Filomat 31:15 (2017), 4703–4713
https://doi.org/10.2298/FIL17154703Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Conservative Matrices in Summability of Series

Dansheng Yu[a]

[a]*Department of Mathematics, Hangzhou Normal University, Hangzhou Zhejiang 310036 China.*

Abstract. In the present paper, we give some sufficient conditions for a matrix belongs to the class $B(\alpha_n, \beta_n; \gamma_n, \delta_n; \varphi)$ when $\varphi \in \Delta(p, q)$. Our results generalize the result of Das [4] and Yu [13]. Some applications of the main results are given.

1. Introduction

Let $\{s_n\}$ be the partial sums of the infinite series $\sum_{n=0}^{\infty} a_n$, The Cesàro means of order α of the series $\sum_{n=0}^{\infty} a_n$ are defined by

$$\sigma_n^\alpha := \frac{1}{A_n^\alpha} \sum_{j=0}^{n} A_{n-j}^{\alpha-1} s_j, \ n = 0, 1, \cdots,$$

where

$$A_n^\alpha := \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \quad n = 0, 1, \cdots.$$

Let (C, α) be the Cesàro matrix of order α, that is, (C, α) be the lower triangular matrix $\left(A_{n-v}^{\alpha-1}/A_n^\alpha\right)$.

Das [4] defined a matrix $T := \left(t_{nj}\right)$ to be absolutely kth-power conservative for $k \geq 1$, denoted by $T \in B(A_k)$, that is, if $\{s_n\}$ satisfies

$$\sum_{n=1}^{\infty} n^{k-1} |s_n - s_{n-1}|^k < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty,$$

2010 *Mathematics Subject Classification*. Primary 40F05, 40D25.
Keywords. conservative matrices, absolute summability.
Received: 09 May 2016; Accepted: 19 December 2016
Communicated by Dragan S. Djordjević
Email address: dsyu_math@163.com (Dansheng Yu)

where

$$t_n = \sum_{j=0}^{n} t_{nj} s_j.$$

Flett [5] introduced the concept of absolute summability of order k. A series $\sum_{n=0}^{\infty} a_n$ is summable $|C,\alpha|_k$, $k \geq 1$, $\alpha > -1$, if

$$\sum_{n=0}^{\infty} n^{k-1} \left|\sigma_{n-1}^{\alpha} - \sigma_n^{\alpha}\right|^k < \infty.$$

Flett [5] established the following inclusion theorem for $|C,\alpha|_k$. If the series $\sum_{n=0}^{\infty} a_n$ is summable $|C,\alpha|_k$, it is also summable for $|C,\alpha|_r$ for each $r \geq k \geq 1$, $\alpha > -1$, $\beta > \alpha + \frac{1}{k} - \frac{1}{r}$. Especially, a series $\sum_{n=0}^{\infty} a_n$ which is $|C,\alpha|_k$ summability is also $\left|C,\beta\right|_k$ summability for $k \geq 1$, $\beta > \alpha > -1$.

If one sets $\alpha = 0$, from the above inclusion result, we have

Theorem A. *Let $k \geq 1$, then $(C,\alpha) \in B(A_k)$ for $\alpha > 0$.*

As we know, the $k-$th power conservative matrices actually are results of comparison of summability fields of absolute summability methods. Many mathematicians have obtained a lot of important results by comparing different absolute summability methods. Here we remind readers some interesting papers of Sarigöl ([8]-[11]). For example, take $a_v = s_v - s_{v-1}$, $s_0 = 0$. Denoted by t_n and T_n the Riesz means (R,p_n) and (R,q_n) of the sequence $\{s_v\}$, respectively. It is called that a series $\sum a_v$ or a sequence s_v is summable $\left|R,p_n\right|_k$ $(k \geq 1)$ (see [10]), if $\{t_n^*\} \in l_k$, where

$$t_n^* := n^{1/k^*}(t_n - t_{n-1}) = \frac{n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v.$$

Since

$$T_n = \frac{1}{Q_n} \sum_{v=1}^{n-1} \Delta\left(\frac{q_v}{p_v}\right) P_v t_v + \frac{q_n P_n}{Q_n p_n} t_n = \sum_{v=1}^{n} d_{nv} t_v,$$

where

$$d_{nv} = \begin{cases} \frac{q_n P_v P_{v-1}}{Q_n Q_{n-1} p_v} \Delta\left(\frac{Q_{v-1}}{P_{v-1}}\right), & 1 \leq v < n, \\ \frac{q_n P_n}{Q_n p_n}, & v = n, \\ 0, & v > n. \end{cases}$$

It is easy to see that $D \in B(A_k)$ iff $\left|R,p_n\right|_k \Rightarrow \left|R,q_n\right|_k$. The author is indebted to Professor Sarigöl in Pamukkale University for providing this nice example.

There are many works have done to generalize the results of Das [4] and Flett [5](see [1]-[3], [12]-[16], for examples). Among them, we [13] generalized the concept of the absolutely kth-power conservative to the following

Definition 1.1. *Let $\varphi(x)$ be a nonnegative function defined on $[0,\infty)$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be nonnegative sequences. We say that a matrix*

$$T := \left(t_{nj}\right) \in B\left(\alpha_n, \beta_n; \gamma_n, \delta_n; \varphi\right),$$

if

$$\sum_{n=1}^{\infty} \alpha_n \varphi\left(\beta_n \left|s_n - s_{n-1}\right|\right) < \infty$$

implies that

$$\sum_{n=1}^{\infty} \gamma_n \varphi\left(\delta_n \left|t_n - t_{n-1}\right|\right) < \infty.$$

If $\alpha_n = \gamma_n = n^{-1},\ \beta_n = \delta_n = n,\ \varphi(x) = x^k,\ k \geq 1$, then $B(\alpha_n, \beta_n; \gamma_n, \delta_n; \varphi)$ reduces to $B(A_k)$.

Let $T := \left(t_{nj}\right)$ be a lower triangular matrix, $\lambda = \{\lambda_n\}$ be a positive sequence. Set

$$\widetilde{t}_{ni} := \begin{cases} \sum_{j=i}^{n} t_{nj} - \sum_{j=i}^{n-1} t_{n-1,j}, & 0 \leq i \leq n-1, \\ t_{nn}, & i = n, \end{cases}$$

$$\widetilde{T}_n(\lambda) := \sum_{j=1}^{n} \frac{\left|\widetilde{t}_{ni}\right|}{\lambda_i}.$$

We [13] established the following general result:

Theorem 1.2. *Let $\varphi(x)$ be a nonnegative convex function defined on $[0, \infty)$, $T := \left(t_{nj}\right)$ be a lower triangular matrix satisfying $\sum_{j=0}^{n} t_{nj} = 1$, and let $\{\alpha_n\}$ be a nonnegative sequence. If $\lambda = \{\lambda_n\}$ is a positive sequence such that*

$$\lambda_n^{-1} \sum_{j=n}^{\infty} \alpha_j \left|\widetilde{t}_{jn}\right| \left(\widetilde{T}_j(\lambda)\right)^{-1} = O(A_n),\ n \geq 1,$$

then

$$T \in B\left(A_n, \lambda_n; \alpha_n, \left(\widetilde{T}_n(\lambda)\right)^{-1}; \varphi\right).$$

Theorem 1.2 can be applied to test whether a Cesàro matrix or a Riesz matrix belong to $B(\alpha_n, \beta_n, \gamma_n, \delta_n, \varphi)$ or not. Especially, we [13] generalized Theorem A by applying Theorem 1.2 (see Theorem 3.3 in [13]).

Denote by $\Delta(p, q)\ (0 \leq q \leq p)$ the set of all nonnegative functions $\varphi(x)$ defined on $[0, \infty)$ such that $\varphi(0) = 0$, $\varphi(x)/x^p$ is nonincreasing and $\varphi(x)/x^q$ is nondecreasing. It is clear that $\Delta(p, q) \subset \Delta(p, 0)$ for $0 < q \leq p$. For example, $\Delta(p, 0)$ contains the function $\varphi(x) = \log(1 + x)$, $t^p \in \Delta(p, p)$ and $t^p \log(1 + t) \subset \Delta(p + 1, p)$ for $p > 0$.

We will establish two general results similar to Theorem 1.2 when $\varphi \in \Delta(p, q)$ in section 2 (Theorem 2.1 and Theorem 2.3), some applications of these two general results will be given in section 3.

Throughout the paper C_α denotes a positive constant depending only on α, their values may be different even in the same line. $\alpha_n \simeq \beta_n$ means that there is a positive constant C such that $C^{-1}\beta_n \leq \alpha_n \leq C\beta_n$.

2. Main Results

Firstly, we have

Theorem 2.1. *Let $\varphi(x) \in \Delta(p, q)\ (0 \leq q \leq p)$, $T := \left(t_{nj}\right)$ be a lower triangular matrix satisfying $\sum_{j=0}^{n} t_{nj} = 1$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\lambda = \{\lambda_n\}$ are positive sequences satisfying the following conditions:*

(A) There is a positive constant K_1 such that at least one of the conditions $\inf \beta_n \geq K_1$ and $\sup \beta_n \leq K_1$ holds;

(B) There is a positive constant K_2 such that $\inf \lambda_n \geq K_2 > 0$;

(C)

$$\sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} \left(\widetilde{T}_n(\lambda)\right)^{p^*-1} \left|\widetilde{t}_{ni}\right|^{q-p^*+1} = O\left(\lambda_i^{q-p^*+1} \gamma_i\right),\ i \geq 1, \tag{1}$$

where $p^* := \max(1, p)$ and

$$\theta := \begin{cases} q, & \text{if } \inf \beta_n = 0, \\ p, & \text{otherwise.} \end{cases}$$

Then

$$T \in B(\alpha_n, \beta_n; \gamma_n, \lambda_n; \varphi). \tag{2}$$

To prove Theorem 2.1, we need some properties of $\Delta(p,q)$:

Lemma 2.2. *([7]) Let $\Phi(x) \in \Delta(p,q)$ $(0 \le q \le p)$ and $t_j \ge 0,\ j = 1, 2, \cdots$. Then*
(a) $\eta^p \Phi(t) \le \Phi(\eta t) \le \eta^q \Phi(t)$ for $0 \le \eta \le 1,\ t \ge 0$;
(b) $\Phi(\eta t) \le \eta^p \Phi(t)$ for $\eta \ge 1,\ t \ge 0$;
(c) $\Phi\left(\sum_{j=1}^n t_j\right) \le \left(\sum_{j=1}^n \Phi^{1/p^*}\left(t_j\right)\right)^{p^*}$;
(d) $\Phi(x)$ is nondecreasing.

Remark. From (a) and (b) in Lemma 2.2, we have

$$\Phi(\beta_n t) = O(1)\beta_n^q \Phi(t),$$

when $\sup \beta_n \le K_1$, and

$$\Phi(\beta_n t) = O(1)\beta_n^p \Phi(t),$$

when $\inf \beta_n \ge K_1$. In other words, we have

$$\Phi(\beta_n t) = O(1)\beta_n^\theta \Phi(t).$$

Proof of Theorem 2.1 Since (set $s_{-1} := 0$)

$$\begin{aligned} t_n &= \sum_{j=0}^n t_{nj} s_j = \sum_{j=0}^n t_{nj} \left(\sum_{i=0}^j (s_i - s_{i-1}) \right) \\ &= \sum_{i=0}^n (s_i - s_{i-1}) \left(\sum_{j=i}^n t_{nj} \right), \end{aligned}$$

then

$$\begin{aligned} t_n - t_{n-1} &= \sum_{i=0}^n (s_i - s_{i-1}) \left(\sum_{j=i}^n t_{nj} \right) - \sum_{i=0}^{n-1} (s_i - s_{i-1}) \left(\sum_{j=i}^{n-1} t_{n-1,j} \right) \\ &= \sum_{i=0}^n \widetilde{t_{ni}} (s_i - s_{i-1}) = \sum_{i=1}^n \widetilde{t_{ni}} (s_i - s_{i-1}), \end{aligned}$$

where in the last inequality, we used the fact $\widetilde{t_{n0}} = 0$, which follows from $\sum_{j=0}^n t_{nj} = 1$ and the definition of $\widetilde{t_{n0}}$.

Since $\inf \lambda_n \ge K_2 > 0$ and T is a lower triangular matrix satisfying $\sum_{j=0}^n t_{nj} = 1$, we see that $\frac{\left|\widetilde{t_{ni}}\right|}{\lambda_i} = O(1)$. Then, by Lemma 2.2 and (1), we get

$$\begin{aligned} \sum_{n=1}^\infty \alpha_n \varphi\left(\beta_n |t_n - t_{n-1}|\right) &\le \sum_{n=1}^\infty \alpha_n \varphi\left(\beta_n \sum_{i=1}^n \lambda_i^{-1} \left|\widetilde{t_{ni}}\right| (\lambda_i |s_i - s_{i-1}|)\right) \\ &= O(1) \sum_{n=1}^\infty \alpha_n \beta_n^\theta \varphi\left(\sum_{i=1}^n \lambda_i^{-1} \left|\widetilde{t_{ni}}\right| (\lambda_i |s_i - s_{i-1}|)\right) \\ &= O(1) \sum_{n=1}^\infty \alpha_n \beta_n^\theta \left(\sum_{i=1}^n \varphi^{1/p^*}\left(\lambda_i^{-1} \left|\widetilde{t_{ni}}\right| (\lambda_i |s_i - s_{i-1}|)\right)\right)^{p^*} \end{aligned}$$

$$= O(1)\sum_{n=1}^{\infty}\alpha_n\beta_n^{\theta}\left(\sum_{i=1}^{n}\left(\frac{\left|\widetilde{t}_{ni}\right|}{\lambda_i}\right)^{q/p^*}\varphi^{1/p^*}\left(\lambda_i\left|s_i-s_{i-1}\right|\right)\right)^{p^*}$$

$$= O(1)\sum_{n=1}^{\infty}\alpha_n\beta_n^{\theta}\left(\sum_{i=1}^{n}\frac{\left|\widetilde{t}_{ni}\right|}{\lambda_i}\right)^{p^*-1}\left(\sum_{i=1}^{n}\left(\frac{\left|\widetilde{t}_{ni}\right|}{\lambda_i}\right)^{q-p^*+1}\varphi\left(\lambda_i\left|s_i-s_{i-1}\right|\right)\right)$$

(by Hölder's inequality)

$$= \sum_{i=1}^{\infty}\varphi\left(\lambda_i\left|s_i-s_{i-1}\right|\right)\lambda_i^{p^*-q-1}\sum_{n=i}^{\infty}\alpha_n\beta_n^{\theta}\left(\widetilde{T}_n(\lambda)\right)^{p^*-1}\left|\widetilde{t}_{ni}\right|^{q-p^*+1}$$

$$= O(1)\sum_{i=1}^{\infty}\gamma_i\varphi\left(\lambda_i\left|s_i-s_{i-1}\right|\right),$$

which implies (2).

Theorem 2.3. *Let $T = \left(t_{nj}\right)$ be a lower triangular matrix with the entries t_{nj} having the form $\frac{p_j}{P_n}$, where $p_j \geq 0$ for $0 \leq j \leq n$ and $P_n = \sum_{j=0}^{n} p_j > 0$. Let $\varphi \in \Delta(p,q)$ $(0 \leq q \leq p)$ and $\{\alpha_n\}$, $\lambda = \{\lambda_n\}$ be positive sequences. If*

$$\sum_{i=n}^{\infty}\alpha_i\left(\widetilde{T}_i(\lambda)\frac{P_iP_{i-1}}{p_i}\right)^{-q} = O\left(n\alpha_n\left(\widetilde{T}_n(\lambda)\frac{P_nP_{n-1}}{p_n}\right)^{-q}\right), \tag{3}$$

then

$$T \in B\left(B_n,\lambda_n;\alpha_n,\left(\widetilde{T}_n(\lambda)\right)^{-1};\varphi\right)$$

where

$$B_n = n^{p^*}\alpha_n\left(\widetilde{T}_n(\lambda)\frac{P_n}{p_n}\right)^{-q}\lambda_n^{-q}.$$

Lemma 2.4. *([6]) Let $p \geq 1$, $\alpha_n \geq 0$, $\lambda_n > 0$, then*

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=1}^{n}\alpha_k\right)^p \leq p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=n}^{\infty}\lambda_k\right)^p\alpha_n^p.$$

Proof of Theorem 2.3 First, we have

$$\begin{aligned}\widetilde{t}_{ni} &= \sum_{j=i}^{n}t_{nj}-\sum_{j=i}^{n-1}t_{nj}\\ &= \frac{p_n}{P_n}+\left(\frac{1}{P_n}-\frac{1}{P_{n-1}}\right)\sum_{j=i}^{n-1}p_j\\ &= \frac{p_n}{P_n}-\frac{p_n}{P_nP_{n-1}}\left(P_{n-1}-P_{i-1}\right)\\ &= \frac{p_nP_{i-1}}{P_nP_{n-1}},\ 1 \leq i \leq n-1,\end{aligned} \tag{4}$$

and

$$\widetilde{t}_{n0} = 0,\ \widetilde{t}_{nn} = \frac{p_n}{P_n}. \tag{5}$$

Noting that

$$\left(\widetilde{T_n}(\lambda)\right)^{-1}\frac{\left|\widetilde{t_{ni}}\right|}{\lambda_i}\le 1,\ 0\le i\le n,$$

by Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\alpha_n\varphi\left(\left(\widetilde{T_n}(\lambda)\right)^{-1}|t_n-t_{n-1}|\right)\\
&\le\sum_{n=1}^{\infty}\alpha_n\varphi\left(\left(\widetilde{T_n}(\lambda)\right)^{-1}\sum_{i=1}^{n}\lambda_i^{-1}\left|\widetilde{t_{ni}}\right|(\lambda_i|s_i-s_{i-1}|)\right)\\
&\le\sum_{n=1}^{\infty}\alpha_n\left(\sum_{i=1}^{n}\varphi^{1/p^*}\left(\left(\widetilde{T_n}(\lambda)\right)^{-1}\lambda_i^{-1}\left|\widetilde{t_{ni}}\right|(\lambda_i|s_i-s_{i-1}|)\right)\right)^{p^*}\\
&\le\sum_{n=1}^{\infty}\alpha_n\left(\sum_{i=1}^{n}\left(\widetilde{T_n}(\lambda)\lambda_i\right)^{-q/p^*}\left|\widetilde{t_{ni}}\right|^{q/p^*}\varphi^{1/p^*}(\lambda_i|s_i-s_{i-1}|)\right)^{p^*}\\
&=O(1)\sum_{n=1}^{\infty}\alpha_n\left(\widetilde{T_n}(\lambda)\frac{P_nP_{n-1}}{p_n}\right)^{-q}\times\left(\sum_{i=1}^{n}\left(\lambda_i^{-1}P_{i-1}\right)^{q/p^*}\varphi^{1/p^*}(\lambda_i|s_i-s_{i-1}|)\right)^{p^*}\\
&=O(1)\sum_{n=1}^{\infty}\left(\alpha_n\left(\widetilde{T_n}(\lambda)\frac{P_nP_{n-1}}{p_n}\right)^{-q}\right)^{1-p^*}\\
&\quad\times\left(\lambda_n^{-1}P_{n-1}\right)^{q}\varphi(\lambda_n|s_n-s_{n-1}|)\times\left(\sum_{i=n}^{\infty}\alpha_i\left(\widetilde{T_i}(\lambda)\frac{P_iP_{i-1}}{p_i}\right)^{-q}\right)^{p^*}\\
&=O(1)\sum_{n=1}^{\infty}n^{p^*}\alpha_n\left(\widetilde{T_n}(\lambda)\frac{P_n}{p_n}\right)^{-q}\lambda_n^{-q}\varphi(\lambda_n|s_n-s_{n-1}|),
\end{aligned}$$

which completes the proof of Theorem 2.3.

3. Applications of The Main Results

We will use the following estimate frequently (see [17]):

$$A_n^{\alpha}\simeq\frac{n^{\alpha}}{\Gamma(\alpha+1)},\ \alpha>-1. \tag{6}$$

Theorem 3.1. *Let $\varphi(x)\in\Delta(p,q)\,(0\le q\le p)$, $\{\alpha_n\}$, $\{\beta_n\}$ be positive sequences satisfying*

(i) There is a positive constant K such that at least one of the conditions $\inf\beta_n\ge K$ and $\sup\beta_n\le K$ holds;

(ii) $\alpha_m\simeq\alpha_n$, $\beta_m\simeq\beta_n$ for any $n\le m\le 2n$;

(iii) $\sum_{n=2i+1}^{\infty}n^{-2(q-p^*+1)+\mu(1-p^*)}\alpha_n\beta_n^{\theta}=O\left(i^{-2(q-p^*+1)+\mu(1-p^*)+1}\alpha_i\beta_i^{\theta}\right)$.

Then

$$(C,\alpha)\in B(\alpha_n,\beta_n;\gamma_n,n^{\mu};\varphi),\quad \alpha>0,\ 0\le\mu<2,$$

where

$$\gamma_n:=\begin{cases} n^{-(1+\mu)q+p^*}\alpha_n\beta_n^{\theta}, & \alpha(p^*-q-1)<p^*-q,\\ n^{-(1+\mu)q+p^*}(\log n)\,\alpha_n\beta_n^{\theta}, & \alpha(p^*-q-1)=p^*-q,\\ n^{\alpha(p^*-q-1)-\mu q}\alpha_n\beta_n^{\theta}, & \alpha(p^*-q-1)>p^*-q.\end{cases}$$

Proof. Let

$$t_{nj} := \frac{A_{n-j}^{\alpha-1}}{A_n^{\alpha}},\ j = 0, 1, \cdots, n;\ \alpha > -1.$$

Then for $0 \le i \le n-1$,

$$\begin{aligned}\widetilde{t_{ni}} &= \frac{1}{A_n^{\alpha}} \sum_{j=i}^{n} A_{n-j}^{\alpha-1} - \frac{1}{A_{n-1}^{\alpha}} \sum_{j=i}^{n} A_{n-1-j}^{\alpha-1} \\ &= \frac{1}{A_n^{\alpha}} \sum_{j=0}^{n-i} A_{j}^{\alpha-1} - \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n-1-i} A_{j}^{\alpha-1} \\ &= \frac{A_{n-i}^{\alpha}}{A_n^{\alpha}} - \frac{A_{n-1-i}^{\alpha}}{A_{n-1}^{\alpha}} = \frac{i}{n}\frac{A_{n-i}^{\alpha-1}}{A_n^{\alpha}},\end{aligned} \tag{7}$$

and

$$\widetilde{t_{nn}} = \frac{A_0^{\alpha-1}}{A_n^{\alpha}} = \frac{1}{A_n^{\alpha}}. \tag{8}$$

Taking $\lambda_n = n^{\mu},\ n \ge 1,\ 0 \le \mu < 2$, by (6)-(8), we have

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^{-1} \left|\widetilde{t_{ni}}\right| = \sum_{i=1}^{n} \frac{\left|\widetilde{t_{ni}}\right|}{i^{\mu}} &= \frac{1}{nA_n^{\alpha}} \sum_{i=1}^{n} i^{1-\mu} A_{n-v}^{\alpha-1} \\ &= O\left(n^{-1-\alpha}\right)\left(n^{\alpha-1}\sum_{i=1}^{n/2} i^{1-\mu} + n^{1-\mu}\sum_{i=n/2+1}^{n}(n-v+1)^{\alpha-1}\right) \\ &= O\left(n^{-\mu}\right).\end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned}\sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} \left(\sum_{v=1}^{n} \frac{\left|\widetilde{t_{nv}}\right|}{\lambda_v}\right)^{p^*-1} \left|\widetilde{t_{ni}}\right|^{q-p^*+1} &= O\left(\sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} n^{-\mu(p^*-1)} \left|\widetilde{t_{ni}}\right|^{q-p^*+1}\right) \\ &= O\left(i^{q-p^*+1}\right)\left(\sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} n^{-\mu(p^*-1)} \left(\frac{\left|A_{n-i}^{\alpha-1}\right|}{nA_n^{\alpha}}\right)^{q-p^*+1}\right) \\ &= O\left(i^{q-p^*+1}\right)\left(\sum_{n=i}^{2i} + \sum_{n=2i+1}^{\infty}\right) \\ &=: I_1 + I_2.\end{aligned} \tag{10}$$

By (6) and (ii), we have

$$\begin{aligned}I_1 &= O\left(i^{q-p^*+1}\right) \sum_{n=i}^{2i} \alpha_n \beta_n^{\theta} n^{-\mu(p^*-1)} \left(\frac{\left|A_{n-i}^{\alpha-1}\right|}{nA_n^{\alpha}}\right)^{q-p^*+1} \\ &= O\left(i^{q-p^*+1} \alpha_i \beta_i^{\theta} i^{-\mu(p^*-1)}\right) \sum_{n=i}^{2i} \left(\frac{\left|A_{n-i}^{\alpha-1}\right|}{nA_n^{\alpha}}\right)^{q-p^*+1} \\ &= O\left(i^{q-p^*+1+(1+\alpha)(p^*-q-1)+\mu(1-p^*)} \alpha_i \beta_i^{\theta}\right) \sum_{n=i}^{2i} (n+1-i)^{(q-p^*+1)(\alpha-1)} \\ &= O\left(i^{\alpha(p^*-q-1)+\mu(1-p^*)} \alpha_i \beta_i^{\theta} A_i\right),\end{aligned} \tag{11}$$

where

$$A_i := \begin{cases} i^{(q-p^*+1)(\alpha-1)+1}, & (q-p^*+1)(\alpha-1) > -1, \\ \log i, & (q-p^*+1)(\alpha-1) = -1, \\ 1 & (q-p^*+1)(\alpha-1) < -1. \end{cases} \tag{12}$$

By (6) and (iii), we have

$$\begin{aligned} I_2 &= O\left(i^{q-p^*+1}\right) \sum_{n=2i+1}^{\infty} \frac{(n-i)^{(q-p^*+1)(\alpha-1)}}{n^{(q-p^*+1)(\alpha+1)}} \alpha_n \beta_n^{\theta} n^{\mu(1-p^*)} \\ &= O\left(i^{q-p^*+1}\right) \sum_{n=2i+1}^{\infty} n^{-2(q-p^*+1)+\mu(1-p^*)} \alpha_n \beta_n^{\theta} \\ &= O\left(i^{p^*-q+\mu(1-p^*)} \alpha_i \beta_i^{\theta}\right). \end{aligned} \tag{13}$$

Therefore, by (10)-(13) and Theorem 2.1, we prove Theorem 3.1. □

A non-negative sequence $\{a_n\}$ is said to be almost decreasing, if there is a positive constant K such that $a_n \geq K a_m$ for all $n \leq m$, and it is said to be quasi-β−power increasing with some real number β, if

Corollary 3.2. *Let $\varphi(x) = x^p$, then*

(a) If $\{\alpha_n\}$ is quasi-ε-power decreasing with some $\varepsilon > 0$ and satisfies the condition (ii) in Theorem 3.1, then

$$(C, \alpha) \in B(\alpha_n, n; \alpha_n, n; \varphi) \tag{14}$$

for $\alpha > 0,\ p \geq 1$. Especially, if $\delta < \frac{1}{p},\ \gamma \in R$, then

$$(C, \alpha) \in B\left(n^{\delta p-1} \log^{\gamma} n, n; n^{\delta p-1} \log^{\gamma} n, n; \varphi\right) \tag{15}$$

for $\alpha > 0,\ p \geq 1$.

(b) If $\{\alpha_n\}$ is quasi-ε-power decreasing for some $\varepsilon > 1-p$ and satisfies the condition (ii) in Theorem 3.1, then (14) holds for $\alpha > 0,\ 0 < p < 1$. Especially, if $\delta < 1, \gamma \in R$, then (15) holds for $\alpha > 0,\ 0 < p < 1$.

Proof. (a) Since $\varphi(x) = x^p,\ p \geq 1$, we may take $q = p = p^*$. To prove (14), by Theorem 2.3, we only need to verify that (iii) in Theorem 3.1 holds with $\beta_n = n, \mu = 1$ ($\theta = p$ in this case). Since $\{\alpha_n\}$ is quasi-ε-power decreasing with $\varepsilon > 0$, then

$$\begin{aligned} \sum_{n=2i+1}^{\infty} n^{-2(q-p^*+1)+\mu(1-p^*)} \alpha_n \beta_n^{\theta} &= O(1) \sum_{n=2i+1}^{\infty} n^{-1-\varepsilon} \alpha_n n^{\varepsilon} \\ &= O\left(\alpha_i i^{\varepsilon}\right) \sum_{n=2i+1}^{\infty} n^{-1-\varepsilon} \\ &= O\left(i^{-1} \alpha_i\right), \end{aligned}$$

which means (iii).

If $\delta < \frac{1}{p}$, then there is an $\varepsilon > 0$ such that $\delta p - 1 + \varepsilon < 0$, hence $\left\{n^{\delta p-1} \log^{\gamma} n\right\}$ is quasi-ε−power decreasing for any $\gamma \in R$. Now, applying (14), we get (15).

(b) Since $\varphi(x) = x^p,\ 0 < p < 1$, we may take $q = p,\ p^* = 1$. Let $\beta_n = n, \mu = 1$ ($\theta = p$ again), then

$$\begin{aligned} \sum_{n=2i+1}^{\infty} n^{-2(q-p^*+1)+\mu(1-p^*)} \alpha_n \beta_n^{\theta} &= O(1) \sum_{n=2i+1}^{\infty} n^{-p-\varepsilon} n^{\varepsilon} \alpha_n \\ &= O\left(i^{\varepsilon} \alpha_i\right) \sum_{n=2i+1}^{\infty} n^{-p-\varepsilon} \\ &= O\left(i^{1-p} a_i\right) \end{aligned}$$

for $\varepsilon > 1 - p$, which implies (iii), and hence (14).

If $\delta < 1$, then $\delta p - 1 + 1 - p < 0$, which implies that there exists an $\varepsilon > 1 - p$ such that $\delta p - 1 + \varepsilon < 0$. Thus $\left\{n^{\delta p-1}\log^{\gamma} n\right\}$ is quasi-ε−power decreasing for any $\gamma \in R$. (15) is proved. □

Remark. Theorem A is the special case when $\delta = \gamma = 0$ in (15).

Theorem 3.3. *Let $\varphi(x) \in \Delta(p,q)\,(0 \le q \le p)$, $\{\alpha_n\}$, $\{\beta_n\}$ be nonnegative sequences satisfying (i), (ii) in Theorem 3.1 and*

(iv) $\sum_{n=2i+1}^{\infty} n^{-2(q-p^*+1)+(\mu+\alpha)(1-p^*)}\alpha_n\beta_n^{\theta} = O\left(i^{-2(q-p^*+1)+(\mu+\alpha)(1-p^*)+1}\alpha_i\beta_i^{\theta}\right)$.

Then

$$(C,\alpha) \in B\left(\alpha_n, \beta_n; \gamma_n, n^{\mu}; \varphi\right), \ \ -1 < \alpha < 0,\ 0 \le \mu < 2,$$

where

$$\gamma_n := \begin{cases} n^{(1-\alpha)p^*-(1+\mu)q+\alpha}\alpha_n\beta_n^{\theta}, & \alpha(p^*-q-1) < p^*-q, \\ n^{(1-\alpha)p^*-(1+\mu)q+\alpha}(\log n)\,\alpha_n\beta_n^{\theta}, & \alpha(p^*-q-1) = p^*-q, \\ n^{-(\mu+\alpha)q}\alpha_n\beta_n^{\theta}, & \alpha(p^*-q-1) > p^*-q. \end{cases}$$

Proof. Similar to (9), we have

$$\sum_{i=1}^{n}\lambda_i^{-1}\left|\widetilde{t_{ni}}\right| = O\left(n^{-1-\alpha}\right)\left(n^{\alpha-1}\sum_{i=1}^{n/2} i^{1-\mu} + n^{1-\mu}\sum_{i=n/2+1}^{n}(n-v+1)^{\alpha-1}\right) = O\left(n^{-\mu-\alpha}\right),$$

hence

$$\begin{aligned}\sum_{n=i}^{\infty}\alpha_n\beta_n^{\theta}\left(\sum_{v=1}^{n}\frac{\left|\widetilde{t_{nv}}\right|}{\lambda_v}\right)^{p^*-1}\left|\widetilde{t_{ni}}\right|^{q-p^*+1} &= O\left(\sum_{n=i}^{\infty}\alpha_n\beta_n^{\theta}n^{-(\mu+\alpha)(p^*-1)}\left|\widetilde{t_{ni}}\right|^{q-p^*+1}\right) \\ &= O\left(i^{q-p^*+1}\right)\left(\sum_{n=i}^{\infty}\alpha_n\beta_n^{\theta}n^{(\mu+\alpha)(1-p^*)}\left(\frac{\left|A_{n-i}^{\alpha-1}\right|}{nA_n^{\alpha}}\right)^{q-p^*+1}\right) \\ &= O\left(i^{q-p^*+1}\right)\left(\sum_{n=i}^{2i} + \sum_{n=2i+1}^{\infty}\right) \\ &=: J_1 + J_2. \end{aligned} \tag{16}$$

Similar to (11) and (13), we have

$$\begin{aligned} J_1 &= O\left(i^{q-p^*+1}\right)\sum_{n=i}^{2i}\alpha_n\beta_n^{\theta}n^{(\mu+\alpha)(1-p^*)}\left(\frac{\left|A_{n-i}^{\alpha-1}\right|}{nA_n^{\alpha}}\right)^{q-p^*+1} \\ &= O\left(i^{q-p^*+1+(\mu+\alpha)(1-p^*)}\alpha_i\beta_i^{\theta}A_i\right), \end{aligned} \tag{17}$$

and

$$\begin{aligned} J_2 &= O\left(i^{q-p^*+1}\right)\sum_{n=2i+1}^{\infty} n^{-2(q-p^*+1)+(\mu+\alpha)(1-p^*)}\alpha_n\beta_n^{\theta} \\ &= O\left(i^{p^*-q+(\mu+\alpha)(1-p^*)}\alpha_i\beta_i^{\theta}\right). \end{aligned} \tag{18}$$

where A_i is defined by (12). Therefore, we prove Theorem 3.3 by (16)-(18) and Theorem 2.3. □

Corollary 3.4. *Let $\varphi(x) = x^p$, then*

(a) If $\{\alpha_n\}$ is quasi-ε-power decreasing for some $\varepsilon > \alpha - \alpha p$ and satisfies the condition (iv) in Theorem 3.3, then (14) holds for all $-1 < \alpha < 0,\ p \geq 1$. Especially, if $\delta < \frac{1+\alpha p-\alpha}{p},\ \gamma \in R$, then (15) holds for all $-1 < \alpha < 0,\ p \geq 1$.

(b) If $\{\alpha_n\}$ is quasi-ε-power decreasing for some $\varepsilon > 1 - \alpha p$ and satisfies the condition (iv) in Theorem 3.3, then (14) holds for $-1 < \alpha < 0,\ 0 < p < 1$. Especially, if $\delta < 1, \gamma \in R$, then (15) holds for $-1 < \alpha < 0,\ 0 < p < 1$.

Proof. It can be proved in a way similar to Corollary 3.2, we omit the details here. □

Theorem 3.5. *Let $\varphi(x) \in \Delta(p,q)\ (0 \leq q \leq p)$, $T = \left(t_{nj}\right)$ be a lower triangular matrix with the members t_{nj} having the form $\frac{p_j}{P_n}$, where $p_j \geq 0$ for $0 \leq j \leq n$ and $P_n = \sum_{j=0}^{n} p_j > 0$, $\lambda = \{\lambda_n\}$ be a positive sequence. If*

(v) $np_n \simeq P_n$,

(vi) $\sum_{v=1}^{n} \frac{P_{v-1}}{\lambda_v} = O\left(\frac{nP_{n-1}}{\lambda_n}\right)$,

(vii) $\sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} \lambda_n^{1-p^*} n^{p^*-q-1} P_{n-1}^{p^*-q-1} = O\left(\alpha_i \beta_i^{\theta} \lambda_i^{1-p^*} i^{p^*-q} P_{i-1}^{p^*-q-1}\right)$,

then

$$T \in B\left(\alpha_n, \beta_n; \gamma_n, \lambda_n; \varphi\right),$$

where $\gamma_n = \alpha_i \beta_i^{\theta} \lambda_i^{-q} i^{p^*-q}$.

Proof. By (4), (5) and (vi), we have

$$\sum_{v=1}^{n} \frac{\left|\widetilde{t}_{nv}\right|}{\lambda_v} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}}{\lambda_v} = O\left(\frac{np_n}{P_n \lambda_n}\right).$$

Therefore, by (v) and (vii), we get

$$\begin{aligned}
&\sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} \left(\sum_{v=1}^{n} \frac{\left|\widetilde{t}_{nv}\right|}{\lambda_v}\right)^{p^*-1} \left|\widetilde{t}_{ni}\right|^{q-p^*+1} \\
&= O\left(P_{i-1}^{q-p^*+1}\right) \sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} \left(\frac{np_n}{P_n \lambda_n}\right)^{p^*-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^{q-p^*+1} \\
&= O\left(P_{i-1}^{q-p^*+1}\right) \sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} \lambda_n^{1-p^*} n^{p^*-q-1} P_{n-1}^{p^*-q-1} \\
&= O\left(\alpha_i \beta_i^{\theta} \lambda_i^{1-p^*} i^{p^*-q}\right) = O\left(\lambda_i^{q-p^*+1} \left(\alpha_i \beta_i^{\theta} \lambda_i^{-q} i^{p^*-q}\right)\right),
\end{aligned}$$

which together with Theorem 2.3 implies Theorem 3.5. □

Corollary 3.6. *Let $\varphi(x) = x^p,\ p > 0$, $T = \left(t_{nj}\right)$ be a lower triangular matrix with the members t_{nj} having the form $\frac{p_j}{P_n}$, where $p_j \geq 0$ for $0 \leq j \leq n$ and $P_n = \sum_{j=0}^{n} p_j > 0$. If $\{p_n\}$ and $\{\alpha_n\}$ satisfy (v), (vi) in Theorem 3.5 with $\lambda_n = n$, and*

$$\sum_{n=i}^{\infty} \alpha_n P_{n-1}^{-1} = O\left(i\alpha_i P_{i-1}^{-1}\right),\ \text{when } p \geq 1, \tag{19}$$

and

$$\sum_{n=i}^{\infty} \alpha_n P_{n-1}^{-q} = O\left(i\alpha_i P_{i-1}^{-q}\right),\ \text{when } 0 < p < 1, \tag{20}$$

Then

$$T \in B\left(\alpha_n, n; \gamma_n, n; \varphi\right), \tag{21}$$

where $\gamma_n = \alpha_n$ when $p \geq 1$ and $\gamma_n = n^{1-p}\alpha_n$.

Proof. We only prove the case when $p \geq 1$, the case when $0 < p < 1$ can be proved similarly. Let $\beta_n = \lambda_n = n$, by (19), we have

$$\sum_{n=i}^{\infty} \alpha_n \beta_n^{\theta} \lambda_n^{1-p^*} n^{p^*-q-1} P_{n-1}^{p^*-q-1} = \sum_{n=i}^{\infty} \alpha_n P_{n-1}^{-1} = O\left(i \alpha_i P_{i-1}^{-1}\right),$$

which means that the condition (vii) of Theorem 3.5 holds, and thus (21) is proved. □

Corollary 3.7. *Under the conditions of Corollary 3.6 with* $p_n = (n+1)^{\alpha}$ *,* $\alpha > -1$*, we have*
(a) if $\delta < \frac{1+\alpha}{p}$*,* $p \geq 1$*, then*

$$T \in B\left(n^{\delta p-1}, n; n^{\delta p-1}, n; \varphi\right).$$

(b) if $\delta < 1 + \alpha, 0 < p < 1$*, then*

$$T \in B\left(n^{\delta p-1}, n; n^{\delta p-p}, n; \varphi\right).$$

Proof. It is easy to verify that (19) and (20) are satisfied under the condition of (a) and the condition of (b) respectively. □

Acknowledgement: The author is very grateful to Professor Ali Sarigöl for his valuable suggestions.

References

[1] H. Bor, A note on two summability methods, Proc. Amer. Math. Soc., 98(1986), 81-84.
[2] H. Bor, A new factor theorem for generalized Cesàro summability, Filomat, 28(2014), 1205-1209.
[3] H. Bor, A new application of quasi power increasing sequences, II, Fixed Point Theory and Applications, 2013:75, doi: 10.1186/1687-1812-2013-75.
[4] G. Das, A tauberian theorem for absolute summability, Proc. Cambridge. Philos. Soc. 67(1970), 321-326.
[5] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc., 7 (1957), 113-141.
[6] L. Leindler, Generalization of inequalities of Hardy and Littlewood, Acta Sci. Math., 31(1970), 279-285.
[7] M. Mateljevic and M. Pavlovic, L^p−behavior of power series with positive coefficients and Hardy spaces, Proc. Amer. Math. Soc., 87 (1983), 309-316.
[8] M. Ali. Sarigol, Necessary and sufficient conditions for the equivalence of the summability methods $\left|\bar{N}, p_n\right|_k$ and $|C, 1|_k$, Indian J. Pure Appl. Math., 22(1991), 483-489.
[9] M. Ali. Sarigol, On absolute weighted mean summability methods, Proc. Amer. Math. Soc., 115(1992), 157-160.
[10] M. Ali. Sarigol, On two absolute Riesz summability factors of infinite series, Proc. Amer. Math. Soc., 118(1993), 485-488.
[11] M. Ali. Sarigol, On inclusion relations for absolute weighted mean summability, J. Math. Anal. Appl., 181(1994), 762-767.
[12] M. Ali. Sarigol, Matrix operators on sequence A_k, Math. Comput. Modelling, 55(2012), 1763-1769.
[13] D. S. Yu, Conservative matrices in summability of series, Filomat, 29(2015), 1085-1092.
[14] D. S. Yu, On a result of Flett for Cesàro matrices, Appl. Math. Letter, 22(2009), 1803-1809.
[15] D. S. Yu and P. Zhou, A new class of Matrices and its applications to absolute summability factors theorem, Math. Comput. Modelling, 57(2013), 401-412.
[16] D. S. Yu and G. Z. Zhou, A summability factors theorem for generalized absolute Cesàro summability, Math. Comput. Modelling, 53(2011), 832-838.
[17] A. Zygmund, Trigonometric Series, Cambridge Univ. Press, Cambridge, 1977.