circ t\Big)\bigcup z\circ z=\Big[(x\circ x)\circ(x\circ x)\Big]\bigcup z\circ z.$$

Therefore, we have

$$(x\circ y)\circ z=\Big[(x\circ x)\circ(x\circ x)\Big]\bigcup z\circ z\bigcup\Big[(y\circ y)\circ(y\circ y)\Big]$$

and

$$x\circ(y\circ z)=\Big[(y\circ y)\circ(y\circ y)\Big]\bigcup x\circ x\bigcup\Big[(z\circ z)\circ(z\circ z)\Big].$$

(2) We have $\emptyset\neq\Big[(y\circ y)\circ(y\circ y)\Big]\subseteq(x\circ y)\circ z\bigcap x\circ(y\circ z)$. This completes the proof. $\square$

Notice that every partial g-hypergroupoid is separable and so we have the following corollary.

Corollary 2.10. *Every g-hypergroupoid is an H_v-semigroup.*

Corollary 2.11. *A partial g-hypergroupoid $X_{\mathcal{G}}$ is an H_v-semigroup if and only if $\mathcal{G}$ is v-linked.*

Theorem 2.12. *Let $\mathcal{G} = (X, R)$ be a v-linked g-hypergraph. Then, the g-hypergroupoid $X_{\mathcal{G}} = (X, \circ)$ is an H_v-group if and only if $\mathcal{G}$ is plenary.*

Proof. Suppose that $X_{\mathcal{G}} = (X, \circ)$ is an H_v-group. It suffices to show that $X \subseteq \bigcup\limits_{A \in Cod(R)} A$. Let $x \in X$ be an arbitrary element. By assumption, we have $x \circ X = X$ and so there is $y \in X$ such that $x \in x \circ y = \mathcal{N}(x) \bigcup \mathcal{N}(y)$. Thus, there is $A \in Cod(R)$ such that $x \in A \subseteq \bigcup\limits_{A \in Cod(R)} A$.

Conversely, let $\mathcal{G}$ be plenary and $x \in X$ be an arbitrary element. By Corollary 2.10, it is sufficient to show that $x \circ X = X \circ x = X$. It is obvious that $x \circ X \subseteq X$. We show that $X \subseteq x \circ X$. Since $\mathcal{G}$ is plenary, if $z \in X$ is an arbitrary element, then there is $A \in Cod(R)$ such that $z \in A$. Since $A \in Cod(R)$, it follows that there is $y \in X$ such that $(y, A) \in R$ and so we have $z \in x \circ y \subseteq x \circ X$. This implies that $X \subseteq x \circ X$ and so $x \circ X = X$. Clearly $X \circ x = X$ since $\circ$ is commutative. Therefore, $X_{\mathcal{G}}$ is an H_v-group. □

Corollary 2.13. *$X_{\mathcal{G}}$ is a reproductive g-hypergroupoid if and only if $\mathcal{G}$ is v-linked and plenary.*

Theorem 2.14. *Let $(X, \circ)$ be a separable hypergroupoid. Then, $\circ$ is associative if and only if the following conditions hold:*

(1) $x \circ x \subseteq (x \circ x) \circ (x \circ x)$, *for all* $x \in X$,

(2) $\Big[(x \circ x) \circ (x \circ x)\Big] - x \circ x \subseteq (y \circ y) \circ (y \circ y)$, *for all* $x, y \in X$.

Proof. Suppose that $\circ$ is associative and x, y are arbitrary elements of X. First, we show that $x \circ x \subseteq (x \circ x) \circ (x \circ x)$. Suppose that $x \circ x = \{x_1, \ldots, x_n\}$ and $x_i \in x \circ x$ is an arbitrary element. Since $x \circ x_i = x \circ x \bigcup x_i \circ x_i$, it follows that $x_i \in x \circ x_i$ and so $x_i \in x \circ (x \circ x_i)$. Associativity of $\circ$ implies that

$$x_i \in (x \circ x) \circ x_i = x_1 \circ x_1 \bigcup \ldots \bigcup x_n \circ x_n = (x \circ x) \circ (x \circ x).$$

Thus (1) holds. Now, to prove the condition (2) we have

$$\begin{aligned}(y \circ y) \circ x &= \bigcup_{t \in y \circ y} t \circ x = \Big(\bigcup_{t \in y \circ y} t \circ t\Big) \bigcup x \circ x = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup x \circ x,\\ y \circ (y \circ x) &= \bigcup_{t \in y \circ x} y \circ t = \bigcup_{t \in y \circ x} (y \circ y \bigcup t \circ t) = y \circ y \bigcup \Big(\bigcup_{t \in y \circ y} t \circ t\Big) \bigcup \Big(\bigcup_{t \in x \circ x} t \circ t\Big)\\ &= \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup \Big[(x \circ x) \circ (x \circ x)\Big].\end{aligned}$$

Consequently, (2) holds.

Conversely, suppose that x, y, z are arbitrary elements of X and the conditions (1) and (2) hold. From point (1) of Lemma 2.9, we have

$$(x \circ y) \circ z = \Big[(x \circ x) \circ (x \circ x)\Big] \bigcup z \circ z \bigcup \Big[(y \circ y) \circ (y \circ y)\Big],$$

and

$$x \circ (y \circ z) = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup x \circ x \bigcup \Big[(z \circ z) \circ (z \circ z)\Big].$$

By setting $A = \Big[(x \circ x) \circ (x \circ x)\Big] \bigcup z \circ z$ and $B = \Big[(z \circ z) \circ (z \circ z)\Big] \bigcup x \circ x$ we have $(x \circ y) \circ z = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup A$ and $x \circ (y \circ z) = \Big[(y \circ y) \circ (y \circ y)\Big] \bigcup B$. By using the conditions (1) and (2) we have

$$\begin{aligned} A &= \Big(\Big[(x \circ x) \circ (x \circ x)\Big] - x \circ x\Big) \bigcup x \circ x \bigcup z \circ z\\ &\subseteq \Big[(z \circ z) \circ (z \circ z)\Big] \bigcup z \circ z \bigcup x \circ x\\ &= \Big[(z \circ z) \circ (z \circ z)\Big] \bigcup x \circ x = B.\end{aligned}$$

In a similar way the inverse inclusion is proved and then $\circ$ is associative. □

Theorem 2.15. *Let $(X, \circ)$ be a separable hypergroupoid. Then, $\circ$ is associative if and only if the following conditions hold:*

(1) $A \circ A \subseteq (A \circ A) \circ (A \circ A)$, *for all* $A \subseteq X$,

(2) $\Big[(A \circ A) \circ (A \circ A)\Big] - A \circ A \subseteq (B \circ B) \circ (B \circ B)$, *for all* $A, B \subseteq X$.

Proof. Suppose that $\circ$ is associative and A, B are arbitrary subsets of X. Then, by using Theorem 2.14 we have

$$\begin{aligned} A \circ A &= \bigcup_{a \in A} a \circ a \subseteq \bigcup_{a \in A} (a \circ a) \circ (a \circ a) = \bigcup_{a \in A} \Big(\bigcup_{t \in a \circ a} t \circ t \Big) = \bigcup_{t \in A \circ A} t \circ t \\ &= (A \circ A) \circ (A \circ A). \end{aligned}$$

Hence, (1) is true. For every $b \in B$ we have

$$\Big[(A \circ A) \circ (A \circ A)\Big] - A \circ A \subseteq \bigcup_{a \in A} \Big[\Big((a \circ a) \circ (a \circ a)\Big) - a \circ a\Big] \subseteq (b \circ b) \circ (b \circ b).$$

On the other hand, we have $(b \circ b) \circ (b \circ b) \subseteq (B \circ B) \circ (B \circ B)$. Hence, the assertion (2) holds too.

Conversely, suppose that the assertions (1) and (2) hold for all subsets A and B of X. Let x, y be arbitrary elements of X. By setting $A = \{x\}$ and $B = \{y\}$, the assertions (1) and (2) of Theorem 2.14 hold and therefore $\circ$ is associative. □

Corollary 2.16. *If a reproductive g-hypergroupoid $X_{\mathcal{G}} = (X, \circ)$ satisfies anyone of the following conditions:*

$$\begin{aligned} &(x \circ x) \circ (x \circ x) = x \circ x, \text{ for all } x \in X, \\ &(x \circ x) \circ (x \circ x) = X, \text{ for all } x \in X, \end{aligned}$$

then it is a hypergroup.

Example 2.17. *The g-hypergroupoid associated with the g-hypergraph of Figure 2 has the following table:*

$\circ$	1	2	3	4
1	$\{2\}$	X	$\{1,2\}$	$\{1,2,3\}$
2	X	X	X	X
3	$\{1,2\}$	X	$\{1,2\}$	$\{1,2,3\}$
4	$\{1,2,3\}$	X	$\{1,2,3\}$	$\{1,2,3\}$

where $X = \{1, 2, 3, 4\}$. It is easy to verify that $(x \circ x) \circ (x \circ x) = X$, for all $x \in X$. On the other hand, for every $x \in X$ we have $x \circ X = X \circ x = X$. So, by Corollary 2.16, $(X, \circ)$ is a hypergroup.

Example 2.18. *Consider the g-hypergroupoid of Example 2.4. By Theorem 2.14, $(X, \circ)$ is a hypergroup. Also, we have $(1 \circ 1) \circ (1 \circ 1) = \{1, 2\}$. This shows that the converse of Corollary 2.16 is not true.*

3. Higher-order Hypergroupoids

Let $(X, \circ)$ be a separable hypergroupoid. We construct a sequence of hypergroupoids $X_0 = (X, \circ_0), X_1 = (X, \circ_1), X_2 = (X, \circ_2), \ldots$ recursively as follows: for all $x, y \in X$ we define $x \circ_0 y = x \circ y$, $x \circ_{k+1} x = (x \circ_k x) \circ_k (x \circ_k x)$ and $x \circ_{k+1} y = x \circ_{k+1} x \bigcup y \circ_{k+1} y$, where $k \geq 0$. Set $\mathcal{N}_k(x) = x \circ_k x$. We define $\mathcal{N}_k(A) = \bigcup_{a \in A} \mathcal{N}_k(a)$, where A is a subset of X. The following properties are immediate:

(1) $\mathcal{N}_k(A) = A \circ_k A$, for all $A \subseteq X$,

(2) $\mathcal{N}_{k+1}(x) = \mathcal{N}_k(\mathcal{N}_k(x))$, for all $x \in X$ and $k \geq 0$,

(3) $\mathcal{N}_k(\mathcal{N}_{k+1}(x)) = \mathcal{N}_{k+1}(\mathcal{N}_k(x))$, for all $x \in X$ and $k \geq 0$,

(4) $\mathcal{N}_{k+1}(A) = \mathcal{N}_k(\mathcal{N}_k(A))$, for all $A \subseteq X$ and $k \geq 0$,

(5) $A \subseteq B$ implies that $\mathcal{N}_k(A) \subseteq \mathcal{N}_k(B)$, for all $A, B \subseteq X$,

(6) $\mathcal{N}_k(x) = \mathcal{N}_{k+1}(x)$ implies that $\mathcal{N}_k(x) = \mathcal{N}_r(x)$, for all $r \geq k$.

By Theorem 2.14, X_k is a semihypergroup if and only if the following conditions hold:

(α) $\mathcal{N}_k(x) \subseteq \mathcal{N}_{k+1}(x)$, for all $x \in X$,

(β) $\mathcal{N}_{k+1}(x) - \mathcal{N}_k(x) \subseteq \mathcal{N}_{k+1}(y)$, for all $x, y \in X$.

Lemma 3.1. *The above hyperoperation $\circ_k$ has the following properties:*

(1) $A \circ_{k+1} A = (A \circ_k A) \circ_k (A \circ_k A)$, for all $A \subseteq X$,

(2) $x \circ_{k+2} x = \big((x \circ_{k+1} x) \circ_k (x \circ_{k+1} x)\big) \circ_k \big((x \circ_{k+1} x) \circ_k (x \circ_{k+1} x)\big)$, for all $x \in X$.

Proof. (1) Let A be a subset of X. Then,

$$A \circ_{k+1} A = \mathcal{N}_{k+1}(A) = \mathcal{N}_k(\mathcal{N}_k(A)) = \mathcal{N}_k(A) \circ_k \mathcal{N}_k(A) = (A \circ_k A) \circ_k (A \circ_k A).$$

(2) The result follows from part (1) and the definition of $\circ_{k+2}$. □

Theorem 3.2. *Let $(X, \circ)$ be a separable hypergroupoid.*

(1) If $X_k = (X, \circ_k)$ satisfies condition (α) for some $k \geq 0$, then $\mathcal{N}_r(x) \subseteq \mathcal{N}_{r+1}(x)$, for all $x \in X$ and $r \geq k$.

(2) If $X_k = (X, \circ_k)$ satisfies condition (β) for some $k \geq 0$, then $\mathcal{N}_{r+1}(x) \subseteq \mathcal{N}_r(x)$, for all $x \in X$ and $r > k$.

Proof. (1) Let $x \in X$ be an arbitrary element. We prove the result by induction on r. If $r = k$, then there is nothing to prove. Assume that $\mathcal{N}_{r-1}(x) \subseteq \mathcal{N}_r(x)$ for $r > k$, the induction hypothesis. Thus we have

$$\begin{aligned}\mathcal{N}_r(x) &= \mathcal{N}_{r-1}(\mathcal{N}_{r-1}(x)) \subseteq \mathcal{N}_{r-1}(\mathcal{N}_r(x)) = \mathcal{N}_r(\mathcal{N}_{r-1}(x)) \\ &\subseteq \mathcal{N}_r(\mathcal{N}_r(x)) = \mathcal{N}_{r+1}(x).\end{aligned}$$

(2) Let $x \in X$ be an arbitrary element. First, we show that $\mathcal{N}_k(\mathcal{N}_{k+1}(x)) \subseteq \mathcal{N}_{k+1}(x)$. Assume to the contrary that $t \in \mathcal{N}_k(\mathcal{N}_{k+1}(x)) - \mathcal{N}_{k+1}(x)$. Then, $t \notin \mathcal{N}_{k+1}(x)$ and there is $a \in \mathcal{N}_{k+1}(x)$ such that $t \in \mathcal{N}_k(a)$. Since $a \in \mathcal{N}_{k+1}(x)$, it follows that there is $b \in \mathcal{N}_k(x)$ such that $a \in \mathcal{N}_k(b)$ and so $t \in \mathcal{N}_k(\mathcal{N}_k(b)) = \mathcal{N}_{k+1}(b)$. On the other hand, $t \notin \mathcal{N}_{k+1}(x)$ implies that $t \notin \mathcal{N}_k(b)$ and so $t \in \mathcal{N}_{k+1}(b) - \mathcal{N}_k(b)$. By hypothesis we have $\mathcal{N}_{k+1}(b) - \mathcal{N}_k(b) \subseteq \mathcal{N}_{k+1}(x)$ which implies that $t \in \mathcal{N}_{k+1}(x)$ contradicting to $t \notin \mathcal{N}_{k+1}(x)$. Now, we prove the result by induction on r. We have $\mathcal{N}_{k+2}(x) = \mathcal{N}_k(\mathcal{N}_k(\mathcal{N}_{k+1}(x))) \subseteq \mathcal{N}_k(\mathcal{N}_{k+1}(x)) \subseteq \mathcal{N}_{k+1}(x)$. So, we are done with the initial step. Assume that $\mathcal{N}_{r+1}(x) \subseteq \mathcal{N}_r(x)$ for $r > k$, the induction hypothesis. We obtain

$$\begin{aligned}\mathcal{N}_{r+2}(x) &= \mathcal{N}_{r+1}(\mathcal{N}_{r+1}(x)) \subseteq \mathcal{N}_{r+1}(\mathcal{N}_r(x)) = \mathcal{N}_r(\mathcal{N}_{r+1}(x)) \\ &\subseteq \mathcal{N}_r(\mathcal{N}_r(x)) = \mathcal{N}_{r+1}(x).\end{aligned}$$

□

Corollary 3.3. *If $(X, \circ_k)$ is a separable semihypergroup, then $\mathcal{N}_r(x) = \mathcal{N}_{r+1}(x)$, for all $x \in X$ and $r > k$.*

Corollary 3.4. *If $(X, \circ_k)$ is a separable semihypergroup, then $\mathcal{N}_r(A) = \mathcal{N}_{r+1}(A)$, for all $A \subseteq X$ and $r > k$.*

Next example shows that the converse of Corollary 3.3 is not true.

Example 3.5. *Let $(X = \{1, 2, 3\}, \circ)$ be a hypergroupoid with the following table:*

$\circ$	1	2	3
1	$\{2\}$	X	$\{2\}$
2	X	$\{1,3\}$	X
3	$\{2\}$	X	$\{2\}$

We can see that $\mathcal{N}_0(1) = \{2\}$ and $\mathcal{N}_1(1) = \{1, 3\}$. Since $\mathcal{N}_0(x) \not\subseteq \mathcal{N}_1(x)$ by Theorem 2.14, $(X, \circ)$ is not a semihypergroup. We can check that $\mathcal{N}_k(x) = \mathcal{N}_{k+1}(x)$, for all $x \in X$ and $k > 0$. This means that the converse of Corollary 3.3 is not true.

Proposition 3.6. *If there exists a natural number k such that $\mathcal{N}_k(x) = \mathcal{N}_{k+1}(x)$, for all $x \in X$, then*

(1) $X_k = (X, \circ_k)$ is a semihypergroup,

(2) $X_r = X_k$, for all $r \geq k$.

At the beginning of this section, we construct a sequence of separable hypergroupoids $X_0 = (X, \circ_0), X_1 = (X, \circ_1), X_2 = (X, \circ_2), \dots$ by a given separable hypergroupoid $(X, \circ)$. Let $\mathcal{G}$ be a v-linked g-hypergraph and $X_{\mathcal{G}}$ be the g-hypergroupoid associated with $\mathcal{G}$. Set $X_0 = X_{\mathcal{G}}$. For $k > 0$, we define an h-relation R_k on X as follows:

$$R_k = \Big\{(x, x \circ_k x) \mid x \in X\Big\}$$

and therefore we have a sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots$ of g-hypergraphs where $\mathcal{G}_0 = \mathcal{G}$ and $\mathcal{G}_k = (X, R_k)$, for $k > 0$. It is easy to verify that X_k is the g-hypergroupoid associated with $\mathcal{G}_k$. Now, by Corollary 3.3 and Proposition 3.6 we conclude that if X_k is an associative g-hypergroupoid, then $\mathcal{G}_r = \mathcal{G}_{k+1}$ and $X_r = X_k$, for all $r > k$. For a given g-hypergraph $\mathcal{G}$ we define

$$n(\mathcal{G}) = \min\Big\{k \mid \mathcal{N}_k(x) = \mathcal{N}_{k+1}(x) \text{ for all } x \in X\Big\}$$

and

$$s(\mathcal{G}) = \min\Big\{k \mid X_k \text{ is a semihypergroup}\Big\}.$$

Obviously, $s(\mathcal{G}) \leq n(\mathcal{G})$. Consider the g-hypergraph $\mathcal{G}$ of Figure 2. In Example 2.17 we showed that $X_{\mathcal{G}}$ is a hypergroup and so we have $s(\mathcal{G}) = 0$ whereas $n(\mathcal{G}) = 1$. This means that the inequality $s(\mathcal{G}) \leq n(\mathcal{G})$ may be hold strictly.

4. Quotient g-Hypergroupoids

In this section, by considering a regular equivalence relation on a g-hypergroupoid, we define a quotient g-hypergroupoid. Next, we investigate some relationships between diagonal direct product of hypergroupoids and direct product of g-hypergraphs. In this regards we recall some definitions and results which we need for the development of the rest of paper.

Let $(X, *)$ be a hypergroupoid and ρ be an equivalence relation on X. If A and B are non-empty subsets of X, then $A\overline{\rho}B$ means that for all $a \in A$, there exists $b \in B$ such that $a\rho b$ and for all $b' \in B$ there exists $a' \in A$ such that $a'\rho b'$. We say that ρ is *regular* if for all $a \in X$ from $x\rho\, y$, it follows that $(a * x)\overline{\rho}(a * y)$ and $(x * a)\overline{\rho}(y * a)$. For an equivalence relation ρ on X, we may use $\rho(x)$ to denote the equivalence class of $x \in X$. Moreover, generally, if A is a non-empty subset of X, then $\rho(A) = \bigcup\{\rho(x) \mid x \in A\}$. Let X/ρ be the family $\{\rho(x) \mid x \in X\}$ of classes of ρ. By Theorem 2.5.2 of [6], if $(X, *)$ is a hypergroupoid and ρ is a regular equivalence relation on X, then the following hyperoperation on X/ρ is well defined:

$$\rho(x) \odot \rho(y) = \Big\{\rho(z) \mid z \in x * y\Big\}.$$

Let $\mathcal{G} = (X, R)$ be a v-linked g-hypergraph and $(X, \circ)$ be the g-hypergroupoid associated with $\mathcal{G}$. We define the relation $\rho_{\mathcal{G}}$ on X as follows:

$$x\, \rho_{\mathcal{G}}\, y \text{ if and only if } x_R = y_R.$$

Lemma 4.1. *The relation $\rho_{\mathcal{G}}$ is a regular equivalence relation.*

Proof. Obviously, $\rho_{\mathcal{G}}$ is an equivalence relation. Let $z \in X$ be an arbitrary element and $x\rho_{\mathcal{G}} y$. First, we show that $x \circ z = y \circ z$ which implies that $(x \circ z)\overline{\rho}_{\mathcal{G}}(y \circ z)$. Let $r \in x \circ z = \mathcal{N}(x) \bigcup \mathcal{N}(z)$ be an arbitrary element. In the case that $r \in \mathcal{N}(z)$, there is nothing to prove. If $r \in \mathcal{N}(x)$, then there is a hyperedge A such that $(x, A) \in R$ and $r \in A$. By assumption, we have $x_R = y_R$ and therefore we have $(y, A) \in R$. This implies that $r \in \mathcal{N}(y)$. Hence $x \circ z \subseteq y \circ z$. The reverse inclusion can be shown similarly. In a similar way we can show that $(z \circ x)\overline{\rho}_{\mathcal{G}}(z \circ y)$. □

Definition 4.2. *Let $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$ be two g-hypergraphs. Then, the direct product of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the g-hypergraph $\mathcal{G}_1 \times \mathcal{G}_2 = (X_1 \times X_2, R_1 \times R_2)$ where $R_1 \times R_2 = \Big\{\Big((x, y), A \times B\Big) \mid (x, A) \in R_1, (y, B) \in R_2\Big\}$.*

Lemma 4.3. *Let $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$ be two g-hypergraphs. Then, for every $(x, y), (u, v) \in X_1 \times X_2$,*

$$(x, y)\rho_{\mathcal{G}_1\times\mathcal{G}_2}(u, v) \Leftrightarrow x\rho_{\mathcal{G}_1} u \text{ and } y\rho_{\mathcal{G}_2} v.$$

Proof. It is obvious. □

Definition 4.4. *Let $(X, *)$ and $(Y, \circ)$ be two hypergroupoids. We define the hyperoperation $\times_d$ on the Cartesian product $X \times Y$ as follows:*

$$(x_1, y_1) \times_d (x_2, y_2) = \Delta\Big((x_1, y_1)\Big) \bigcup \Delta\Big((x_2, y_2)\Big),$$

where $\Delta\Big((a, b)\Big) = \Big\{(x, y) \mid x \in a * a \text{ and } y \in b \circ b\Big\}$. The hypergroupoid $(X \times Y, \times_d)$ is called the diagonal direct product of $(X, *)$ and $(Y, \circ)$.

Theorem 4.5. *Let $(X_1, *)$ and $(X_2, \circ)$ be the g-hypergroupoids associated with the v-linked g-hypergraphs $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$, respectively. Then, the diagonal direct product of $(X_1, *)$ and $(X_2, \circ)$ is the g-hypergroupoid associated with $\mathcal{G}_1 \times \mathcal{G}_2$.*

Proof. Let $(X_1 \times X_2, \times_d)$ be the diagonal direct product of $(X_1, *)$ and $(X_2, \circ)$. It suffices to show that

$$(x, y) \times_d (x, y) = \bigcup \Big\{A \times B \mid \Big((x, y), A \times B\Big) \in R_1 \times R_2\Big\},$$

where (x, y) is an arbitrary element of $X_1 \times X_2$. This can be seen by the following argument. Let $(r, s) \in (x, y) \times_d (x, y)$ be an arbitrary element. Then, $r \in x * x$ and $s \in y \circ y$. Since $(X_1, *)$ and $(X_2, \circ)$ are the g-hypergroupoids associated with the g-hypergraphs $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$, respectively, there are hyperedges A and B such that $(x, A) \in R_1$, $(y, B) \in R_2$ and $(r, s) \in A \times B$. By the definition of $R_1 \times R_2$ we have $\Big((x, y), A \times B\Big) \in R_1 \times R_2$ and therefore $(r, s) \in \bigcup \Big\{A \times B \mid \Big((x, y), A \times B\Big) \in R_1 \times R_2\Big\}$. Hence $(x, y) \times_d (x, y) \subseteq \bigcup \Big\{A \times B \mid \Big((x, y), A \times B\Big) \in R_1 \times R_2\Big\}$. The reverse inclusion can be shown similarly. □

Definition 4.6. *Let $(X_1, *)$ and $(X_2, \circ)$ be two hypergroupoids. A map $\varphi : X_1 \to X_2$ is called a homomorphism if for all $x, y \in X_1$ we have $\varphi(x * y) = \varphi(x) \circ \varphi(y)$. If φ is one to one (onto) we say that φ is a monomorphism (epimorphism). If there exists a one to one epimorphism from X_1 onto X_2 we say that X_1 is isomorphic to X_2 and we write $X_1 \cong X_2$.*

Theorem 4.7. *Let $(X_1, *)$ and $(X_2, \circ)$ be the g-hypergroupoids associated with the v-linked g-hypergraphs $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$, respectively. Then,*

$$X_1/\rho_{\mathcal{G}_1} \times_d X_2/\rho_{\mathcal{G}_2} \cong (X_1 \times_d X_2)/\rho_{\mathcal{G}_1\times\mathcal{G}_2}.$$

Proof. We equip $X_1/\rho_{\mathcal{G}_1}$, $X_2/\rho_{\mathcal{G}_2}$ and $(X_1 \times_d X_2)/\rho_{\mathcal{G}_1\times\mathcal{G}_2}$ with hyperoperations $\odot$, $\boxdot$ and $\circledcirc$, respectively. Define

$$\varphi : X_1/\rho_{\mathcal{G}_1} \times_d X_2/\rho_{\mathcal{G}_2} \to (X_1 \times_d X_2)/\rho_{\mathcal{G}_1\times\mathcal{G}_2}$$

by

$$\varphi\Big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))\Big) = \rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((x, y)\Big), \text{ for all } (x, y) \in X_1 \times X_2.$$

First, we prove φ is well defined. Consider

$$(\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y)) = (\rho_{\mathcal{G}_1}(x'), \rho_{\mathcal{G}_2}(y')).$$

Hence, we have $x_{R_1} = x'_{R_1}$ and $y_{R_2} = y'_{R_2}$. Since

$$\begin{aligned} A \times B \in (x, y)_{R_1\times R_2} &\Leftrightarrow A \in x_{R_1}, B \in y_{R_2} \Leftrightarrow A \in x'_{R_1}, B \in y'_{R_2} \\ &\Leftrightarrow A \times B \in (x', y')_{R_1\times R_2}, \end{aligned}$$

we obtain $\rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((x, y)\Big) = \rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((x', y')\Big)$, i.e., φ is well defined. Now, we check that φ is one to one. Suppose that $\rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((x, y)\Big) = \rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((x', y')\Big)$. We obtain

$$\begin{aligned} A \in x_{R_1}, B \in y_{R_2} &\Leftrightarrow A \times B \in (x, y)_{R_1\times R_2} \Leftrightarrow A \times B \in (x', y')_{R_1\times R_2} \\ &\Leftrightarrow A \in x'_{R_1}, B \in y'_{R_2}. \end{aligned}$$

This implies that $(\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y)) = (\rho_{\mathcal{G}_1}(x'), \rho_{\mathcal{G}_2}(y'))$. Clearly φ is onto. We need only to show that φ is a homomorphism. Before that we show that $\varphi\Big(\Delta\Big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))\Big)\Big) = \rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big(\Delta(x, y)\Big)$, for all $(x, y) \in X_1 \times X_2$. We know that

$$\begin{aligned} &\Delta\Big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))\Big) \\ &= \Big\{(\rho_{\mathcal{G}_1}(r), \rho_{\mathcal{G}_2}(s)) \mid \rho_{\mathcal{G}_1}(r) \in \rho_{\mathcal{G}_1}(x) \odot \rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(s) \in \rho_{\mathcal{G}_2}(y) \boxdot \rho_{\mathcal{G}_2}(y)\Big\}, \end{aligned}$$

and so we have

$$\begin{aligned} &\varphi\Big(\Delta\Big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))\Big)\Big) \\ &= \Big\{\rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((r, s)\Big) \mid \rho_{\mathcal{G}_1}(r) \in \rho_{\mathcal{G}_1}(x) \odot \rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(s) \in \rho_{\mathcal{G}_2}(y) \boxdot \rho_{\mathcal{G}_2}(y)\Big\}. \end{aligned}$$

But $\rho_{\mathcal{G}_1}(r) \in \rho_{\mathcal{G}_1}(x) \odot \rho_{\mathcal{G}_1}(x)$ if and only if there is $u \in x * x$ such that $\rho_{\mathcal{G}_1}(r) = \rho_{\mathcal{G}_1}(u)$ and $\rho_{\mathcal{G}_2}(s) \in \rho_{\mathcal{G}_2}(y) \boxdot \rho_{\mathcal{G}_2}(y)$ if and only if there is $v \in y \circ y$ such that $\rho_{\mathcal{G}_2}(s) = \rho_{\mathcal{G}_2}(v)$. Now, by using Lemma 4.3 we have

$$\begin{aligned} \varphi\Big(\Delta\Big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))\Big)\Big) &= \Big\{\rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((r, s)\Big) \mid r \in x * x, s \in y \circ y\Big\} \\ &= \Big\{\rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big((r, s)\Big) \mid (r, s) \in \Delta\Big((x, y)\Big)\Big\} \\ &= \rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big(\Delta(x, y)\Big). \end{aligned}$$

Now, by using the above argument, for every elements $(x, y), (x', y') \in X_1 \times X_2$ we obtain

$$\begin{aligned}
&\varphi\Big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y)) \times_d (\rho_{\mathcal{G}_1}(x'), \rho_{\mathcal{G}_2}(y'))\Big)\\
&= \varphi\Big(\Delta\big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))\big) \bigcup \Delta\big((\rho_{\mathcal{G}_1}(x'), \rho_{\mathcal{G}_2}(y'))\big)\Big)\\
&= \rho_{\mathcal{G}_1\times\mathcal{G}_2}\big(\Delta(x, y)\big) \bigcup \rho_{\mathcal{G}_1\times\mathcal{G}_2}\big(\Delta(x', y')\big)\\
&= \rho_{\mathcal{G}_1\times\mathcal{G}_2}\Big(\Delta\big((x, y)\big) \bigcup \Delta\big((x', y')\big)\Big)\\
&= \rho_{\mathcal{G}_1\times\mathcal{G}_2}\big((x, y) \times_d (x', y')\big)\\
&= \big\{\rho_{\mathcal{G}_1\times\mathcal{G}_2}\big((r, s)\big) \mid (r, s) \in (x, y) \times_d (x', y')\big\}\\
&= \rho_{\mathcal{G}_1\times\mathcal{G}_2}\big((x, y)\big) \circledcirc \rho_{\mathcal{G}_1\times\mathcal{G}_2}\big((x', y')\big)\\
&= \varphi\Big((\rho_{\mathcal{G}_1}(x), \rho_{\mathcal{G}_2}(y))\Big) \circledcirc \varphi\Big((\rho_{\mathcal{G}_1}(x'), \rho_{\mathcal{G}_2}(y'))\Big).
\end{aligned}$$

Hence, φ is an isomorphism. □

Theorem 4.8. *Let $\mathcal{G}$ be a v-linked g-hypergraph and $(X_2, \circ)$ be the g-hypergroupoid associated with $\mathcal{G}$. If $(X_1, *)$ is a separable hypergroupoid and $\varphi : X_1 \to X_2$ is an epimorphism, then there exists a regular equivalence relation μ on X_1 such that*

$$X_1/\mu \cong X_2/\rho_{\mathcal{G}}.$$

Proof. Suppose that the relation μ on X_1 is defined by $x\mu y \Leftrightarrow \varphi(x)\rho_{\mathcal{G}}\varphi(y)$, for all $x, y \in X_1$. Since $\rho_{\mathcal{G}}$ is an equivalence relation on X_2, then it is easy to check that μ is an equivalence relation on X_1. Let $x, y, z \in X_1$ be arbitrary elements such that $x\mu y$. We show that $(x * z)\overline{\mu}(y * z)$. From $x\mu y$ it follows that $\varphi(x) \circ \varphi(x) = \varphi(y) \circ \varphi(y)$ which implies that $\varphi(x * x) = \varphi(y * y)$. Let $r \in x * z$ be an arbitrary element. Then, we have $\varphi(r) \in \varphi(x * z) = \varphi(x * x) \bigcup \varphi(z * z) = \varphi(y * y) \bigcup \varphi(z * z) = \varphi(y * z)$. Therefore, there is $t \in y * z$ such that $\varphi(r) = \varphi(t)$. This means that $r\mu t$ and so $(x * z)\overline{\mu}(y * z)$. In a similar way we can show that $(z * x)\overline{\mu}(z * y)$. Thus μ is regular. Now, let $\psi : X_1/\mu \to X_2/\rho_{\mathcal{G}}$ is defined by $\psi\big(\mu(x)\big) = \rho_{\mathcal{G}}\big(\varphi(x)\big)$. Suppose that $x, y \in X_1$. Then,

$$\mu(x) = \mu(y) \Leftrightarrow \varphi(x)\rho_{\mathcal{G}}\varphi(y) \Leftrightarrow \rho_{\mathcal{G}}(\varphi(x)) = \rho_{\mathcal{G}}(\varphi(y)) \Leftrightarrow \psi\big(\mu(x)\big) = \psi\big(\mu(y)\big).$$

Thus ψ is well defined and one to one. Since φ is onto, it follows that ψ is onto. We equip X_1/μ and $X_1/\rho_{\mathcal{G}}$ with the hyperoperations $\odot$ and $\boxdot$, respectively. Let $x, y \in X_1$. The following argument shows that ψ is a homomorphism.

$$\begin{aligned}
\psi\big(\mu(x)\big) \boxdot \psi\big(\mu(y)\big) &= \rho_{\mathcal{G}}(\varphi(x)) \boxdot \rho_{\mathcal{G}}(\varphi(y))\\
&= \big\{\rho_{\mathcal{G}}(\varphi(z)) \mid \varphi(z) \in \varphi(x) \circ \varphi(y)\big\}\\
&= \big\{\rho_{\mathcal{G}}(\varphi(z)) \mid \varphi(z) = \varphi(t) \text{ for some } t \in x * y\big\}\\
&= \big\{\rho_{\mathcal{G}}(\varphi(t)) \mid t \in x * y\big\}\\
&= \psi\big(\{\mu(t) \mid t \in x * y\}\big)\\
&= \psi\big(\mu(x) \odot \mu(y)\big).
\end{aligned}$$

□

Theorem 4.9. *Let $\mathcal{G}$ be a v-linked g-hypergraph and $(X_1, *)$ be the g-hypergroupoid associated with $\mathcal{G}$. If $(X_2, \circ)$ is a separable hypergroupoid and $\varphi : X_1 \to X_2$ is a monomorphism, then there exists a regular equivalence relation μ' on $\varphi(X_1)$ such that*

$$X_1/\rho_{\mathcal{G}} \cong \varphi(X_1)/\mu'.$$

Proof. Suppose that the relation μ' on $\varphi(X_1)$ is defined by $\varphi(x)\mu'\varphi(y) \Leftrightarrow x\rho_{_{\mathcal{G}}} y$, for all $x, y \in X_1$. It is easy to see that μ' is a regular equivalence relation. Define $\psi : X_1/\rho_{_{\mathcal{G}}} \to \varphi(X_1)/\mu'$ by $\psi\big(\rho_{_{\mathcal{G}}}(x)\big) = \mu'(\varphi(x))$. One can easily checks that ψ is an isomorphism. $\square$

Lemma 4.10. *Let ρ be a regular equivalence relation on a hypergroupoid $(X, \circ)$. Then, $\pi : X \to X/\rho$ which is defined by $\pi(x) = \rho(x)$, for all $x \in X$, is an epimorphism which is called canonical epimorphism.*

Proof. The proof is straightforward. $\square$

Theorem 4.11. *Let $(X_1, *)$ and $(X_2, \circ)$ be g-hypergroupoids associated with the v-linked g-hypergraphs $\mathcal{G}_1 = (X_1, R_1)$ and $\mathcal{G}_2 = (X_2, R_2)$, respectively. Let $\varphi : X_1 \to X_2$ be an epimorphism such that $\varphi(x)\rho_{_{\mathcal{G}_2}}\varphi(y)$ implies $x\rho_{_{\mathcal{G}_1}} y$. If $\mu = \big\{(x, y) \in X_1^2 \mid \varphi(x)\rho_{_{\mathcal{G}_2}}\varphi(y)\big\}$ and $\mu' = \big\{(\varphi(x), \varphi(y)) \in X_2^2 \mid x\rho_{_{\mathcal{G}_1}} y\big\}$, then there exists a unique homomorphism $\varphi^* : X_1/\mu \to X_2/\mu'$ such that the following diagram is commutative; i.e., $\pi' \circ \varphi = \varphi^* \circ \pi$, where π and π' denote the*

$$\begin{tikzcd}
X_1 \arrow[r, "\varphi"] \arrow[d, "\pi"'] & X_2 \arrow[d, "\pi'"] \\
X_1/\mu \arrow[r, "\varphi^*"] & X_2/\mu'
\end{tikzcd}$$

canonical epimorphisms.

Proof. The proof of the fact that μ and μ' are regular equivalence relations is analogous to the corresponding part of the proof of Theorem 4.8 and we omit the details. We equip X_1/μ and X_2/μ' with the hyperoperations $\odot$ and $\boxdot$, respectively. Let $\varphi^* : X_1/\mu \to X_2/\mu'$ is defined by $\varphi^*\big(\mu(x)\big) = \mu'(\varphi(x))$, for all $x \in X_1$. First, we show that φ^* is well defined. Let $x, y \in X_1$ and $\mu(x) = \mu(y)$. Then, $\varphi(x)\rho_{_{\mathcal{G}_2}}\varphi(y)$ and so $x\rho_{_{\mathcal{G}_1}} y$. Therefore, φ^* is well defined. Moreover, it is easy to prove that $\varphi^*\big(\mu(x) \odot \mu(y)\big) = \varphi^*\big(\mu(x)\big) \boxdot \varphi^*\big(\mu(y)\big)$ and $\pi' \circ \varphi = \varphi^* \circ \pi$. Now, we show that φ^* is unique. Let $g : X_1/\mu \to X_2/\mu'$ be a homomorphism such that $\pi' \circ \varphi = g \circ \pi$. Then, for all $x \in X_1$, $g(\mu(x)) = g(\pi(x)) = \pi' \circ \varphi(x) = \varphi^* \circ \pi(x) = \varphi^*\big(\mu(x)\big)$. $\square$

5. Fundamental Relation on a g-Hypergroupoid

One of the main tool to study hyperstructures is the fundamental relation β^* in an H_v-semigroup $(X, \circ)$ as the smallest equivalence relation so that the quotient X/β^* would be a semigroup. The relation β^* was introduced on hypergroups by M. Koskas in 1970 [9] and was mainly studied intensively and in depth by Corsini [3], also see [7].

For a relation β on a non-empty set X, we denote by $\widehat{\beta}$ the transitive closure of β and define it as follows:

$$x\widehat{\beta}y \quad \text{if and only if} \quad \begin{array}{l} \text{there exists a natural number } k \text{ and elements} \\ x = a_1, a_2, \ldots, a_{k-1}, a_k = y \text{ in } X \text{ such that} \\ a_1\beta a_2, a_2\beta a_3, \ldots, a_{k-1}\beta a_k. \end{array}$$

Obviously, $\beta = \widehat{\beta}$ if β is transitive.

The proof of following theorem is similar to the proof of Theorem 1.2.2 of [14].

Theorem 5.1. *Let $(X, \circ)$ be an H_v-semigroup and denote $\mathbf{U}$ the set of all finite products of elements of X. We define the relation β on X by setting $x\beta y$ if and only if $x = y$ or $\{x, y\} \subseteq u$ where $u \in \mathbf{U}$. Then, β^* is the transitive closure of β.*

Theorem 5.2. *Let $\mathcal{G} = (X, R)$ be a v-linked g-hypergraph and $X_{\mathcal{G}} = (X, \circ)$ be the g-hypergroupoid associated with $\mathcal{G}$. Then, we have*

$$\beta^*(x) = \begin{cases} S & \text{if } x \in S \\ \{x\} & \text{if } x \notin S, \end{cases}$$

where $S = \bigcup_{A \in Cod(R)} A$ and β is the relation defined in Theorem 5.1.

Proof. First, we show that $\bigcup\{u \mid u \in \mathbf{U}\} = S$. Let $x \in S$ be an arbitrary element. Then, there exist $A \in Cod(R)$ and $a \in X$ such that $(a, A) \in R$ and $x \in A$. Therefore, by setting $u = a \circ a$ we have $x \in u$. Hence $S \subseteq \bigcup\{u \mid u \in \mathbf{U}\}$. The reverse inclusion is obvious. We conclude that if $x, y \in X$ and $x\beta y$, then $x \in S$ if and only if $y \in S$. Now, let $x, y \in S$ be arbitrary elements. Then, there exist $A, B \in Cod(R)$ and $a, b \in X$ such that $(a, A) \in R$, $(b, B) \in R$, $x \in A$ and $y \in B$. Therefore, $\{x, y\} \subseteq a \circ b$ which implies that $x\beta y$. Consequently, for every $x, y \in S$ we have $x\beta y$. By the above argument, we obtain

$$\beta(x) = \begin{cases} S & \text{if } x \in S \\ \{x\} & \text{if } x \notin S. \end{cases}$$

It is easy to see that β is transitive and so we have $\widehat{\beta} = \beta$. By using Theorem 5.1, we have $\beta^* = \widehat{\beta}$ which completes the proof. □

Corollary 5.3. *A v-linked g-hypergraph $\mathcal{G} = (X, R)$ is plenary if and only if $\beta^* = X \times X$.*

By using the above corollary and Remark 2.7, we obtain the following corollary.

Corollary 5.4. *For every separable H_v-group, we have $\beta^* = \beta$.*

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