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A New Version of the Real Paley-Wiener Theorems

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Abstract. In this paper, we establish new real Paley-Wiener theorems for the Dunkl transform.

1. Introduction

Let R be a finite root system in $\mathbb{R}^d$ with associated Coxeter-Weyl group W (see [1] for details on root systems) and for $j = 1, ..., d$, let T_j be the Dunkl operators defined by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle},$$

where R_+ is a subsystem of positive roots, σ_α is the reflection directed by the root $\alpha \in R_+$, k is a nonnegative multiplicity function defined on R.

The Dunkl theory is based on the Dunkl kernel $K(i\lambda, .)$, $\lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = i\lambda_j u(x), \quad j = 1, 2, ..., d,$$

satisfying the normalizing condition $u(0) = 1$.

With the Dunkl kernel $K(i\lambda, .)$, Dunkl have defined in [2] the Dunkl transform $\mathcal{F}_D$ by

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(-iy, x)\omega_k(x)dx, \quad \text{for all } y \in \mathbb{R}^d,$$

where

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \quad \text{and} \quad c_k = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}}\omega_k(x)\, dx,$$

and have established that the Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. The Plancherel's and inversion theorems are also established for this transform.

Very recently, many authors have been investigating the behavior of the Dunkl transform with respect to several problems already studied for the Fourier transform; for instance, uncertainty [4], Besov spaces [5], real Paley-Wiener theorems [6], generalized Sonine-type integral transforms [7], heat equation [8], maximal function [10], and so on.

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In this article we prove a new version of the real Paley-Wiener theorems for the Dunkl transform. We recall that, in [6] we have provide a general theory for the real Paley-Wiener theorems associated with the Dunkl transform. Finally, we note that in [3] the authors have studied a family of Wiener transforms associated with a pair of operators on Hilbert space.

The remaining part of the paper is organized as follows. The §2 is devoted to characterize the support for the Dunkl transform on the generalized Lebesgue spaces. The last section is devoted to characterize the support of the Dunkl transform under the behaviour of the generalized Lebesgue norms of iterated Dunkl potentials.

2. New Version of Real Paley-Wiener Theorem for the Dunkl Transform

Let $\Phi : [0,\infty) \longrightarrow [0,\infty)$ be a non-zero concave function which is non-decreasing and $\Phi(0) = 0$. Denote $N_{\Phi,k}(\mathbb{R}^d)$ the set of all measurable functions f such that

$$\|f\|_{N_{\Phi,k}(\mathbb{R}^d)} := \int_0^\infty \Phi(mes_k(f)(y))dy < \infty,$$

where

$$mes_k(f)(y) = \int_{\{x\in\mathbb{R}^d:\ |f(x)|>y\}} \omega_k(x)dx,$$

and by $M_{\Phi,k}(\mathbb{R}^d)$ the set of all measurable function g such that

$$\|g\|_{M_{\Phi,k}(\mathbb{R}^d)} := \sup\Big\{\frac{1}{\Phi(mes_k(I))}\int_I |g(x)|\omega_k(x)dx :\ I \subset \mathbb{R}^d, 0 < mes_k(I) < \infty\Big\} < \infty,$$

where

$$mes_k(I) = \int_I \omega_k(x)dx.$$

Note that if $\Phi(t) = t$ then $N_{\Phi,k}(\mathbb{R}^d) = L^1_k(\mathbb{R}^d) := L^1(\mathbb{R}^d, \omega_k(x)dx)$ and $M_{\Phi,k}(\mathbb{R}^d) = L^\infty_k(\mathbb{R}^d) := L^\infty(\mathbb{R}^d, \omega_k(x)dx)$.

We proceed as in [9], we prove.

Lemma 1. *There exists a positive constant C, such that for all $f \in N_{\Phi,k}(\mathbb{R}^d)$ and $g \in M_{\Phi,k}(\mathbb{R}^d)$, we have*

$$\Big|\int_{\mathbb{R}^d} f(x)g(x)\omega_k(x)dx\Big| \leq C\|f\|_{N_{\Phi,k}(\mathbb{R}^d)}\|g\|_{M_{\Phi,k}(\mathbb{R}^d)}.$$

Definition 1. *Let u be a distribution on $\mathbb{R}^d$ and P a polynomial. Then we let*

$$R(P,u) = \sup\Big\{|P(y)| :\quad y \in supp u\Big\} \in [0,\infty],$$

where by convention $R(P,u) = 0$ if $u = 0$.

Theorem 1. *Let P be a non-constant polynomial with complex coefficients on $\mathbb{R}^d$, and f in $\mathcal{S}(\mathbb{R}^d)$. Then in the extended positive real numbers*

$$\lim_{n\to\infty} \|P^n(-iT)f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R}^d)} = \sup_{y\in supp\mathcal{F}_D(f)} |P(y)|. \tag{1}$$

with $T = (T_1, ..., T_d)$.

For prove this theorem we need the following lemmas.

Lemma 2. *Let P be a non-constant polynomial with complex coefficients on $\mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Then in the extended positive real numbers*

$$\limsup_{n\to\infty} \|P^n(-iT)f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R}^d)} \leq \sup_{y\in supp\mathcal{F}_D(f)} |P(y)|. \tag{2}$$

Proof. Suppose firstly that $R(P,\mathcal{F}_D(f)) = 0$. Then $\mathcal{F}_D(f) = 0$, and as the Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, then $f = 0$. Thus (2) is immediately.

Moreover, the inequality (2), is clear when $R(P,\mathcal{F}_D(f)) = \infty$. So we can assume that

$$0 < R(P,\mathcal{F}_D(f)) < \infty.$$

Using the hypothesis on Φ, it is easy to see that there exist a positive constant C

$$\|f\|_{M_{\Phi,k}(\mathbb{R}^d)} \leq C\sup_{x\in\mathbb{R}^d} |(1+\|x\|^2)^m f(x)|,$$

for $m \geq \frac{d+1+2\sum_{\alpha\in R_+} k(\alpha)}{2}$.

Consequently for all $n \in \mathbb{N}$, we deduce that

$$\|P^n(-iT)f\|_{M_{\Phi,k}(\mathbb{R}^d)} \leq C\sup_{x\in\mathbb{R}^d} |(1+\|x\|^2)^m[(\mathcal{F}_D)^{-1}(P^n(\xi)\mathcal{F}_D(f))](x)|.$$

Using the continuity of $(\mathcal{F}_D)^{-1}$ we can show that

$$\|P^n(-iT)f\|_{M_{\Phi,k}(\mathbb{R}^d)} \leq C\sup_{\xi\in\mathbb{R}^d}\Big|\sum_{1\leq|l|,j\leq M}(1+\|\xi\|^2)^j D^l\Big[P^n(\xi)\mathcal{F}_D(f)(\xi)\Big]\Big|, \tag{3}$$

with positive constants C and integer M, independent of n. Using Leibniz's rule we deduce that

$$\|P^n(-iT)f\|_{M_{\Phi,k}(\mathbb{R}^d)} \leq Cn^M \sup_{y\in supp\mathcal{F}_D(f)} |P(y)|^{n-M},$$

with C is a constant independent of n. Hence, from the previous inequalities we obtain

$$\limsup_{n\to\infty} \|P^n(-iT)f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R}^d)} \leq \sup_{y\in supp\mathcal{F}_D(f)} |P(y)|.$$

Lemma 3. *Let P be a polynomial. Suppose that $P^n(-iT)f \in M_{\Phi,k}(\mathbb{R}^d)$ for all $n \in \mathbb{N}_0$. Then in the extended positive real numbers*

$$\liminf_{n\to\infty} \|P^n(-iT)f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R}^d)} \geq R(P,\mathcal{F}_D(f)). \tag{4}$$

Proof. Fix $\xi_0 \in supp\,\mathcal{F}_D(f)$. We can assume that $P(\xi_0) \neq 0$. We will show that

$$\liminf_{n\to\infty} \|P^n(-iT)f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R}^d)} \geq |P(\xi_0)| - \varepsilon,$$

for any fixed $\varepsilon > 0$ such that $0 < 2\varepsilon < |P(\xi_0)|$.
To this end, choose and fix $\chi \in D(\mathbb{R}^d)$ such that $\langle\mathcal{F}_D(f),\chi\rangle \neq 0$, and

$$supp\,\chi \subset \Big\{\xi \in \mathbb{R}^d : \ |P(\xi_0)| - \varepsilon < |P(\xi)| < |P(\xi_0)| + \varepsilon\Big\}.$$

For $n \in \mathbb{N}$, let $\chi_n(\xi) = P^{-n}(\xi)\chi(\xi)$.

As above, using the continuity of $(\mathcal{F}_D)^{-1}$ and Leibniz's rule we can show that

$$\forall\, x \in \mathbb{R}^d, \quad (1+\|x\|^2)^m|(\mathcal{F}_D)^{-1}(\chi_n)(x)| \leq C_1 n^M(|P(\xi_0)| - \varepsilon)^{-n}, \tag{5}$$

with positive constants C_1 and integer M, independent of n.

Using the hypothesis on Φ and the relation (5), it is easy to see that

$$\|(\mathcal{F}_D)^{-1}(\chi_n)\|_{N_{\Phi,k}(\mathbb{R}^d)} \quad \leq \quad C_2 n^M(|P(\xi_0)| - \varepsilon)^{-n}.$$

Then, since

$$\langle \mathcal{F}_D(f), \chi\rangle \quad = \quad \langle P^n(-iT)f, (\mathcal{F}_D)^{-1}(\chi_n)\rangle,$$

we use the Lemma 1, to obtain

$$\begin{aligned} |\langle \mathcal{F}_D(f), \chi\rangle| &\leq C\|P^n(-iT)f\|_{M_{\Phi,k}(\mathbb{R}^d)}\|(\mathcal{F}_D)^{-1}(\chi_n)\|_{N_{\Phi,k}(\mathbb{R}^d)} \\ &\leq C_3 n^M(|P(\xi_0)| - \varepsilon)^{-n}\|P^n(-iT)f\|_{M_{\Phi,k}(\mathbb{R}^d)}. \end{aligned}$$

Since $|\langle \mathcal{F}_D(f), \chi\rangle| > 0$, we deduce that

$$\liminf_{n\to\infty} \|P^n(-iT)f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R}^d)} \geq |P(\xi_0)| - \varepsilon.$$

Thus

$$\liminf_{n\to\infty} \|P^n(-iT)f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R}^d)} \geq \sup_{y\in supp\,\mathcal{F}_D(f)} |P(y)|.$$

Proof. of Theorem 1. Combining Lemma 2 and Lemma 3 together, we get the result.

3. Characterization for the Support of the Dunkl Transform on $M_{\Phi,k}(\mathbb{R})$ via the Dunkl Potential

In this section, we consider for $d = 1$, the root system $\mathcal{R} = \{\pm\alpha, \pm 2\alpha\}$, with $\alpha = 2$. Here $\mathcal{R}_+ = \{\alpha, 2\alpha\}$, and the reflection group $W = \mathbb{Z}_2$. The multiplicity function is a single positive parameter denoted by k.

The Dunkl operator T_k is defined for f of class C^1 on $\mathbb{R}$, and $x \in \mathbb{R}\backslash\{0\}$, by

$$T_k f(x) = \frac{d}{dx} f(x) + \frac{k}{x}\{f(x) - f(-x)\}.$$

Definition 2. *Let f be in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$. The tempered generalized function $R_0 f$ is termed the Dunkl potential of f if $T_k(R_0 f) = f$, that is*

$$\langle R_0 f, T_k\varphi\rangle = -\langle f, \varphi\rangle, \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

Theorem 2. *Let $R_0^n f \in M_{\Phi,k}(\mathbb{R})$ for all $n \in \mathbb{N}_0$, then*

$$\lim_{n\to\infty} \|R_0^n f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R})} = \frac{1}{\sigma_0}, \tag{6}$$

where

$$\sigma_0 = \inf\Big\{|\xi| \ : \ \xi \in supp\mathcal{F}_D(f)\Big\}.$$

For prove this theorem we need the following lemmas.

Lemma 4. *If $\sigma_0 > 0$, then*

$$supp\,\mathcal{F}_D\Big(R_0^n f\Big) = supp\,\mathcal{F}_D(f), \quad n = 1, ... \tag{7}$$

Proof. As

$$T_k^n(R_0^n f) = f$$

we deduce that

$$\mathcal{F}_D(f) = (i\xi)^n \mathcal{F}_D\Big(R_0^n f\Big).$$

Therefore,

$$supp\,\mathcal{F}_D(f) \subset supp\,\mathcal{F}_D\Big(R_0^n f\Big) \subset \mathcal{F}_D(f) \cup \Big\{0\Big\}.$$

So, to obtain (7), it is enough to show that $0 \notin supp\,\mathcal{F}_D\Big(R_0^n f\Big)$.
We choose numbers $a, b : 0 < a < b < \sigma_0$ and a function $h \in D(\mathbb{R})$ such that $supp\,h \subset (-b, b)$ and $h(x) \equiv 1$ in $(-a, a)$. Then

$$supp\Big(h\mathcal{F}_D(R_0^n f)\Big) \subset \Big\{0\Big\}.$$

Suppose that $supp\Big(h\mathcal{F}_D(R_0^n f)\Big) = \Big\{0\Big\}$, then there is a numbers $N(n) \in \mathbb{N}$ such that

$$h\mathcal{F}_D\Big(R_0^n f\Big) = \sum_{j=0}^{N(n)} C_j(N(n)) T_k^j \delta.$$

Hence,

$$\forall\, x \in \mathbb{R}, \quad \mathcal{F}_D^{-1}(h) *_k R_0^n f(x) = \sum_{j=0}^{N(n)} C_j(N(n))(-ix)^j,$$

where $*_k$ is the Dunkl convolution product.
As $R_0^n f \in M_{\Phi,k}(\mathbb{R})$ and $\mathcal{F}_D^{-1}(h) \in N_{\Phi,k}(\mathbb{R})$, we get $\mathcal{F}_D^{-1}(h) *_k R_0^n f \in L_k^\infty(\mathbb{R})$.
Therefore

$$\forall\, x \in \mathbb{R}, \quad \mathcal{F}_D^{-1}(h) *_k R_0^n f(x) = C_0(N(n)), \; n \in \mathbb{N}.$$

Note that

$$\begin{aligned} C_0(N(n)) &= \mathcal{F}_D^{-1}(h) *_k R_0^n f(x) = \mathcal{F}_D^{-1}(h) *_k \Big(T_k\Big) R_0^{n+1} f(x) \\ &= \Big(T_k\Big)\Big(\mathcal{F}_D^{-1}(h) *_k R_0^{n+1} f(x)\Big) = \Big(T_k\Big)(C_0(N(n+1))) = 0. \end{aligned}$$

Thus we deduce that $C_0(N(n)) = 0$. So $h\mathcal{F}_D\Big(R_0^n f\Big) = 0$.
Assume now the contrary that

$$\Big\{0\Big\} \subset supp\; \mathcal{F}_D\Big(R_0^n f\Big).$$

Then there is a function $\chi \in D(\mathbb{R})$, with $supp\,\chi \subset (-a, a)$ and such that

$$\langle \mathcal{F}_D\Big(R_0^n f\Big), \chi \rangle \neq 0.$$

So, as $h(x) = 1$ for $|x| < a$, we get

$$0 \neq \langle \mathcal{F}_D\Big(R_0^n f\Big), \chi \rangle = \langle \mathcal{F}_D\Big(R_0^n f\Big), h\chi \rangle = \langle h\mathcal{F}_D\Big(R_0^n f\Big), \chi \rangle = 0,$$

which is impossible. Thus we have proved (7).

Lemma 5. *If $\sigma_0 > 0$, then*

$$\limsup_{n\to\infty} \|R_0^n f\|_{M_{\Phi,k}(\mathbb{R})}^{\frac{1}{n}} \leq \frac{1}{\sigma_0}. \tag{8}$$

Proof. From (7) we have

$$supp\,\mathcal{F}_D\Big(R_0^n f\Big) \subset \mathbb{R}\backslash(-\sigma_0, \sigma_0). \tag{9}$$

For any $\varepsilon > 0$, $\varepsilon < \frac{\sigma_0}{2}$ we choose a function $h \in C^\infty(\mathbb{R})$ satisfying

$$h(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \geq \sigma_0 - \varepsilon \\ 0 & \text{if} \quad |\xi| < \sigma_0 - 2\varepsilon. \end{cases}$$

Let χ be an arbitrary element in $\mathcal{S}(\mathbb{R})$. Then it follow from (9) that

$$\langle R_0^n f, \chi\rangle = \langle \mathcal{F}_D\Big(R_0^n f\Big), \mathcal{F}_D^{-1}(\chi)\rangle = \langle \mathcal{F}_D\Big(R_0^n f\Big), h\mathcal{F}_D^{-1}(\chi)\rangle = \langle R_0^n f, \mathcal{F}_D\Big(h\mathcal{F}_D^{-1}(\chi)\Big)\rangle.$$

Therefore,

$$\langle R_0^n f, \chi\rangle = \langle R_0^n f, \varphi\rangle, \tag{10}$$

where

$$\varphi = \mathcal{F}_D\Big(h\mathcal{F}_D^{-1}(\chi)\Big).$$

We put

$$\varphi_n = \mathcal{F}_D\Big(\frac{h(\xi)}{\xi^n}\mathcal{F}_D^{-1}(\chi)\Big).$$

Then $\varphi_n \in \mathcal{S}(\mathbb{R})$ and

$$|\langle f, \varphi_n\rangle| = |\langle (T_k)^n R_0^n f, \varphi_n\rangle| = |\langle R_0^n f, (T_k)^n \varphi_n\rangle| = |\langle R_0^n f, \varphi\rangle|. \tag{11}$$

Combining (10) and (11), we get

$$|\langle R_0^n f, \chi\rangle| = |\langle f, \varphi_n\rangle| = |\langle f, \chi *_k \mathcal{F}_D(\frac{h(\xi)}{\xi^n})\rangle|. \tag{12}$$

Therefore, we have

$$\begin{aligned}
\|R_0^n f\|_{M_{\Phi,k}(\mathbb{R})} &= \sup_{\left\{\chi\in\mathcal{S}(\mathbb{R}):\ \|\chi\|_{N_{\Phi,k}(\mathbb{R})}\le 1\right\}} \Big|\langle f, \chi *_k \mathcal{F}_D(\frac{h(\xi)}{\xi^n})\rangle\Big| \\
&\le \sup_{\left\{\chi\in\mathcal{S}(\mathbb{R}):\ \|\chi\|_{N_{\Phi,k}(\mathbb{R})}\le 1\right\}} \|f\|_{M_{\Phi,k}(\mathbb{R})}\|\chi *_k \mathcal{F}_D(\frac{h(\xi)}{\xi^n})\|_{N_{\Phi,k}(\mathbb{R})} \\
&\le C\|f\|_{M_{\Phi,k}(\mathbb{R})}\|\mathcal{F}_D(\tfrac{h(\xi)}{\xi^n})\|_{L_k^1(\mathbb{R})}.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \|R_0^n f\|_{N_{\Phi,k}(\mathbb{R})}^{\frac{1}{n}} \le \limsup_{n\to\infty} \|\mathcal{F}_D(\frac{h(\xi)}{\xi^n})\|_{L_k^1(\mathbb{R})}^{\frac{1}{n}}. \tag{13}$$

Moreover by a simple calculation we prove

$$\limsup_{n\to\infty} \|\mathcal{F}_D(\frac{h(\xi)}{\xi^n})\|_{L_k^1(\mathbb{R})}^{\frac{1}{n}} \le \frac{1}{\sigma_0 - 2\varepsilon}. \tag{14}$$

Combining (13) and (14), we get

$$\limsup_{n\to\infty} \|R_0^n f\|_{M_{\Phi,k}(\mathbb{R})}^{\frac{1}{n}} \le \frac{1}{\sigma_0 - 2\varepsilon}$$

and then (8) by letting $\varepsilon \to 0$.

Lemma 6. *If $\sigma_0 > 0$, then*

$$\liminf_{n\to\infty} \|R_0^n f\|_{M_{\Phi,k}(\mathbb{R})}^{\frac{1}{n}} \ge \frac{1}{\sigma_0}. \tag{15}$$

Proof. From the definition of σ_0 there exists a function $\chi \in D(\mathbb{R})$ such that

$$supp\chi \subset \left\{\xi : \sigma_0 - \varepsilon < |\xi| < \sigma_0 + \varepsilon\right\} \quad \text{and} \quad \langle \mathcal{F}_D(f), \chi\rangle \neq 0.$$

Therefore,

$$\begin{aligned} 0 \neq |\langle f, \chi\rangle| &= |\langle (T_k)^n R_0^n f, \chi\rangle| = |\langle R_0^n f, T_k^n \chi\rangle| \\ &\leq C\|R_0^n f\|_{M_{\Phi,k}(\mathbb{R})}\|T_k^n \chi\|_{N_{\Phi,k}(\mathbb{R})}. \end{aligned} \tag{16}$$

We proceed as in the Lemma 3, we prove that

$$\liminf_{n\to\infty} \|R_0^n f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R})} \geq \frac{1}{(\sigma_0 + \varepsilon)}, \quad \varepsilon > 0,$$

and then (15).

Proof. of Theorem 2.

We divide our proof into two cases.

Case 1. $\sigma_0 = 0$. We have $\xi_0 \in supp\mathcal{F}_D(f)$. Hence, for any $\varepsilon > 0$ there is a function $\chi \in D(\mathbb{R})$ such that $supp\,\chi \subset (-\varepsilon, \varepsilon)$ such that $\langle \mathcal{F}_D(f), \chi\rangle \neq 0$. Arguing as above we obtain

$$\liminf_{n\to\infty} \|R_0^n f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R})} \geq \frac{1}{\limsup\limits_{n\to\infty} \|T_k^n(\chi)\|^{\frac{1}{n}}_{N_{\Phi,k}(\mathbb{R})}} \geq \frac{1}{\varepsilon}.$$

Therefore

$$\liminf_{n\to\infty} \|R_0^n f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R})} = \infty.$$

So we always have

$$\lim_{n\to\infty} \|R_0^n f\|^{\frac{1}{n}}_{M_{\Phi,k}(\mathbb{R})} = \frac{1}{\sigma_0}.$$

Case 2. $\sigma_0 > 0$. Combining (8) and (15), we arrive to (6).

References

[1] C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Am. Math. Soc. 311, 1989, 167-183.

[2] C.F. Dunkl, Hankel transforms associated to finite reflection groups, Contemp. Math. 138, 1992, 123-138.

[3] N. Hayek, H. M. Srivastava, B. J. González and E. R. Negrín, A family of Wiener transforms associated with a pair of operators on Hilbert space, Integral Transforms Spec. Funct. 24 (2013), 1-8.

[4] T. Kawazoe and H. Mejjaoli, Uncertainty principles for the Dunkl transform, Hiroshima Math. J. Volume 40, Number 2 (2010), 241-268.

[5] T. Kawazoe and H. Mejjaoli, Generalized Besov spaces and their applications, Tokyo J. of Math. Volume 35, Number 2 (2012), 297-320.

[6] H. Mejjaoli and K. Trimèche, Spectrum of functions for the Dunkl transform on $\mathbb{R}^d$. Fract. Calc. Appl. Anal. 10 (2007), no. 1, 19-38.

[7] M. A. Mourou and H. M. Srivastava, Generalized Sonine-type integral transforms on the real line, Integral Transform. Spec. Funct. 21, No. 10, (2010), 713-724.

[8] M. Rösler, Hermite polynomials and the heat equation for Dunkl operators. Comm.Math Phys.192 (1998), 519-542.

[9] M. S. Steigerwalt and A. J. White, Some function spaces related to L^p, Proc. London. Math. Soc. 22 (1971), 137-163.

[10] S. Thangavelu and Y. Xu, Convolution operator and maximal functions for Dunkl transform, J. d'Analyse Mathematique **97** (2005), 25-56.