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Inequalities for H-invex Functions with Applications for Uniformly Convex and Superquadratic Functions

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Abstract. In this paper, we introduce and study H-invex functions including the classes of convex, η-invex, (F,G)-invex, c-strongly convex, φ-uniformly convex and superquadratic functions, respectively. Each H-invex function attains its global minimum at an H-stationary point. For H-invex functions we prove Jensen, Sherman and Hardy-Littlewood-Pólya-Karamata type inequalities, respectively. We also analyze such inequalities when the control function H is convex. As applications, we give interpretations of the obtained results for uniformly convex and superquadratic functions, respectively.

1. Introduction and Summary

In this paper we propose systematic study of the class of H-invex functions and its subclasses from the point of view of majorization and weighted majorization. (See below and Section 2 for relevant definitions.) It is essential for an H-invex function f that an *H-stationary point* of f is a *global minimizer* of f.

Our aim is to establish Jensen, Sherman and HLPK type inequalities for H-invex functions. Consequently, we will obtain such inequalities for all classes **(i)-(vii)** of functions defined in Section 2.

We say that a vector $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is *majorized* by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, written as $\mathbf{y} \prec \mathbf{x}$, if

$$\sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]} \quad \text{for } k = 1, 2, \ldots, n$$

with equality for $k = n$ (see [16, p. 8]). Throughout the symbols $x_{[i]}$ and $y_{[i]}$ stand for the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively.

By Birkhoff's and Rado's Theorems [16, pp. 10,34,162], $\mathbf{y} \prec \mathbf{x}$ if and only if $\mathbf{y} = \mathbf{x}P$ for some $n \times n$ doubly stochastic matrix P.

In the forthcoming theorem we demonstrate Hardy-Littlewood-Pólya-Karamata's result showing a relationship between majorization and convexity [15, 16].

Theorem A [15, p. 75], [16, p. 92] (HLPK's inequality) *Let $f : I \to \mathbb{R}$ be a convex continuous function on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$.*

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If $\mathbf{y} \prec \mathbf{x}$*, then*

$$\sum_{k=1}^{n} f(y_k) \leq \sum_{k=1}^{n} f(x_k). \tag{1}$$

If f is concave, then the inequality (1) is reversed.

We now present Sherman's inequality (3) (cf. [22], see also [9, 11, 17]).

Theorem B [22] (Sherman's inequality) *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in I^m$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$, $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}_+^m$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$.*

If

$$\mathbf{y} = \mathbf{x}P \quad \text{and} \quad \mathbf{a} = \mathbf{b}P^T \tag{2}$$

for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) \leq \sum_{i=1}^{m} a_i f(x_i). \tag{3}$$

If f is concave, then the inequality (3) is reversed.

The relation (2) is called *weighted majorization* of pairs $(\mathbf{x}, \mathbf{b})$ and $(\mathbf{y}, \mathbf{a})$ (see [9, 11]).

The paper is organized as follows. In Section 2 we introduce the class of H-invex functions. We also present its subclasses as convex, η-invex, (F, G)-invex, c-strongly convex, uniformly convex with modulus φ, and superquadratic functions, respectively. Thus Section 2 collects some important examples of H-invex functions.

In Section 3 we establish some Jensen type inequalities for H-invex functions. Specifications for (F, G)-invex and η-invex functions are also provided. In Section 4 we deal with Sherman type inequalities for H-invex functions. We also show corollaries to (F, G)-invex and η-invex functions. As special case, we demonstrate Hardy-Littlewood-Pólya-Karamata like theorems in Section 5. Sections 6 and 7 are devoted to applications. Here we interpret the obtained results for uniformly convex and superquadratic functions, respectively.

We end this summary with the remark that the results obtained for the class of H-invex functions (class **(iv)** in the paper) introduced in Section 2 includes all the other results dealt with in this paper.

2. *H*-invex Functions

We begin with a review of some important classes of functions.

(i). **Convex functions**.

A function $f : I \to \mathbb{R}$ defined on a convex set $I \subset \mathbb{R}^n$ is said to be *convex* on I, if for any points $x_i \in I$ and scalars $p_i \geq 0$, $i = 1, \ldots, m$, with $\sum_{i=1}^{m} p_i = 1$ and $\bar{x} = \sum_{i=1}^{m} p_i x_i$, the following *Jensen's inequality* holds:

$$f(\bar{x}) \leq \sum_{i=1}^{m} p_i f(x_i). \tag{4}$$

It is well known that if $f : I \to \mathbb{R}$ is a differentiable convex function then *(sub)differential inequality* holds, as follows:

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle \quad \text{for } x, y \in I, \tag{5}$$

where the symbol ∇ stands for the gradient, and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$.

It is a consequence of (5) that each *stationary point* $y \in I$ of f (i.e., $\nabla f(y) = 0$) is a *global minimizer* of f, that is

$$f(x) \geq f(y) \quad \text{for } x \in I.$$

(ii). **η-invex functions**.

Let $\eta : I \times I \to \mathbb{R}^n$ be a continuous function, where $I \subset \mathbb{R}^n$ is an (open) convex set. A differentiable function $f : I \to \mathbb{R}$ is called *η-invex*, if for all $x, y \in I$,

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y) \rangle \tag{6}$$

(see [8, p. 1]).

It follows from (6) that each *stationary point* $y \in I$ of f (i.e., $\nabla f(y) = 0$) is a global minimizer of f.

For applications of invex functions in optimization and mathematical programming, see [5, 8, 12, 14, 20].

(iii). **(F, G)-invex functions**.

Let V be a real linear space, and $I, J \subset V$ be two convex sets in V. Let $F : I \times I \to J$ and $G : J \times I \to \mathbb{R}$ be two continuous functions, where $I \subset \mathbb{R}^n$ is a convex set and $J \subset \mathbb{R}^n$. A function $f : I \to \mathbb{R}$ is said to be *(F, G)-invex*, if for all $x, y \in I$,

$$f(x) - f(y) \geq G_y(F(x, y)), \tag{7}$$

where $G_y(z) = G(z, y)$ for $y, z \in I$.

By virtue of (7), if $y \in I$ is a *G-stationary point* of f (i.e., $G_y(\cdot) \equiv 0$) then f has a global minimum at y.

It is readily seen that an η-invex function is (F, G)-invex for $F(x, y) = \eta(x, y)$ and $G_y(\cdot) = \langle \nabla f(y), \cdot \rangle$ with $V = J = \mathbb{R}^n$.

(iv). **H-invex functions**.

Let $H : I \times I \to \mathbb{R}$ be a continuous function, where $I \subset V$ is a convex set in a real linear space V. A function $f : I \to \mathbb{R}$ is called *H-invex*, if for all $x, y \in I$,

$$f(x) - f(y) \geq H_y(x), \tag{8}$$

where $H_y(x) = H(x, y)$ for $x, y \in I$.

Similarly as above, if f is H-invex and $y \in I$ is an *H-stationary point* of f (i.e., $H_y(\cdot) \equiv 0$), then f has a global minimum at y.

Evidently, each (F, H)-invex function is H-invex for H_y being the composition $G_y \circ F_y$, i.e.,

$$H(x, y) = H_y(x) = (G_y \circ F_y)(x) = G(F(x, y), y) \quad \text{for } x, y \in I.$$

We now present some further special cases of the notion of H-invexity.

(v). **Uniformly convex functions with modulus φ**.

A function $f : I \to \mathbb{R}$ defined on a convex set $I \subset V = \mathbb{R}^n$ is said to be uniformly convex with modulus $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, if the following inequality holds for all points $x, y \in I$ and $p \in [0, 1]$:

$$f(px + (1 - p)y) + p(1 - p)\varphi(||x - y||) \leq pf(x) + (1 - p)f(y), \tag{9}$$

where $|| \cdot || = \langle \cdot, \cdot \rangle^{1/2}$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $V = \mathbb{R}^n$ (see [23]).

In the case of differentiable f, condition (9) amounts to

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle + \varphi(||x - y||) \quad \text{for } x, y \in I. \tag{10}$$

Thus a uniformly convex function with modulus φ is H-invex with

$$H(x, y) = \langle \nabla f(y), x - y \rangle + \varphi(||x - y||) \quad \text{for } x, y \in I.$$

(vi). **c-strongly convex functions**.

A function $f : I \to \mathbb{R}$ defined on a convex set $I \subset V = \mathbb{R}^n$ is said to be *c-strongly convex* on I, where $c \in \mathbb{R}_+$, if the following inequality holds for all points $x, y \in I$ and $p \in [0, 1]$:

$$f(px + (1 - p)y) + \frac{c}{2}p(1 - p)||x - y||^2 \leq pf(x) + (1 - p)f(y), \tag{11}$$

where $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $V = \mathbb{R}^n$ (see [6, 21]).

For differentiable f, condition (11) becomes

$$f(x) - f(y) \geq \langle \nabla f(y), x - y\rangle + \frac{c}{2}\|x-y\|^2 \quad \text{for } x, y \in I.$$

(see [6, p. 684]).

Therefore a c-strongly convex function f is H-invex with

$$H(x,y) = \langle \nabla f(y), x - y\rangle + \frac{c}{2}\|x-y\|^2 \quad \text{for } x, y \in I.$$

(vii). Superquadratic functions.

A function $f : \mathbb{R}^m_+ \to \mathbb{R}$ defined on the convex cone $I = \mathbb{R}^m_+$ is said to be *superquadratic*, if for each point $y \in \mathbb{R}^m_+$ there exists a vector $C(y) \in \mathbb{R}^n$ such that the following condition is fulfilled:

$$f(x) - f(y) \geq \langle C(y), x - y\rangle + f(|x-y|) \quad \text{for } x, y \in I, \tag{12}$$

where $|z| = (|z_1|, \dots, |z_m|)$ for $z = (z_1, \dots, z_m) \in \mathbb{R}^m = V$ (see [1]).

It is easy to check that a superquadratic function is H-invex with

$$H(x,y) = \langle C(y), x - y\rangle + f(|x-y|) \quad \text{for } x, y \in I = \mathbb{R}^m_+.$$

For further information on superquadratic functions, consult [1–3, 7].

3. Jensen Like Inequality for *H*-invex Functions

Unless stated otherwise, V is a real linear space, and $I, J \subset V$ are (nonempty) convex sets in V. Additionally, $\leq$ is a preorder on J. The assumptions just made shall be in force throughout the paper.

We say that a map $\Phi : I \to J$ is *$\leq$-convex* if

$$\Phi(\alpha v + (1-\alpha)w) \leq \alpha\Phi(v) + (1-\alpha)\Phi(w) \quad \text{for } v, w \in I \text{ and } \alpha \in [0,1]. \tag{13}$$

Likewise, we say that a map $\Psi : J \to \mathbb{R}$ is *$\leq$-increasing* if

$$v \leq w \quad \text{implies} \quad \Psi(v) \leq \Psi(w) \quad \text{for } v, w \in J. \tag{14}$$

In what follows, the notation $(w_1, \dots, w_k) = (v_1, \dots, v_m)S$ for an $m \times k$ real matrix $S = (s_{il})$ and vectors $w_1, \dots, w_k, v_1, \dots, v_m \in V$ means that (see [18])

$$w_l = \sum_{i=1}^m s_{il} v_i \quad \text{for } l = 1, \dots, k.$$

In this section we are interested in some results for H-invex functions $f : I \to \mathbb{R}$ extending the classical Jensen's inequality for convex functions (see (4)). Special attention is paid to the case when the control function H has the property that

$$\text{the map } x \to H(x,y),\ x \in I, \text{ is convex}$$

for any (or fixed) point $y \in I$.

Theorem 3.1. *Let $H : I \times I \to \mathbb{R}$ be a continuous function. Let $f : I \to \mathbb{R}$ be an H-invex function. Let $x_i \in I$ and $p_i \geq 0$, $i = 1, \dots, m$, with $\sum\limits_{i=1}^m p_i = 1$. Denote $\bar{x} = \sum\limits_{i=1}^m p_i x_i$.*

Then the following Jensen type inequality holds:

$$f(\bar{x}) + \sum_{i=1}^{m} p_i H(x_i, \bar{x}) \le \sum_{i=1}^{m} p_i f(x_i). \tag{15}$$

If in addition the function $H(\cdot, \bar{x})$ is convex on I, then

$$f(\bar{x}) + H(\bar{x}, \bar{x}) \le \sum_{i=1}^{m} p_i f(x_i). \tag{16}$$

Proof. Since $f : I \to \mathbb{R}$ is H-invex, it follows that

$$f(x_i) - f(\bar{x}) \ge H_{\bar{x}}(x_i) = H(x_i, \bar{x})$$

(see (8) in the definition **(iv)** in Section 2). This implies

$$\sum_{i=1}^{m} p_i f(x_i) - f(\bar{x}) = \sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} p_i f(\bar{x})$$

$$= \sum_{i=1}^{m} p_i [f(x_i) - f(\bar{x})] \ge \sum_{i=1}^{m} p_i H(x_i, \bar{x}).$$

This completes the proof of (15).

If in addition the function $H(\cdot, \bar{x})$ is convex on I, then

$$\sum_{i=1}^{m} p_i H(x_i, \bar{x}) \ge H\left(\sum_{i=1}^{m} p_i x_i, \bar{x}\right) = H(\bar{x}, \bar{x}). \tag{17}$$

By combining (15) and (17) one gets (16). ■

Many inequalities of the form (15) and (16) have proved their worth, with f assumed H-invex.

We now interpret Theorem 3.1 for (F, G)-invex functions.

Proposition 3.2. *Let $F : I \times I \to J$ and $G : J \times I \to \mathbb{R}$ be continuous functions. Let $f : I \to \mathbb{R}$ be an (F, G)-invex function. Let $x_i \in I$ and $p_i \ge 0$, $i = 1, \dots, m$, with $\sum_{i=1}^{m} p_i = 1$. Denote $\bar{x} = \sum_{i=1}^{m} p_i x_i$.*

Then the following Jensen type inequality holds:

$$f(\bar{x}) + \sum_{i=1}^{m} p_i G(F(x_i, \bar{x}), \bar{x}) \le \sum_{i=1}^{m} p_i f(x_i). \tag{18}$$

If in addition the function $F(\cdot, \bar{x})$ is $\preceq$-convex and $G(\cdot, \bar{x})$ is convex and $\preceq$-increasing, then

$$f(\bar{x}) + G(F(\bar{x}, \bar{x}), \bar{x}) \le \sum_{i=1}^{m} p_i f(x_i).$$

Proof. It is now enough to use Theorem 3.1 with $H(x, y) = G(F(x, y), y)$ for $x, y \in I$.

The composition $H(\cdot, \bar{x}) = H_{\bar{x}}(\cdot) = G_{\bar{x}} \circ F_{\bar{x}}(\cdot)$ is convex, whenever $F_{\bar{x}}$ is $\preceq$-convex and $G_{\bar{x}}$ is convex and $\preceq$-increasing.

In fact, the $\preceq$-convexity of $F_{\bar{x}}$ means that (see (13))

$$F_{\bar{x}}(\alpha v + (1 - \alpha) w) \preceq \alpha F_{\bar{x}}(v) + (1 - \alpha) F_{\bar{x}}(w) \quad \text{for } v, w \in I \text{ and } \alpha \in [0, 1].$$

Then the $\preceq$-increasity of $G_{\bar{x}}$ ensures that (see (14))

$$G_{\bar{x}}F_{\bar{x}}(\alpha v + (1-\alpha)w) \le G_{\bar{x}}\left(\alpha F_{\bar{x}}(v) + (1-\alpha)F_{\bar{x}}(w)\right) \quad \text{for } v, w \in I \text{ and } \alpha \in [0,1].$$

Furthermore, the convexity of $G_{\bar{x}}$ on J quarantees that

$$G_{\bar{x}}\left(\alpha F_{\bar{x}}(v) + (1-\alpha)F_{\bar{x}}(w)\right) \le \alpha G_{\bar{x}}F_{\bar{x}}(v) + (1-\alpha)G_{\bar{x}}F_{\bar{x}}(w) \quad \text{for } v, w \in I \text{ and } \alpha \in [0,1].$$

All of this shows the standard convexity of $H_{\bar{x}} = G_{\bar{x}} \circ F_{\bar{x}}$, as claimed. ■

Remark 3.3. In Proposition 3.2, in order to obtain (18), we do not need to assume that $G(\cdot, \bar{x})$ is $\preceq$-increasing provided that the function $F(\cdot, \bar{x})$ is affine.

Remind that a subset $D \subset V$ is said to be a *convex cone* if $v, w \in D$ implies $v + w \in D$, and if $0 \le \alpha \in \mathbb{R}$ and $v \in D$ imply $\alpha v \in D$.

If D is a convex cone in V, then the cone preorder $\preceq_D$ is defined for $v, w \in V$ by

$$v \preceq_D w \quad \text{iff} \quad w - v \in D.$$

If V is equipped with a real inner product $\langle \cdot, \cdot \rangle$, and D is a convex cone in V, then the dual cone of D is defined as follows:

$$\operatorname{dual} D = \{v \in V : \langle u, v \rangle \ge 0 \text{ for all } u \in D\}.$$

Therefore,

$$v \preceq_{\operatorname{dual} D} w \quad \text{iff} \quad w - v \in \operatorname{dual} D \quad \text{iff} \quad \langle u, w - v \rangle \ge 0 \quad \text{for all } u \in D.$$

The $\preceq_{\operatorname{dual} D}$-convexity of a map $\Phi : I \to J$ means that

$$\Phi(\alpha v + (1-\alpha)w) \preceq_{\operatorname{dual} D} \alpha\Phi(v) + (1-\alpha)\Phi(w) \quad \text{for } v, w \in I \text{ and } \alpha \in [0,1]. \tag{19}$$

Equivalently,

$$\langle u, \Phi(\alpha v + (1-\alpha)w)\rangle \preceq_{\operatorname{dual} D} \alpha\langle u, \Phi(v)\rangle + (1-\alpha)\langle u, \Phi(w)\rangle \tag{20}$$

for $v, w \in I$, $\alpha \in [0,1]$ and $u \in D$.

A specification of Proposition 3.2 for η-invex functions and $V = J = \mathbb{R}^n$ is demonstrated below.

Corollary 3.4 (Cf. Craven and Dragomir [13, Proposition 1]). *Let $\eta : I \times I \to \mathbb{R}^n$ be a continuous function, where $I \subset \mathbb{R}^n$ is an open convex set. Let $f : I \to \mathbb{R}$ be an η-invex function. Let $x_i \in I$ and $p_i \ge 0$, $i = 1, \ldots, m$, with $\sum\limits_{i=1}^m p_i = 1$. Denote $\bar{x} = \sum\limits_{i=1}^m p_i x_i$.*

Then the following Jensen type inequality holds:

$$f(\bar{x}) + \sum_{i=1}^m p_i \langle \nabla f(\bar{x}), \eta(x_i, \bar{x})\rangle \le \sum_{i=1}^m p_i f(x_i).$$

If $\nabla f(\bar{x}) \in D$, where D is a convex cone in $\mathbb{R}^n$, and the function $\eta(\cdot, \bar{x})$ is convex with respect to dual D*, then*

$$f(\bar{x}) + \langle \nabla f(\bar{x}), \eta(\bar{x}, \bar{x})\rangle \le \sum_{i=1}^m p_i f(x_i).$$

Proof. In order to prove Corollary 3.4, it suffices to apply Proposition 3.2 with

$$F(x, y) = \eta(x, y) \quad \text{and} \quad G_y(\cdot) = G(\cdot, y) = \langle \nabla f(y), \cdot \rangle \quad \text{for } x, y \in I,$$

and with $\preceq$ being the cone preorder $\preceq_{\operatorname{dual} D}$ induced by the dual cone of D.

Indeed, the function $F(\cdot, \bar{x}) = \eta(\cdot, \bar{x})$ is convex with respect to dual D, i.e., it is $\preceq_{\operatorname{dual} D}$-convex. Since $\nabla f(\bar{x}) \in D$, the function $G_{\bar{x}}(\cdot) = \langle \nabla f(\bar{x}), \cdot \rangle$ is $\preceq_{\operatorname{dual} D}$-increasing. Simultaneously, $G_{\bar{x}}(\cdot)$ is linear, so it is also convex.

In conclusion, the function $H_{\bar{x}}(\cdot) = G_{\bar{x}} \circ F_{\bar{x}}(\cdot) = \langle \nabla f(\bar{x}), \eta((\cdot), \bar{x})\rangle$ is convex in the standard sense. ■

4. Sherman Like Theorem for *H*-invex Functions

As previously, V is a real linear space, and $I, J \subset V$ are convex sets, and $\preceq$ is a preorder on J (unless otherwise specified).

The following theorem is of fundamental importance.

Theorem 4.1. *Let $H : I \times I \to \mathbb{R}$ be a continuous function. Let $f : I \to \mathbb{R}$ be an H-invex function. Let $\mathbf{x} = (x_1, \dots, x_m) \in I^m$, $\mathbf{y} = (y_1, \dots, y_n) \in I^n$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m_+$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n_+$.*

If

$$\mathbf{y} = \mathbf{x}P \quad \text{and} \quad \mathbf{a} = \mathbf{b}P^T \tag{21}$$

for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} H(x_i, y_j) \le \sum_{i=1}^{m} a_i f(x_i). \tag{22}$$

If in addition for each $y \in I$ the function $H_y(\cdot)$ is convex, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j H(y_j, y_j) \le \sum_{i=1}^{m} a_i f(x_i). \tag{23}$$

Proof. Taking into account (21) we get

$$(y_1, \dots, y_n) = (x_1, \dots, x_m)P,$$

that is, $y_j = \sum\limits_{i=1}^{m} p_{ij} x_i$ with $\sum\limits_{i=1}^{m} p_{ij} = 1$, $j = 1, \dots, n$, and $p_{ij} \ge 0$. So, the H-invexity of f implies

$$f(y_j) + \sum_{i=1}^{m} p_{ij} H(x_i, y_j) \le \sum_{i=1}^{m} p_{ij} f(x_i) \quad \text{for } j = 1, 2, \dots, n$$

(see (15) in Theorem 3.1). Therefore we can write

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} H(x_i, y_j) \le \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} f(x_i).$$

Equivalently,

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} H(x_i, y_j) \le \sum_{i=1}^{m} \sum_{j=1}^{n} b_j p_{ij} f(x_i).$$

It follows from (21) that $\mathbf{a} = \mathbf{b}P^T$. Hence $a_i = \sum\limits_{j=1}^{n} b_j p_{ij}$, $i = 1, 2, \dots, m$.

In summary, we obtain

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} H(x_i, y_j) \le \sum_{i=1}^{m} \left(\sum_{j=1}^{n} b_j p_{ij} \right) f(x_i) = \sum_{i=1}^{m} a_i f(x_i),$$

which is (22), as required.

To see (23), we observe that the functions $H_{y_j}(\cdot)$ are convex. So, for each $j = 1, \dots, n$,

$$H(y_j, y_j) = H\left(\sum_{i=1}^{m} p_{ij} x_i, y_j \right) \le \sum_{i=1}^{m} p_{ij} H(x_i, y_j),$$

and further

$$\sum_{j=1}^{n} b_j H(y_j, y_j) \le \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} H(x_i, y_j).$$

This and (22) quarantee that (23) holds, as was to be proved. ∎

Theorem 4.1 has an immediate consequence for (F, G)-invex functions.

Proposition 4.2. *Let $F : I \times I \to J$ and $G : J \times I \to \mathbb{R}$ be continuous functions. Let $f : I \to \mathbb{R}$ be an (F, G)-invex function. Let $\mathbf{x} = (x_1, \dots, x_m) \in I^m$, $\mathbf{y} = (y_1, \dots, y_n) \in I^n$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}_+^m$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}_+^n$.*

If $\mathbf{y} = \mathbf{x}P$ and $\mathbf{a} = \mathbf{b}P^T$ for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} G(F(x_i, y_j), y_j) \le \sum_{i=1}^{m} a_i f(x_i).$$

If in addition for each $y \in I$ the function $F_y(\cdot)$ is $\preceq$-convex and $G_y(\cdot)$ is convex and $\preceq$-increasing, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j G(F(y_j, y_j), y_j) \le \sum_{i=1}^{m} a_i f(x_i).$$

Proof. Clearly, it suffices to use Theorem 4.1 with convex $H_y(x) = G_y(F_y(x))$ for $x, y \in I$.

In the situation that for each $y \in I$ the function $F_y(\cdot)$ is $\preceq$-convex and $G_y(\cdot)$ is convex and $\preceq$-increasing, each composition $H_y(\cdot) = G_y(F_y(\cdot))$, $y \in I$, is convex. The reason is that the $\preceq$-convexity of F_y gives (see (13))

$$F_y(\alpha v + (1-\alpha) w) \preceq \alpha F_y(v) + (1-\alpha) F_y(w) \quad \text{for } v, w \in I \text{ and } \alpha \in [0, 1].$$

Next, the $\preceq$-increasity of G_y implies that (see (14))

$$G_y F_y(\alpha v + (1-\alpha) w) \le G_y\left(\alpha F_y(v) + (1-\alpha) F_y(w)\right) \quad \text{for } v, w \in I \text{ and } \alpha \in [0, 1].$$

The convexity of G_y on J leads to

$$G_y\left(\alpha F_y(v) + (1-\alpha) F_y(w)\right) \le \alpha G_y F_y(v) + (1-\alpha) G_y F_y(w) \quad \text{for } v, w \in I \text{ and } \alpha \in [0, 1].$$

Combining the last two inequalities yields the desired standard convexity of $H_{\bar{x}} = G_{\bar{x}} \circ F_{\bar{x}}$. ∎

In the next result we illustrate Proposition 4.2 for η-invex functions. Here $J = V = \mathbb{R}^n$.

Corollary 4.3. *Let $\eta : I \times I \to \mathbb{R}^n$ be a continuous function, where $I \subset \mathbb{R}^n$ is an open convex set. Let $f : I \to \mathbb{R}$ be an η-invex function. Let $\mathbf{x} = (x_1, \dots, x_m) \in I^m$, $\mathbf{y} = (y_1, \dots, y_n) \in I^n$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}_+^m$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}_+^n$.*

If $\mathbf{y} = \mathbf{x}P$ and $\mathbf{a} = \mathbf{b}P^T$ for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} \langle \nabla f(y_j), \eta(x_i, y_j) \rangle \le \sum_{i=1}^{m} a_i f(x_i).$$

If $\nabla f(\bar{x}) \in D$, where D is a convex cone in $\mathbb{R}^n$, and the function $\eta(\cdot, \bar{x})$ is convex with respect to dual D, *then*

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \langle \nabla f(y_j), \eta(y_j, y_j) \rangle \le \sum_{i=1}^{m} a_i f(x_i).$$

Proof. It is evident that an η-function f is (F, G)-invex with

$$F(x, y) = \eta(x, y) \quad \text{and} \quad G(\cdot, y) = G_y(\cdot) = \langle \nabla f(y), \cdot \rangle \quad \text{for } x, y \in I.$$

Now, it remains to employ Proposition 4.2. ∎

5. HLPK Like Theorem for H-invex Functions

We now specialize the results of the previous section to obtain some HLPK like inequalities for H-invex functions.

Theorem 5.1. *Let $H : I \times I \to \mathbb{R}$ be a continuous function. Let $f : I \to \mathbb{R}$ be an H-invex function. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in I^n$.*

If $\mathbf{y} \prec \mathbf{x}$, that is, $y = xP$ for some $n \times n$ doubly stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij} H(x_i, y_j) \le \sum_{i=1}^{n} f(x_i).$$

If in addition for each $y \in I$ the function $H_y(\cdot)$ is convex, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} H(y_j, y_j) \le \sum_{i=1}^{n} f(x_i).$$

Proof. It is sufficient to apply Theorem 4.1 for $m = n$ and $\mathbf{a} = \mathbf{b} = (1, \dots, 1) \in \mathbb{R}^n$. ■

Below we have a counterpart of the previous result for (F, G)-invex functions.

Proposition 5.2. *Let $F : I \times I \to J$ and $G : J \times I \to \mathbb{R}$ be continuous functions. Let $f : I \to \mathbb{R}$ be an (F, G)-invex function. Let $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and $\mathbf{y} = (y_1, \dots, y_n) \in I^n$.*

If $\mathbf{y} \prec \mathbf{x}$, that is, $y = xP$ for some $n \times n$ doubly stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij} G(F(x_i, y_j), y_j) \le \sum_{i=1}^{n} f(x_i).$$

If in addition for each $y \in I$ the function $F_y(\cdot)$ is $\preceq$-convex and $G_y(\cdot)$ is convex and $\preceq$-increasing, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} G(F(y_j, y_j), y_j) \le \sum_{i=1}^{n} f(x_i).$$

Proof. It is enough to utilize Proposition 4.2 for $m = n$ and $\mathbf{a} = \mathbf{b} = (1, \dots, 1) \in \mathbb{R}^n$. ■

We are now ready to give a version of Proposition 5.2 for η-invex functions.

Corollary 5.3. *Let $\eta : I \times I \to \mathbb{R}^n$ be a continuous function, where $I \subset \mathbb{R}^n$ is an open convex set. Let $f : I \to \mathbb{R}$ be an η-invex function. Let $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and $\mathbf{y} = (y_1, \dots, y_n) \in I^n$.*

If $\mathbf{y} \prec \mathbf{x}$, that is, $y = xP$ for some $n \times n$ doubly stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij} \langle \nabla f(y_j), \eta(x_i, y_j) \rangle \le \sum_{i=1}^{n} f(x_i).$$

If $\nabla f(\bar{x}) \in D$, where D is a convex cone in $\mathbb{R}^n$, and the function $\eta(\cdot, \bar{x})$ is convex with respect to dual D, *then*

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} \langle \nabla f(y_j), \eta(y_j, y_j) \rangle \le \sum_{i=1}^{n} f(x_i).$$

Proof. Making use of Corollary 4.3 with $m = n$ and $\mathbf{a} = \mathbf{b} = (1, \dots, 1) \in \mathbb{R}^n$ yields the desired inequalities. ■

6. Applications for Uniformly Convex Functions

In this section we utilize the previous results for uniformly convex functions with modulus $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. By recalling the definition **(v)** in Section 2, we see that each uniformly convex function f with modulus φ is H-invex for

$$H(x, y) = \langle \nabla f(y), x - y \rangle + \varphi(\|x - y\|) \quad \text{for } x, y \in I \tag{24}$$

with a convex set $I \subset V = \mathbb{R}^n$.

Therefore we can state the following.

Proposition 6.1. *Let $f : I \to \mathbb{R}$ be a differentiable uniformly convex function with convex modulus φ. Let $x_i \in I$ and $p_i \geq 0$, $i = 1, \ldots, m$, with $\sum_{i=1}^{m} p_i = 1$. Denote $\bar{x} = \sum_{i=1}^{m} p_i x_i$.*

Then the following Jensen type inequality holds:

$$f(\bar{x}) + \sum_{i=1}^{m} p_i \varphi(\|x_i - \bar{x}\|) \leq \sum_{i=1}^{m} p_i f(x_i). \tag{25}$$

If in addition the modulus φ is convex and increasing, then

$$f(\bar{x}) + \varphi(0) \leq \sum_{i=1}^{m} p_i f(x_i). \tag{26}$$

Proof. Because of (24) and Theorem 3.1 the following holds:

$$f(\bar{x}) + \sum_{i=1}^{m} p_i \left(\langle \nabla f(\bar{x}), x_i - \bar{x} \rangle + \varphi(\|x_i - \bar{x}\|) \right) \leq \sum_{i=1}^{m} p_i f(x_i). \tag{27}$$

However,

$$\sum_{i=1}^{m} p_i \langle \nabla f(\bar{x}), x_i - \bar{x} \rangle = \langle \nabla f(y), \sum_{i=1}^{m} p_i (x_i - \bar{x}) \rangle \tag{28}$$

$$= \langle \nabla f(y), \sum_{i=1}^{m} p_i x_i - \sum_{i=1}^{m} p_i \bar{x}) \rangle = \langle \nabla f(y), \bar{x} - \bar{x}) \rangle = 0.$$

So, it follows from (27) and (28) that

$$f(\bar{x}) + \sum_{i=1}^{m} p_i \varphi(\|x_i - \bar{x}\|) \leq \sum_{i=1}^{m} p_i f(x_i),$$

which proves (25).

If φ is convex and increasing, then it is not hard to check that

$$\varphi(0) = \varphi\left(\| \sum_{i=1}^{m} p_i (x_i - \bar{x}) \| \right) \leq \varphi\left(\sum_{i=1}^{m} p_i \|x_i - \bar{x}\| \right) \leq \sum_{i=1}^{m} p_i \varphi(\|x_i - \bar{x}\|). \tag{29}$$

So, the usage of (25) and (29) leads to (26), as desired. ∎

Note that inequalities (25)-(26) are refinements of the standard Jensen's inequality for uniformly convex functions, since their moduli are nonnegative.

A version of Sherman type inequality for uniformly convex functions is incorporated in the next theorem.

Theorem 6.2. *Let $f : I \to \mathbb{R}$ be a differentiable uniformly convex function with modulus φ. Let $\mathbf{x} = (x_1, \dots, x_m) \in I^m$, $\mathbf{y} = (y_1, \dots, y_n) \in I^n$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m_+$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n_+$.*

If

$$\mathbf{y} = \mathbf{x}P \quad \text{and} \quad \mathbf{a} = \mathbf{b}P^T \tag{30}$$

for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij}\varphi(\|x_i - y_j\|) \le \sum_{i=1}^{m} a_i f(x_i). \tag{31}$$

If in addition the modulus φ is convex and increasing, then

$$\sum_{j=1}^{n} b_j f(y_j) + \varphi(0) \sum_{j=1}^{n} b_j \le \sum_{i=1}^{m} a_i f(x_i). \tag{32}$$

Proof. Similarly as in the proof of Proposition 6.1, we see that f is H-invex with $H(x, y)$ given by (24). Accordingly, we have

$$\sum_{i=1}^{m} p_{ij} H(x_i, y_j) = \sum_{i=1}^{m} p_{ij} \left(\langle \nabla f(y_j), x_i - y_j \rangle + \varphi(\|x_i - y_j\|) \right)$$

$$= \sum_{i=1}^{m} p_{ij} \langle \nabla f(y_j), x_i - y_j \rangle + \sum_{i=1}^{m} p_{ij}\varphi(\|x_i - y_j\|) \tag{33}$$

$$= \langle \nabla f(y_j), \sum_{i=1}^{m} p_{ij}x_i - \sum_{i=1}^{m} p_{ij}y_j \rangle + \sum_{i=1}^{m} p_{ij}\varphi(\|x_i - y_j\|).$$

Thanks to (30) we get

$$(y_1, \dots, y_n) = (x_1, \dots, x_m)P$$

for some $m \times n$ column stochastic matrix $P = (p_{ij})$, Equivalently,

$$y_j = \sum_{i=1}^{m} p_{ij}x_i \quad \text{for } j = 1, \dots, n.$$

This implies

$$\langle \nabla f(y_j), \sum_{i=1}^{m} p_{ij}x_i - \sum_{i=1}^{m} p_{ij}y_j \rangle = \langle \nabla f(y_j), y_j - y_j \rangle = 0,$$

since $\sum\limits_{i=1}^{m} p_{ij} = 1$.

For this reason (33) gives

$$\sum_{i=1}^{m} p_{ij} H(x_i, y_j) = \sum_{i=1}^{m} p_{ij}\varphi(\|x_i - y_j\|).$$

Therefore inequality (31) is fulfilled by Theorem 4.1.

If φ is convex and increasing, then it is not hard to check that

$$\varphi(0) = \varphi\left(\| \sum_{i=1}^{m} p_{ij}(x_i - y_j) \| \right) \le \varphi\left(\sum_{i=1}^{m} p_{ij} \|x_i - y_j\| \right) \le \sum_{i=1}^{m} p_{ij}\varphi(\|x_i - y_j\|). \tag{34}$$

Now, it is a consequence of (31) and (34) that (32) is satisfied. ∎

The standard Sherman's inequality (see Theorem B in Section 1) is refined by inequalities (31)-(32) for uniformly convex functions.

We provide the following corollary for uniformly convex functions.

Corollary 6.3. *Let $f : I \to \mathbb{R}$ be a differentiable uniformly convex function with modulus φ. Let $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and $\mathbf{y} = (y_1, \dots, y_n) \in I^n$.*

If $\mathbf{y} \prec \mathbf{x}$, that is, $\mathbf{y} = \mathbf{x}P$ for some $n \times n$ doubly stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}\varphi(\|x_i - y_j\|) \le \sum_{i=1}^{n} f(x_i). \tag{35}$$

If in addition the modulus φ is convex and increasing, then

$$\sum_{j=1}^{n} f(y_j) + n\varphi(0) \le \sum_{i=1}^{n} f(x_i).$$

Proof. Apply Theorem 6.2 for $m = n$ and $\mathbf{a} = \mathbf{b} = (1, \dots, 1) \in \mathbb{R}^n$. ■

Let f be a c-strongly convex function. Then f is an uniformly convex function with modulus $\varphi(t) = \frac{c}{2}t^2$. Therefore it is obvious that inequalities (25), (31) and (35) take the following form, respectively,

$$f(\bar{x}) + \frac{c}{2}\sum_{i=1}^{m} p_i\|x_i - \bar{x}\|^2 \le \sum_{i=1}^{m} p_i f(x_i),$$

$$\sum_{j=1}^{n} b_j f(y_j) + \frac{c}{2}\sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij}\|x_i - y_j\|^2 \le \sum_{i=1}^{m} a_i f(x_i),$$

$$\sum_{j=1}^{n} f(y_j) + \frac{c}{2}\sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}\|x_i - y_j\|^2 \le \sum_{i=1}^{n} f(x_i).$$

7. Applications for Superquadratic Functions

Recall that a superquadratic function $f : \mathbb{R}_+^m \to \mathbb{R}$ on $I = \mathbb{R}_+^m$ is H-invex for

$$H(x, y) = \langle C(y), x - y\rangle + f(|x - y|) \quad \text{for } x, y \in I \tag{36}$$

(see (12)).

Proposition 7.1 (Cf. [1, Theorem 1]). *Let $f : \mathbb{R}_+^m \to \mathbb{R}$ be superquadratic on $I = \mathbb{R}_+^m$. Let $x_i \in I$ and $p_i \ge 0$, $i = 1, \dots, m$, with $\sum_{i=1}^{m} p_i = 1$. Denote $\bar{x} = \sum_{i=1}^{m} p_i x_i$.*

Then the following Jensen type inequality holds:

$$f(\bar{x}) + \sum_{i=1}^{m} p_i f(|x_i - \bar{x}|) \le \sum_{i=1}^{m} p_i f(x_i). \tag{37}$$

If in addition the function f is convex on I, then

$$f(\bar{x}) + f\left(\sum_{i=1}^{m} p_i|x_i - \bar{x}|\right) \le \sum_{i=1}^{m} p_i f(x_i). \tag{38}$$

Proof. Since f is H-invex function with H defined by (36), we find that

$$H(x_i, \bar{x}) = \langle C(\bar{x}), x_i - \bar{x}\rangle + f(|x_i - \bar{x}|).$$

So we can write

$$\sum_{i=1}^{m} p_i H(x_i, \bar{x}) = \sum_{i=1}^{m} p_i \langle C(\bar{x}), x_i - \bar{x} \rangle + \sum_{i=1}^{m} p_i f(|x_i - \bar{x}|)$$

$$= \langle C(\bar{x}), \sum_{i=1}^{m} p_i x_i - \bar{x} \rangle + \sum_{i=1}^{m} p_i f(|x_i - \bar{x}|) = \sum_{i=1}^{m} p_i f(|x_i - \bar{x}|).$$

Thus we obtain

$$\sum_{i=1}^{m} p_i H(x_i, \bar{x}) = \sum_{i=1}^{m} p_i f(|x_i - \bar{x}|).$$

Now, the required inequality (37) is due to (15) in Theorem 3.1.

If f is convex then (37) forces (38) via the standard Jensen inequality for f. ■

It follows from (37) that nonnegative superquadratic functions must be convex. This is in accordance with [3, Lemma 2.1]. In this case, statements (37)-(38) are refinements of the standard Jensen's inequality for convex functions.

We are now in a position to state a Sherman like inequality for superquadratic functions.

Theorem 7.2. *Let $f : \mathbb{R}_+^m \to \mathbb{R}$ be superquadratic on $I = \mathbb{R}_+^m$. Let $\mathbf{x} = (x_1, \ldots, x_m) \in I^m$, $\mathbf{y} = (y_1, \ldots, y_n) \in I^n$, $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}_+^m$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}_+^n$.*

If $\mathbf{y} = \mathbf{x}P$ and $\mathbf{a} = \mathbf{b}P^T$ for some $m \times n$ column stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j \sum_{i=1}^{m} p_{ij} f(|x_i - y_j|) \leq \sum_{i=1}^{m} a_i f(x_i). \tag{39}$$

If in addition the function f is convex, then

$$\sum_{j=1}^{n} b_j f(y_j) + \sum_{j=1}^{n} b_j f\left(\sum_{i=1}^{m} p_{ij} |x_i - y_j|\right) \leq \sum_{i=1}^{m} a_i f(x_i). \tag{40}$$

Proof. In light of the remark at the beginning of this section we find that $f : I \to \mathbb{R}$ is H-invex with H given by (36). Hence, by a similar proof to that of Proposition 7.1, we obtain

$$\sum_{i=1}^{m} p_{ij} H(x_i, y_j) = \sum_{i=1}^{m} p_{ij} f(|x_i - y_j|). \tag{41}$$

By making use of (41) and Theorem 4.1, eq. (22), we establish (39) and (40), completing the proof. ■

A HLPK type result for superquadratic functions is as follows.

Corollary 7.3. *Let $f : \mathbb{R}_+^n \to \mathbb{R}$ be superquadratic on $I = \mathbb{R}_+^n$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in I^n$.*

If $\mathbf{y} \prec \mathbf{x}$, that is, $\mathbf{y} = \mathbf{x}P$ for some $n \times n$ doubly stochastic matrix $P = (p_{ij})$, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij} f(|x_i - y_j|) \leq \sum_{i=1}^{n} f(x_i). \tag{42}$$

If in addition the function f is convex, then

$$\sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} f\left(\sum_{i=1}^{n} p_{ij} |x_i - y_j|\right) \leq \sum_{i=1}^{n} f(x_i). \tag{43}$$

Proof. In order to prove the required inequalities, it is sufficient to appeal to Theorem 7.2 with $m = n$ and $\mathbf{a} = \mathbf{b} = (1, \ldots, 1) \in \mathbb{R}^n$. ■

We conclude this section with the observation that if f is nonnegative superquadratic (and therefore, convex) then statements (39)-(40) and (42)-(43) are refinements of the standard Sherman's and HLPK's inequalities, respectively (see Theorem B and Theorem A in Section 1).

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