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Fixed Points of (α, ψ, φ)-Generalized Weakly Contractive Maps and Property(P) in S-Metric Spaces

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Abstract. In this paper, we introduce a notion of (α, ψ, φ)-generalized weakly contractive maps in S-metric spaces and prove the existence of fixed points for such maps. We also prove that these maps satisfy property(P). The results presented in this paper extend the results of Khandaqji, Al-Sharif and Al-Khaleel [15] from G-metric spaces to S-metric spaces. We provide examples in support of our results.

1. Introduction and Preliminaries

In 2006, Mustafa and Sims [17] introduced G-metric spaces as a generalization of metric spaces and introduced contraction maps in G-metric spaces and proved the existence of fixed points of various contraction mappings. For more works on the existence of fixed points in G-metric spaces we refer Babu, Babu, Rao and Kumar[5], Chugh, Kadian Anju and Rhoades [8] and Mustafa, Obiedat, Awawdeh [18] and the related references cited in these papers. In 2007, Sedghi [21] introduced D^*-metric spaces which is a probable modification of the definition of D-metric spaces introduced by Dhage [9] and proved some basic properties of D^*-metric spaces [20, 21]. In 2012, Sedghi, Shobe and Aliouche [22] introduced a new concept on metric spaces, namely S-metric spaces and studied some properties of these spaces. Sedghi, Shobe and Aliouche [22] asserted that S-metric space is a generalization of G-metric spaces. But, very recently Dung, Hieu and Radojevic [11] verified by examples (Example 2.1 and Example 2.2) that S-metric space is not a generalization of G- metric space or vise versa . Therefore the class of G- metric spaces and the class of S-metric spaces are different. Some papers dealing with fixed point theorems for mappings satisfying certain contractive conditions on S-metric spaces can be referred in [1, 2, 11, 23].

The study of fixed points of nonlinear mappings satisfying certain contractive conditions has been the center of rigorous research activities. In this direction, Alber and Guerre-Delabriere [3] introduced weakly contractive maps in Hilbert spaces as a generalization of contraction maps and established a fixed point theorem in the Hilbert spaces. Rhoades [19] extended this idea to Banach spaces and proved existence of fixed points of weakly contractive self maps in Banach space setting. Dutta and Choudhury [12] proved the existence and uniqueness of fixed points and generalized the results of Alber and Guerre-Delabriere [3] and Khan Swaleh and Sessa [16]. Since then different types of weakly contractive maps have been considered

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in several works to establish the existence of fixed points. For more works on weakly contractive maps we refer [7, 10, 24].

Now we provide some preliminaries and basic definitions which we use throughout this paper.

Definition 1.1. [16] An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which satisfies (i) ψ is continuous (ii) ψ is non-decreasing and (iii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions by Ψ.

We denote $\Phi = \{\varphi : [0,\infty) \to [0,\infty) \mid (i)\ \varphi \text{ is continuous and } (ii)\ \varphi(t) = 0 \text{ if and only if } t = 0\}$.

In the following, Dutta and Choudhury [12] established the existence of fixed points of (ψ, φ)-weakly contractive maps involving two altering distance functions ψ and φ in complete metric spaces.

Theorem 1.2. [12] *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-map of X. If there exist $\psi, \varphi \in \Psi$ such that*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Definition 1.3. [17] Let X be a non-empty set, $G : X^3 \to [0,\infty)$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(z, x, y) = \ldots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric (G-metric) and the pair (X, G) is called a G-metric space.

For more details on G-metric spaces we refer [17].

Definition 1.4. [24] Let (X, G) be a G-metric space. A self mapping f of X is said to be weakly contractive if for all $x,\ y,\ z \in X$

$$G(fx, fy, fz) \leq G(x, y, z) - \psi(G(x, y, z)),$$

where ψ is an altering distance function.

Definition 1.5. [13] Let X be a nonempty set and T be a self map of X with nonempty fixed point set $F(T)$. Then T is said to satisfy property(P) if $F(T) = F(T^n)$ for all $n \in \mathbb{N}$.

For more details on Property(P) we refer [6, 8, 14].

In 2012, Khandaqji, Al-Sharif and Al-Khaleel [15] proved the following for weakly contractive maps in $G-$metric spaces.

Theorem 1.6. [15] *Let (X, G) be a complete G-metric space and $f : X \to X$ be a self map. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\begin{aligned}\psi(G(fx, fy, fz)) \leq \psi\Big(\max\Big\{&G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \alpha G(fx, fx, y)\\ &+ (1-\alpha)G(fy, fy, z), \beta G(x, fx, fx) + (1-\beta)G(y, fy, fy)\Big\}\Big)\\ &- \phi\Big(\max\Big\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \alpha G(fx, fx, y)\\ &+ (1-\alpha)G(fy, fy, z), \beta G(x, fx, fx) + (1-\beta)G(y, fy, fy)\Big\}\Big) \qquad (1)\end{aligned}$$

for all $x, y, z \in X$, where $\alpha, \beta \in (0, 1)$. Then f has a unique fixed point u (say) and f is G-continuous at u. Further, f has property(P).

Remark 1.7. Since $\beta G(x, fx, fx) + (1-\beta)G(y, fy, fy)\Big\} \leq \max\Big\{G(x, fx, fx), G(y, fy, fy)\Big\}$, the term $\beta G(x, fx, fx) + (1-\beta)G(y, fy, fy)$ is redundant in the inequality (1).

In 2012, Sedghi, Shobe and Aliouche [22] introduced S-metric spaces as follows:

Definition 1.8. [22] Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$

(S1) $S(x, y, z) \geq 0$,

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$ and

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Example 1.9. (Example 2.4 [22]) Let (X, d) be a metric space. Define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then, S is an S-metric on X. This S-metric is called the S-metric induced by the metric d.

Example 1.10. (Example 1.9 [11]) Let $X = \mathbb{R}$ and let $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is an S-metric space.

Example 1.11. [23] Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Remark 1.12. (Example 2.2 [11]) There exists an S-metric which is not a G-metric.

Remark 1.13. (Example 2.1, [11]) There exists a G-metric which is not an S-metric .

Remark 1.14. From Remark 1.12 and Remark 1.13 , we can conclude that the class of S-metrics and the class of G-metrics are distinct.

The following lemmas are very useful in our subsequent discussion and in proving our main results.

Lemma 1.15. [22] In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Lemma 1.16. [11] Let (X, S) be an S-metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$.

Definition 1.17. [22] Let (X,S) be an S-metric space. We define the following:

(i) a sequence $\{x_n\}$ in X converge to a point $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \epsilon$ and we denote it by $\lim\limits_{n\to\infty} x_n = x$.

(ii) a sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \epsilon$ for all $n,m \geq n_0$.

(iii) The S-metric space (X,S) is said to be complete if each Cauchy sequence in X is convergent.

Definition 1.18. Let (X,S) and (Y,S') be two S-metric spaces. Then the function $f : X \to Y$ is S-continuous at a point $x \in X$ if it is S-sequentially continuous at x, that is, whenever $\{x_n\}$ is S-convergent to x, we have $f(x_n)$ is S'-convergent to $f(x)$.

Lemma 1.19. [22] Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 1.20. [22] Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then $\lim\limits_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$.

Lemma 1.21. [1] Any S-metric space is Hausdorff.

In 2012, Sedghi [22] proved an analogue of Banach's contraction principle in S-metric space.

Definition 1.22. [22] Let (X,S) be an $S-$metric space. A map $f : X \to X$ is said to be an S-contraction if there exists a constant $0 \leq \lambda < 1$ such that

$$S(f(x),f(x),f(y)) \leq \lambda S(x,x,y) \text{ for all } x,y \in X.$$

Theorem 1.23. [22] *Let (X,S) be a complete $S-$metric space and $f : X \to X$ be a S-contraction. Then f has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have $\lim\limits_{n\to\infty} f^n(x) = u$ with*

$$S(f^n(x),f^n(x),u) \leq \frac{2\lambda^n}{1-\lambda}(Sx,x,f(x)).$$

We now introduce the following definition and support it with a subsequent example.

Definition 1.24. Let (X,S) be an S-metric space. Let $f : X \to X$ be a self map of X. If there exist $\alpha \in (0,1)$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(S(fx,fy,fz)) \leq \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)), \tag{2}$$

where $M_\alpha(x,y,z) = \max\Big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\Big\}$

for all $x,y,z \in X$, then f is called an (α,ψ,φ)-generalized weakly contractive map on X.

Example 1.25. Let $X = [0, \frac{13}{12}]$. We define $S : X^3 \to [0,\infty)$ by $S(x,y,z) = \max\{|x-z|, |y-z|\}$ for all $x,y,z \in X$. Then S is an S-metric on X.

We define $f : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } x \in [0,1] \\ \frac{7}{6} - x & \text{if } x \in (1, \frac{13}{12}]. \end{cases}$$

We define $\psi,\ \varphi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = t \text{ for all } t \geq 0 \quad \text{and} \quad \varphi(t) = \frac{t}{1+t} \text{ for all } t \geq 0.$$

Now, we verify that f is an (α,ψ,φ)-generalized weakly contractive map on X.
<u>Case</u> (i): Let $x,y,z \in [0,1]$.
We assume, without loss of generality, that $x > y > z$.

$$S(fx,fy,fz) = S\Big(\frac{1}{4},\frac{1}{4},\frac{1}{4}\Big) = 0. \text{ Then trivially the inequality (2) holds.}$$

<u>Case</u> (ii): Let $x,y,z \in (1,\frac{13}{12}]$.
We assume, without loss of generality, that $x > y > z$.

$$S(fx,fy,fz) = S\Big(\frac{7}{6}-x,\frac{7}{6}-y,\frac{7}{6}-z\Big) = \max\{|\frac{7}{6}-x-(\frac{7}{6}-z)|,|\frac{7}{6}-y-(\frac{7}{6}-z)|\}$$

$$\begin{aligned}
&= \max\{|z-x|,|z-y|\} = x-z \leq \frac{1}{12} \leq \frac{25}{66} = \frac{5}{6}-\frac{5}{11}\\
&\leq S(x,x,fx) - \frac{S(x,x,fx)}{1+S(x,x,fx)} = \frac{(S(x,x,fx))^2}{1+S(x,x,fx)}\\
&\leq \frac{(M_\alpha(x,y,z))^2}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \frac{M_\alpha(x,y,z)}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \varphi\Big(M_\alpha(x,y,z)\Big).
\end{aligned}$$

<u>Case</u> (iii): Let $y,z \in [0,1]$ and $x \in (1,\frac{13}{12}]$.
We assume, without loss of generality, that $y > z$.

$$\begin{aligned}
S(fx,fy,fz) &= S(\frac{7}{6}-x,\frac{1}{4},\frac{1}{4}) = \max\{|\frac{7}{6}-x-\frac{1}{4}|,|\frac{1}{4}-\frac{1}{4}|\} = x-\frac{11}{12} \leq \frac{1}{6} \leq \frac{25}{66} = \frac{5}{6}-\frac{5}{11}\\
&\leq S(x,x,fx) - \frac{S(x,x,fx)}{1+S(x,x,fx)} = \frac{(S(x,x,fx))^2}{1+S(x,x,fx)} \leq \frac{(M_\alpha(x,y,z))^2}{1+M_\alpha(x,y,z)}\\
&= M_\alpha(x,y,z) - \frac{M_\alpha(x,y,z)}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \varphi\Big(M_\alpha(x,y,z)\Big).
\end{aligned}$$

<u>Case</u> (iv): Let $z \in [0,1]$ and $x,y \in (1,\frac{13}{12}]$.
We assume, without loss of generality, that $x > y$.

$$\begin{aligned}
S(fx,fy,fz) &= S(\frac{7}{6}-x,\frac{7}{6}-y,\frac{1}{4}) = \max\{|\frac{7}{6}-x-\frac{1}{4}|,|\frac{7}{6}-y-\frac{1}{4}|\} = \max\{|\frac{11}{12}-x|,|\frac{11}{12}-y|\} = x-\frac{11}{12}\\
&\leq \frac{1}{6} \leq \frac{25}{66} = \frac{5}{6}-\frac{5}{11} \leq S(y,y,fy) - \frac{S(y,y,fy)}{1+S(y,y,fy)} = \frac{(S(y,y,fy))^2}{1+S(y,y,fy)}\\
&\leq \frac{(M_\alpha(x,y,z))^2}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \frac{M_\alpha(x,y,z)}{1+M_\alpha(x,y,z)}\\
&= M_\alpha(x,y,z) - \varphi\Big(M_\alpha(x,y,z)\Big).
\end{aligned}$$

<u>Case</u> (v): Let $x,y \in [0,1]$ and $z \in (1,\frac{13}{12}]$.
We assume, without loss of generality, that $x > y$.

$$S(fx,fy,fz) = S(\frac{1}{4},\frac{1}{4},\frac{7}{6}-z) = \max\{|\frac{1}{4}-(\frac{7}{6}-z)|,|\frac{1}{4}-(\frac{7}{6}-z)|\} = z-\frac{11}{12} \leq \frac{1}{6} \leq \frac{25}{66} = \frac{5}{6}-\frac{5}{11}$$

$$\leq S(z,z,fz) - \frac{S(z,z,fz)}{1+S(z,z,fz)} = \frac{(S(z,z,fz))^2}{1+S(z,z,fz)} \leq \frac{(M_\alpha(x,y,z))^2}{1+M_\alpha(x,y,z)}$$
$$= M_\alpha(x,y,z) - \frac{M_\alpha(x,y,z)}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \varphi\Big(M_\alpha(x,y,z)\Big).$$

Case (*vi*): Let $x \in [0,1]$ and $z, y \in (1, \frac{13}{12}]$.
We assume, without loss of generality, that $z > y$.

$$S(fx,fy,fz) = S(\frac{1}{4}, \frac{7}{6}-y, \frac{7}{6}-z) = \max\{|\frac{1}{4}-(\frac{7}{6}-z)|, |\frac{7}{6}-y-(\frac{7}{6}-z)|\} = \max\{|z-\frac{11}{12}|, |z-y|\}$$

$$= z - \frac{11}{12} \leq \frac{1}{6} \leq \frac{25}{66} = \frac{5}{6} - \frac{5}{11} \leq S(y,y,fy) - \frac{S(y,y,fy)}{1+S(y,y,fy)}$$
$$= \frac{(S(y,y,fy))^2}{1+S(y,y,fy)} \leq \frac{(M_\alpha(x,y,z))^2}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \frac{M_\alpha(x,y,z)}{1+M_\alpha(x,y,z)}$$
$$= M_\alpha(x,y,z) - \varphi\Big(M_\alpha(x,y,z)\Big).$$

Case (*vii*): Let $y \in [0,1]$ and $z, x \in (1, \frac{13}{12}]$.
We assume, without loss of generality, that $z > x$.

$$S(fx,fy,fz) = S(\frac{7}{6}-x, \frac{1}{4}, \frac{7}{6}-z) = \max\{|\frac{7}{6}-x-(\frac{7}{6}-z)|, |\frac{1}{4}-(\frac{7}{6}-z)|\} = \max\{|z-x|, |z-\frac{11}{12}|\}$$

$$= z - \frac{11}{12} \leq \frac{1}{6} \leq \frac{25}{66} = \frac{5}{6} - \frac{5}{11} \leq S(y,y,fy) - \frac{S(y,y,fy)}{1+S(y,y,fy)}$$
$$= \frac{(S(y,y,fy))^2}{1+S(y,y,fy)} \leq \frac{(M_\alpha(x,y,z))^2}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \frac{M_\alpha(x,y,z)}{1+M_\alpha(x,y,z)}$$
$$= M_\alpha(x,y,z) - \varphi\Big(M_\alpha(x,y,z)\Big).$$

Hence f is an (α, ψ, φ)-generalized weakly contractive map on X.

Here we observe that f is not an S-contraction. For, let $x,\ y \in (1, \frac{13}{12}]$ with $x \neq y$. Then $S(fx,fx,fy) = S(\frac{7}{6}-x, \frac{7}{6}-x, \frac{7}{6}-y)) = |x-y| \nleq \lambda|x-y| = \lambda S(x,x,y)$ for any $\lambda \in [0,1)$.

We state the following lemma which is useful in proving our main results.

Lemma 1.26. [4] Let (X,S) be an S-metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{3}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon,\ \ S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon \tag{4}$$

and

(*i*) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon$, (*iii*) $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) = \epsilon$,

(*ii*) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \epsilon$ (*iv*) $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon$.

In Section 2, we prove the existence and uniqueness of fixed points of (α, ψ, ϕ) - generalized weakly contractive self map f in a complete S-metric space and study the S-continuity of f at the fixed point. Also, we study property(P) of these maps. In section 3, we draw some corollaries and provide examples in support of our main results.

2. Main Results

Theorem 2.1. *Let (X, S) be a complete S-metric space and let f be an (α, ψ, φ)-generalized weakly contractive map. Then f has a unique fixed point u (say) and f is S-continuous at u.*

proof: Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and we are through.

Now we assume that $x_n \neq x_{n+1}$ for all n. By substituting $x = y = x_{n-1}$, $z = x_n$ in (2), we have

$$\begin{aligned}\psi(S(x_n, x_n, x_{n+1})) = \psi(S(fx_{n-1}, fx_{n-1}, fx_n)) &\leq \psi\Big(\max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, fx_{n-1}),\\ &S(x_{n-1}, x_{n-1}, fx_{n-1}), S(x_n, x_n, fx_n), \alpha S(fx_{n-1}, fx_{n-1}, x_{n-1}) + (1-\alpha)S(fx_{n-1}, fx_{n-1}, x_n)\Big\}\Big)\\ &- \varphi\Big(\max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, fx_{n-1}), S(x_{n-1}, x_{n-1}, fx_{n-1}), S(x_n, x_n, fx_n),\\ &\alpha S(fx_{n-1}, fx_{n-1}, x_{n-1}) + (1-\alpha)S(fx_{n-1}, fx_{n-1}, x_n)\Big\}\Big),\\ &= \psi\Big(\max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\Big\}\Big)\\ &-\varphi\Big(\max\Big\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\Big\}\Big). \end{aligned} \tag{5}$$

Let $M_n = \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}$. Here, we have two cases, either $M_n = S(x_{n-1}, x_{n-1}, x_n)$ or $M_n = S(x_n, x_n, x_{n+1})$. Suppose that, for some n, $M_n = S(x_n, x_n, x_{n+1})$. Therefore from (5), it follows $\varphi\,(S(x_n, x_n, x_{n+1})) = 0$. Hence $x_n = x_{n+1}$, a contradiction since x_n and x_{n+1} are distinct elements. Thus, $M_n = S(x_{n-1}, x_{n-1}, x_n)$. Hence, from (5), we have

$$\psi(S(x_n, x_n, x_{n+1})) \leq \psi(S(x_{n-1}, x_{n-1}, x_n)) - \varphi(S(x_{n-1}, x_{n-1}, x_n)) < \psi(S(x_{n-1}, x_{n-1}, x_n)) \tag{6}$$

Now, by the non-decreasing property of ψ, it follows that $S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n)$ for all $n \in \mathbb{N}$.

Therefore $\{S(x_n, x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = r. \tag{7}$$

On letting $n \to \infty$ in (6) and using (7), we obtain

$$\psi(r) \leq \psi(r) - \varphi(r), \tag{8}$$

so that $\varphi(r) = 0$. Hence $r = 0$.

We now show that $\{x_n\}$ is an S-Cauchy sequence. If possible, suppose that $\{x_n\}$ is not S-Cauchy. Therefore by Lemma 1.26, there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that $S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon$, $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon$ satisfying the identities (*i*) to (*iv*) of Lemma 1.26. Taking $x = y = x_{m_k-1}, z = x_{n_k-1}$ and applying the inequality (2), we have

$$\begin{aligned}\psi(S(x_{m_k}, x_{m_k}, x_{n_k})) = \psi(S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k-1})) &\leq \psi\Big(\max\Big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}),\\ &S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}), S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}), S(x_{n_k-1}, x_{n_k-1}, fx_{n_k-1}),\\ &\alpha S(fx_{m_k-1}, fx_{m_k-1}, x_{m_k-1}) + (1-\alpha)S(fx_{m_k-1}, fx_{m_k-1}, x_{n_k-1})\Big\}\Big)\end{aligned}$$

$$- \varphi\Big(\max\Big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}), S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}),$$
$$S(x_{n_k-1}, x_{n_k-1}, fx_{n_k-1}), \alpha S(fx_{m_k-1}, fx_{m_k-1}, x_{m_k-1}) + (1-\alpha)S(fx_{m_k-1}, fx_{m_k-1}, x_{n_k-1})\Big\}\Big),$$

$$= \psi\Big(\max\Big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k-1}, x_{m_k-1}, x_{m_k}), S(x_{m_k-1}, x_{m_k-1}, x_{m_k}),$$
$$S(x_{n_k-1}, x_{n_k-1}, x_{n_k}), \alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k-1}, x_{m_k-1}, x_{m_k}), S(x_{m_k-1}, x_{m_k-1}, x_{m_k}),$$
$$S(x_{n_k-1}, x_{n_k-1}, x_{n_k}), \alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\Big\}\Big)$$
$$= \psi\Big(\max\Big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k-1}, x_{m_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, x_{n_k}),$$
$$\alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k-1}, x_{m_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, x_{n_k}),$$
$$\alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\Big\}\Big). \tag{9}$$

On letting $k \to \infty$ in (9) and using Lemma 1.26, we get

$$\psi(\epsilon) \leq \psi\Big(\max\Big\{\epsilon, 0, 0, \alpha(0) + (1-\alpha)\epsilon\Big\}\Big) - \varphi\Big(\max\Big\{\epsilon, 0, \alpha(0) + (1-\alpha)\epsilon\Big\}\Big) = \psi(\epsilon) - \varphi(\epsilon) < \psi(\epsilon),$$

a contradiction. Hence $\{x_n\}$ is S-Cauchy. Since X is complete there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

We now show that u is a fixed point of f. Here by Lemma 1.20 we note that $\lim_{n\to\infty} S(x_n, x_n, fu) = S(u, u, fu)$. We now consider

$$\psi(S(fu, fu, x_n)) = \psi(S(fu, fu, fx_{n-1})) \leq \psi\Big(\max\Big\{S(u, u, x_{n-1}), S(u, u, fu), S(u, u, fu),$$
$$S(x_{n-1}, x_{n-1}, fx_{n-1}), \alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1})\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(u, u, x_{n-1}), S(u, u, fu), S(u, u, fu), S(x_{n-1}, x_{n-1}, fx_{n-1}),$$
$$\alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1})\Big\}\Big)$$
$$= \psi\Big(\max\Big\{S(u, u, x_{n-1}), S(u, u, fu), S(x_{n-1}, x_{n-1}, x_n),$$
$$\alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1})\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(u, u, x_{n-1}), S(u, u, fu), S(x_{n-1}, x_{n-1}, x_n),$$
$$\alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1})\Big\}\Big). \tag{10}$$

On letting $n \to \infty$ in (10), we get

$$\psi(S(fu, fu, u)) \leq \psi\Big(\max\Big\{S(u, u, u), S(u, u, fu), S(u, u, u), \alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, u)\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(u, u, u), S(u, u, fu), Su, u, u), \alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, u)\Big\}\Big). \tag{11}$$

From (11), we have

$\psi(S(fu, fu, u)) \leq \psi(S(fu, fu, u)) - \varphi(S(fu, fu, u)),$ so that $\varphi(S(fu, fu, u)) = 0.$

Hence $u = fu$.

We now prove uniqueness of fixed point. If u and v are two fixed points of f with $u \neq v$, then we consider

$$\begin{aligned}\psi(S(u,u,v)) = \psi(S(fu,fu,fv)) &\leq \psi\Big(\max\Big\{S(u,u,v), S(u,u,fu), S(u,u,fu), S(v,v,fv),\\ &\qquad \alpha S(fu,fu,u) + (1-\alpha)S(fu,fu,v)\Big\}\Big)\\ &\quad -\varphi\Big(\max\Big\{S(u,u,v), S(u,u,fu), S((u,u,fu), S(v,v,fv),\\ &\qquad \alpha S(fu,fu,u) + (1-\alpha)S(fu,fu,v)\Big\}\Big),\\ &= \psi\Big(\max\Big\{S(u,u,v), S(u,u,fu), S((u,u,u), S(v,v,v), \alpha S(u,u,u) + (1-\alpha)S(u,u,v)\Big\}\Big)\\ &\quad -\varphi\Big(\max\Big\{S(u,u,v), S(u,u,fu), S((u,u,u), S(v,v,v), \alpha S(u,u,u) + (1-\alpha)S(u,u,v)\Big\}\Big),\\ &= \psi(\max\{S(u,u,v),(1-\alpha)S(u,u,v)\}) - \varphi(\max\{S(u,u,v),(1-\alpha)S(u,u,v)\})\\ &= \psi(S(u,u,v)) - \varphi(S(u,u,v)) < \psi(S(u,u,v)),\end{aligned}$$

a contradiction. Therefore $u = v$.

Finally we prove that f is S-continuous at u. Let $\{x_n\}$ be a sequence in X such that $x_n \to u$ as $n \to \infty$. We consider

$$\begin{aligned}\psi(S(u,u,fx_n)) = \psi(S(fu,fu,fx_n)) &\leq \psi\Big(\max\Big\{S(u,u,x_n), S(u,u,fu), S(u,u,fu), S(x_n,x_n,fx_n),\\ &\qquad \alpha S(fu,fu,u) + (1-\alpha)S(fu,fu,x_n)\Big\}\Big) - \varphi\Big(\max\Big\{S(u,u,x_n), S(u,u,fu),\\ &\qquad S(u,u,fu), S(x_n,x_n,fx_n), \alpha S(fu,fu,u) + (1-\alpha)S(fu,fu,x_n),\\ &= \psi\Big(\max\Big\{S(u,u,x_n), S(u,u,u), S(x_n,x_n,x_{n+1}), \alpha S(u,u,u) + (1-\alpha)S(u,u,x_n)\Big\}\Big)\\ &\quad -\varphi\Big(\max\Big\{S(u,u,x_n), S(u,u,u), S(x_n,x_n,x_{n+1}), \alpha S(u,u,u) + (1-\alpha)S(u,u,x_n)\Big\}\Big). \qquad (12)\end{aligned}$$

Now by taking the limits on both sides of (12), we have

$$\lim_{n\to\infty} \psi(S(fu,fu,fx_n)) = 0.$$

This implies that $\psi(\lim_{n\to\infty} S(fu,fu,fx_n)) = 0$, since ψ is continuous.
Now by property of ψ we have $\lim_{n\to\infty} S(fu,fu,fx_n)) = 0$. Hence $fx_n \to fu$ as $n \to \infty$. Therefore, f is S-continuous at u. This completes the proof of the theorem.

We prove the following theorem by using the technique of Khandaqji, Al-Sharif and Al-Khaleel [15].

Theorem 2.2. *Under the hypotheses of Theorem 2.1, we have f has Property(P).*

proof: In view of the proof of Theorem 2.1, f has a fixed point. Therefore, $F(f^n) \neq \emptyset$. Now, we fix $n > 1$ and assume that $u \in F(f^n)$. That is, $f^n u = u$. We show that $u \in F(f)$. We consider

$$\begin{aligned}\psi(S(u,u,fu)) &= \psi(S(f^n u, f^n u, f^{n+1}u)) = \psi(S(ff^{n-1}u, ff^{n-1}u, ff^n u))\\ &\leq \psi\Big(\max\Big\{S(f^{n-1}u, f^{n-1}u, f^n u), S(f^{n-1}u, f^{n-1}u, ff^{n-1}u), S(f^{n-1}u, f^{n-1}u, ff^{n-1}u),\\ &\qquad S(f^n u, f^n u, ff^n u), \alpha S(ff^{n-1}u, ff^{n-1}u, f^{n-1}u) + (1-\alpha)S(ff^{n-1}u, ff^{n-1}u, f^n u)\Big\}\Big)\\ &\quad -\varphi\Big(\max\Big\{S(f^{n-1}u, f^{n-1}u, f^n u), S(f^{n-1}u, f^{n-1}u, ff^{n-1}u), S(f^{n-1}u, f^{n-1}u, ff^{n-1}u),\\ &\qquad S(f^n u, f^n u, ff^n u), \alpha S(ff^{n-1}u, ff^{n-1}u, f^{n-1}u) + (1-\alpha)S(ff^{n-1}u, ff^{n-1}u, f^n u)\Big\}\Big)\end{aligned}$$

$$= \psi\Big(\max\Big\{S(f^{n-1}u, f^{n-1}u, f^n u), S(f^{n-1}u, f^{n-1}u, f^n u), S(f^{n-1}u, f^{n-1}u, f^n u),$$
$$S(f^n u, f^n u, f^{n+1}u), \alpha S(f^n u, f^n u, f^{n-1}u) + (1-\alpha)S(f^n u, f^n u, f^n u)\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(f^{n-1}u, f^{n-1}u, f^n u), S(f^{n-1}u, f^{n-1}u, f^n u), S(f^{n-1}u, f^{n-1}u, f^n u),$$
$$S(f^n u, f^n u, f^{n+1}u), \alpha S(f^n u, f^n u, f^{n-1}u) + (1-\alpha)S(f^n u, f^n u, f^n u)\Big\}\Big)$$
$$= \psi\Big(\max\Big\{S(f^{n-1}u, f^{n-1}u, f^n u), S(f^n u, f^n u, f^{n+1}u)\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(f^{n-1}u, f^{n-1}u, f^n u), S(f^n u, f^n u, f^{n+1}u)\Big\}\Big). \tag{13}$$

If $S(f^n u, f^n u, f^{n+1}u)$ is the maximum in (13), then

$$\psi(S(u,u,fu)) \le \psi(S(f^n u, f^n u, f^{n+1}u)) - \varphi(S(f^n u, f^n u, f^{n+1}u)) = \psi(S(u,u,fu)) - \varphi(S(u,u,fu))$$

and hence $\varphi(S(u,u,fu)) = 0$. Hence $fu = u$ so that $u \in F(f)$ and we are through.
If $S(f^{n-1}u, f^{n-1}u, f^n u)$ is the maximum in (13), we have

$$\psi(S(u,u,fu)) = \psi(S(f^n u, f^n u, f^{n+1}u)) \le \psi(S(f^{n-1}u, f^{n-1}u, f^n u)) - \varphi(S(f^{n-1}u, f^{n-1}u, f^n u))$$
$$\le \psi(S(f^{n-2}u, f^{n-2}u, f^{n-1}u)) - \varphi(S(f^{n-2}u, f^{n-2}u, f^{n-1}u)) - \varphi(S(f^{n-1}u, f^{n-1}u, f^n u))$$
$$\vdots$$
$$\le \psi(S(u,u,fu)) - \sum_{k=0}^{n-1} \varphi(S(f^{n-k-1}u, f^{n-k-1}u, f^{n-k}u).$$

This implies that $\sum_{k=0}^{n-1} \varphi(S(f^{n-k-1}u, f^{n-k-1}u, f^{n-k}u)) = 0$. Hence $\varphi(S(f^{n-k-1}u, f^{n-k-1}u, f^{n-k}u)) = 0$ for all $0 \le k \le n-1$. Therefore, it follows that $\varphi(S(u,u,fu) = 0$ and hence $fu = u$. Thus we have $u \in F(f)$ and f has property(P).

3. Corollaries and Examples

If ψ is the identity mapping on $[0,\infty)$ in Theorem 2.1, we have the following.

Corollary 3.1. *Let (X,S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exist $\alpha \in (0,1)$ such that*

$$S(fx,fy,fz) \le \max\Big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)$$
$$-\varphi\Big(\max\Big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\Big\}\Big),$$

for all $x,y,z \in X$. Then f has a unique fixed point u (say) and f is S-continuous at u.

Corollary 3.2. *Let (X,S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exist $\lambda, \alpha \in (0,1)$ such that*

$$S(fx,fy,fz) \le \lambda\max\Big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\Big\},$$

for all $x,y,z \in X$. Then f has a unique fixed point u (say) and f is S-continuous at u.

If $\alpha = \frac{1}{2}$ in the inequality (2), we have the following corollary.

Corollary 3.3. *Let (X,S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exist* $\psi \in \Psi$ and $\varphi \in \Phi$ *such that*

$$\psi(S(fx,fy,fz)) \leq \psi\Big(\max\Big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \frac{1}{2}[S(fx,fx,y)+S(fy,fy,z)]\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \frac{1}{2}[S(fx,fx,y)+S(fy,fy,z)]\Big\}\Big)$$

for all $x,y,z \in X$. Then f has a unique fixed point u (say) and f is S-continuous at u.

The following example is in support of Theorem 2.1.

Example 3.4. Let $X = [0, \frac{6}{5}]$. We define $f : X \to X$ by

$$f(x) = \begin{cases} \frac{x}{8} & \text{if } x \in [0,1] \\ x - \frac{4}{5} & \text{if } x \in (1, \frac{6}{5}]. \end{cases}$$

We define $S : X^3 \to [0,\infty)$ by
$S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in X$. Then (X,S) is a complete S-metric space.
We now define functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = t \quad \text{for all } t \geq 0 \quad \text{and} \qquad \varphi(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0,1] \\ \frac{t}{t+1} & \text{if } t \geq 1. \end{cases}$$

We now show that f satisfies the inequality (2).
Case (*i*): Let $x,y,z \in [0,1]$.
We assume, without loss of generality, that $x > y > z$.

$$S(fx,fy,fz) = S\Big(\frac{x}{8}, \frac{y}{8}, \frac{z}{8}\Big) = \frac{1}{8}\Big(|x-z| + |y-z|\Big) \text{ and } S(x,y,z) = |x-z| + |y-z|$$

Sub case (*i*): $|x-z| + |y-z| \in [0,1]$.
In this case,

$$S(fx,fy,fz) = \frac{1}{8}\Big(|x-z|+|y-z|\Big) \leq \frac{1}{2}\Big(|x-z|+|y-z|\Big) = \frac{1}{2}S(x,y,z) \leq \frac{1}{2}M_\alpha(x,y,z).$$
$$= M_\alpha(x,y,z) - \frac{1}{2}M_\alpha(x,y,z) = M_\alpha(x,y,z) - \varphi(M_\alpha(x,y,z)).$$

Sub case (*ii*): $|x-z| + |y-z| \geq 1$,
In this case,

$$S(fx,fy,fz) = \frac{1}{8}\Big(|x-z|+|y-z|\Big) \leq |x-z|+|y-z| - \frac{|x-z|+|y-z|}{1+|x-z|+|y-z|} = S(x,y,z) - \frac{S(x,y,z)}{1+S(x,y,z)}$$
$$= \frac{(S(x,y,z))^2}{1+S(x,y,z)} \leq \frac{(M_\alpha(x,y,z))^2}{1+M_\alpha(x,y,z)} = M_\alpha(x,y,z) - \frac{M_\alpha(x,y,z)}{1+M_\alpha(x,y,z)}$$
$$= M_\alpha(x,y,z) - \varphi\Big(M_\alpha(x,y,z)\Big).$$

Case (*ii*): Let $x,y,z \in (1, \frac{6}{5}]$.
We assume, without loss of generality, that $x > y > z$.

$$S(fx,fy,fz) = S\Big(x-\frac{4}{5}, y-\frac{4}{5}, z-\frac{4}{5})\Big) = |x-\frac{4}{5}-(z-\frac{4}{5})| + |y-\frac{4}{5}-(z-\frac{4}{5})|$$

$$= |x - z| + |y - z| \leq \frac{2}{5} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(x, x, fx) - \frac{S(x, x, fx)}{1 + S(x, x, fx)} = \frac{(S(x, x, fx))^2}{1 + S(x, x, fx)}$$
$$\leq \frac{(M_\alpha(x, y, z))^2}{1 + M_\alpha(x, y, z)} = M_\alpha(x, y, z) - \frac{M_\alpha(x, y, z)}{1 + M_\alpha(x, y, z)} = M_\alpha(x, y, z) - \varphi\Big(M_\alpha(x, y, z)\Big).$$

<u>Case</u> (*iii*): Let $y, z \in [0, 1]$ and $x \in (1, \frac{6}{5}]$.
We assume, without loss of generality, that $y > z$.

$$S(fx, fy, fz) = S(x - \frac{4}{5}, \frac{y}{8}, \frac{z}{8}) = |x - \frac{4}{5} - \frac{z}{8}| + |\frac{y}{8} - \frac{z}{8}| = x - \frac{z}{8} - \frac{4}{5} + \frac{y - z}{8} = x + \frac{y}{8} - \frac{z}{4} - \frac{4}{5}$$
$$\leq \frac{21}{40} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(x, x, fx) - \frac{S(x, x, fx)}{1 + S(x, x, fx)} = \frac{(S(x, x, fx))^2}{1 + S(x, x, fx)} \leq \frac{(M_\alpha(x, y, z))^2}{1 + M_\alpha(x, y, z)}$$
$$= M_\alpha(x, y, z) - \frac{M_\alpha(x, y, z)}{1 + M_\alpha(x, y, z)} = M_\alpha(x, y, z) - \varphi\Big(M_\alpha(x, y, z)\Big).$$

<u>Case</u> (*iv*): Let $z \in [0, 1]$ and $x, y \in (1, \frac{6}{5}]$.
We assume, without loss of generality, that $x > y$.

$$S(fx, fy, fz) = S(x - \frac{4}{5}, y - \frac{4}{5}, \frac{z}{8}) = |x - \frac{4}{5} - \frac{z}{8}| + |y - \frac{4}{5} - \frac{z}{8}| = x - \frac{4}{5} - \frac{z}{8} + y - \frac{4}{5} - \frac{z}{8}$$
$$= x + y - \frac{z}{4} - \frac{8}{5} \leq \frac{4}{5} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(y, y, fy) - \frac{S(y, y, fy)}{1 + S(y, y, fy)} = \frac{(S(y, y, fy))^2}{1 + S(y, y, fy)}$$
$$\leq \frac{(M_\alpha(x, y, z))^2}{1 + M_\alpha(x, y, z)} = M_\alpha(x, y, z) - \frac{M_\alpha(x, y, z)}{1 + M_\alpha(x, y, z)} = M_{\alpha,}(x, y, z) - \varphi\Big(M_{\alpha,}(x, y, z)\Big).$$

<u>Case</u> (*v*): Let $x, y \in [0, 1]$ and $z \in (1, \frac{6}{5}]$.
We assume, without loss of generality, that $x > y$.

$$S(fx, fy, fz) = S(\frac{x}{8}, \frac{y}{8}, z - \frac{4}{5}) = |\frac{x}{8} - (z - \frac{4}{5})| + |\frac{y}{8} - (z - \frac{4}{5})| = |\frac{4}{5} - (z - \frac{x}{8})| + |\frac{4}{5} - (z - \frac{y}{8})|$$
$$= z - \frac{x}{8} - \frac{4}{5} + z - \frac{y}{8} - \frac{4}{5} = 2z - \frac{x + y}{8} - \frac{8}{5} \leq \frac{4}{5} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(z, z, fz) - \frac{S(z, z, fz)}{1 + S(z, z, fz)}$$
$$= \frac{(S(z, z, fz))^2}{1 + S(z, z, fz)} \leq \frac{(M_\alpha(x, y, z))^2}{1 + M_\alpha(x, y, z)} = M_\alpha(x, y, z) - \frac{M_\alpha(x, y, z)}{1 + M_\alpha(x, y, z)} = M_\alpha(x, y, z) - \varphi\Big(M_\alpha(x, y, z)\Big).$$

<u>Case</u> (*vi*): Let $x \in [0, 1]$ and $z, y \in (1, \frac{6}{5}]$.
We assume, without loss of generality, that $z > y$.

$$S(fx, fy, fz) = S(\frac{x}{8}, y - \frac{4}{5}, z - \frac{4}{5}) = |\frac{x}{8} - z + \frac{4}{5}| + |y - z| = z - \frac{x}{8} - \frac{4}{5} + z - y = 2z - \frac{x}{8} - \frac{4}{5} - y$$
$$\leq \frac{3}{5} \leq \frac{64}{65} = \frac{8}{5} - \frac{8}{13} = S(z, z, fz) - \frac{S(z, z, fz)}{1 + S(z, z, fz)} = \frac{(S(z, z, fz))^2}{1 + S(z, z, fz)} \leq \frac{(M_\alpha(x, y, z))^2}{1 + M_\alpha(x, y, z)}$$
$$= M_\alpha(x, y, z) - \frac{M_\alpha(x, y, z)}{1 + M_\alpha(x, y, z)} = M_\alpha(x, y, z) - \varphi\Big(M_\alpha(x, y, z)\Big).$$

From all the above cases, we conclude that f is an (α, ψ, φ)-generalized weakly contractive map on X. Therefore, f, ψ and φ satisfy all the hypotheses of Theorem 2.1 and f has a unique fixed point $u = 0$.

Here we observe that the S-metric defined in this example is not a G-metric. Hence Theorem 1.6 is not applicable.

In view of Remark 1.14, our results extend Theorem 1.6 to S- metric spaces.

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