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On the Equivalence and Qualitative Behavior of the Rational Differential Equations

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Abstract. Recent study has shown great interest in the existence of center for planar differential systems. Sufficient conditions have been given for a critical point of some polynomial systems to be a center. However, it is challenging to generalize these results to higher-order or non-polynomial systems by traditional methods. In this paper, we tackle the problem by studying the equivalence of differential systems using the Mironenko's method. Our method includes the construction of RF- integrals for the rational differential equations, we give some new criterions for two differential equations to be equivalent. Applying these results to the study of center of planar differential systems, we generalize the conclusions made in existing literature for a specific polynomial system to a wide range of higher-order polynomial or non-polynomial systems.

1. Introduction and Preliminaries

In this paper, we consider the rational differential equations of the form

$$\frac{dr}{d\theta} = \frac{\sum_{i=0}^{n} b_i(\theta) r^i}{\sum_{i=0}^{n} a_i(\theta) r^i} r. \tag{1.1}$$

The main reason why we are interested in these equations is that they are closely related to planar polynomial system:

$$\begin{cases} x' = P(x, y) \\ y' = Q(x, y), \end{cases} \tag{1.2}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials of degree $n + 1$ with respect to x, y. To see this, let us note that the phase curves of (1.2) near the origin $(0, 0)$ in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ are determined by (1.1), where $a_i(\theta)$ and $b_i(\theta)$ $(i = 0, 1, 2, ..., n)$ are polynomials in $\cos\theta$ and $\sin\theta$. Since the limit cycles of (1.2) correspond to 2π-periodic solutions of (1.1), the planar vector field (1.2) has a center at (0,0) if and only if equation (1.1) has a center at $r = 0$, i.e., all the solutions nearby are 2π-periodic[1, 3–5, 11]. As usual, we

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often apply the method of Lyapunov to study center-focus problem. A new technique recently developed by Christopher. Invariant algebraic curves are sought and appropriate Dulac functions constructed [3, 11]. But for high-order polynomial systems and non-polynomial systems, to give the center conditions is very difficult. Necessary and sufficient conditions are known for very few classes of systems. There are well-known conditions for quadratic systems [4] and the problem has been resolved for systems in which P and Q are cubic polynomials without quadratic terms [1, 5].

In the following, we will apply a new method (Mironenko's reflecting function [7]) to study the qualitative behavior of solutions of the differential systems, and derive some sufficient conditions for a critical point of the certain non-polynomial differential systems and some high-order polynomial systems, to be a center. According to our method, if we know the qualitative behavior of the periodic solutions of (1.2), then we know the qualitative behavior of the periodic solutions of the equivalent system:

$$\begin{cases} x' = D(x,y)P(x,y) + \alpha(x,y)R(x,y)x, \\ y' = D(x,y)Q(x,y) + \alpha(x,y)R(x,y)y. \end{cases} \tag{1.3}$$

In this paper, in sections 2, we improve the conclusion of Mironenko [8] by using the even and odd algebraic integrals and give some new sufficient conditions under which two differential systems are equivalent in the sense of coincident reflecting function. In section 3, we study the structure of the RF-integrals of the rational differential equations and get their RF-integrals which are rational fractional functions not polynomial functions. In section 4, we present the sufficient conditions for two differential equations to be equivalent and derive some sufficient conditions for a critical point of the certain polynomial differential systems and their infinite equivalent planar systems, to be a center.

Now, we simply introduce the concept of the reflecting function, which will be used throughout the rest of this article.

Consider differential system

$$x' = X(t,x), \ (t \in I \subset R, \ x \in D \subset R^n, \ 0 \in I) \tag{1.4}$$

which has a continuously differentiable right-hand side and general solution $\phi(t; t_0, x_0)$.

Definition 1.1. *[7] For system (1.4), $F(t,x) := \phi(-t,t,x)$ is called its* **Reflecting function(RF)**.

If system (1.4) is 2ω-periodic with respect to t, and $F(t,x)$ is its reflecting function, then $T(x) := F(-\omega,x) = \phi(\omega; -\omega, x)$ is the Poincaré mapping of (1.4) over the period $[-\omega,\omega]$. Thus, the solution $x = \phi(t; -\omega, x_0)$ of (1.4) defined on $[-\omega,\omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$.

Lemma 1.2. *[7] A differentiable function $F(t,x)$ is a reflecting function of system (1.4) if and only if it is a solution of the Cauchy problem*

$$F_t + F_x X(t,x) + X(-t,F) = 0, F(0,x) = x. \tag{1.5}$$

This implies that sometimes for non-integrable periodic systems we can find out its Poincaré mapping. If, for example, $X(t,x) + X(-t,x) = 0$, then $T(x) = x$.

Definition 1.3. *[7] If the reflecting functions of two differential systems are coincident in their common domain, then these systems are said to be* **Equivalent**.

All the equivalent 2ω-periodic systems have a common Poincaré mapping over the period $[-\omega,\omega]$, and the qualitative behavior of the periodic solutions of these systems are the same. By this one can study the qualitative behavior of the solutions of a complicated system by using a simple differential system. Unfortunately, in general, it is very difficult to find out the reflecting function of (1.4). How to judge two systems are equivalent when we do not know their reflecting function? This is a very important and interesting problem! Mironenko in [7–9] has studied it and obtained some valuable and interesting conclusions. In this paper, we will improve their results and get some new sufficient conditions under which the differential equations are equivalent.

Definition 1.4. *If $\Delta(t,x)$ is a non-zero solution of the partial differential system*

$$\Delta_t(t,x) + \Delta_x(t,x)X(t,x) - X_x(t,x)\Delta(t,x) = 0, \tag{1.6}$$

then $\Delta(t,x)$ is called **RF-integral** *of (1.4).*

By [8], we know

Theorem 1.5. *[8] If $\Delta(t,x)$ is the RF-integral of (1.4), then the system (1.4) is equivalent to the system*

$$x' = X(t,x) + \beta(t)\Delta(t,x),$$

where $\beta(t)$ is an arbitrary odd continuous scalar function.

There are many papers which are also devoted to investigations of qualitative behavior of solutions of the differential systems by help of reflecting functions[2, 6–10, 12–16].

In this paper, we will discuss the qualitative behavior of solutions of the fractional differential equation (1.2) which has a continuously differentiable right-hand side, and has a unique solution for their initial value problem in a neighborhood of the origin.

2. Odd and Even Algebraic Integrals and Equivalence

Definition 2.1. *If $\delta(t,x)$ is a continuously differentiable scalar function and for any solution $x(t)$ of (1.4), $\delta(t,x(t)) = \delta(-t,x(-t))$, then this function is called* **Even algebraic integral** *of (1.4). If $\delta(t,x(t)) + \delta(-t,x(-t)) = 0$, then $\delta(t,x)$ is called* **Odd algebraic integral** *of (1.4).*

Remark 2.2. *If $u(t,x) = c$ (c is a constant) is the first-integral of (1.4), then for any differentiable function $\phi(t,x)$ such that $\phi(t,x) = \phi(-t,x)$, $\phi(t,u)$ is the even algebraic integral of (1.4) and $\alpha(t)\phi(t,u(t,x))$ is the odd algebraic integral of (1.4), where $\alpha(t)$ is an arbitrary odd function.*

Lemma 2.3. *Suppose that*

$$\delta_t(t,x) + \delta_x X(t,x) = f(t,\delta(t,x)) \tag{2.1}$$

and $f(t,\delta)$ is a continuously differentiable odd function with respect to t. Then $\delta(t,x)$ is the even algebraic integral of (1.4).

Proof. Denoting $\delta := \delta(t,x(t))$, $\bar{\delta} := \delta(-t,x(-t))$. It is easy to check that δ and $\bar{\delta}$ are the solutions of (2.1) and at $t = 0$ they are equal, by the uniqueness of the solutions of the initial value problem of (2.1), we get $\delta = \bar{\delta}$. Thus the proof is finished. □

Theorem 2.4. *If $\Delta(t,x)$ is the RF-integral of (1.4), $\delta(t,x)$ is an odd algebraic integral of (1.4), then (1.4) is equivalent to the system*

$$x' = X(t,x) + \delta(t,x)\Delta(t,x). \tag{2.2}$$

Proof. Let $F(t,x)$ be the reflecting function of (1.4). By (1.5) we have

$$F_t + F_x X(t,x) + X(-t,F) = 0,\ F(0,x) = x.$$

As $\Delta(t,x)$ is the RF-integral of (1.4), by lemma 1 of [8], we get $F_x\Delta(t,x) = \Delta(-t,F(t,x))$, thus

$$\begin{aligned} F_t + F_x(X + \delta(t,x)\Delta(t,x)) + X(-t,F) + \delta(-t,F)\Delta(-t,F) = \\ = F_x\delta(t,x)\Delta(t,x) + \delta(-t,F)\Delta(-t,F) = \\ = \delta(t,x)(F_x\Delta(t,x) - \Delta(-t,F(t,x)) = 0, \end{aligned}$$

hence, $F(t,x)$ is the reflecting function of (2.2), too. Therefore, the systems (1.4) and (2.2) are equivalent. □

Similarly, we get the following conclusions.

Theorem 2.5. *If $\Delta(t,x)$ is the RF-integral of (1.4), $\delta(t,x)$ is an even algebraic integral of (1.4), $\alpha(t,x)$ is an arbitrary continuously differentiable scalar odd function with respect to t, then the system (1.4) is equivalent to the system*

$$x' = X(t,x) + \alpha(t,\delta(t,x))\Delta(t,x).$$

Remark 2.6. *By lemma 2.3 follows that if $\Delta(t,x)$ is the RF-integral of (1.4) and $\delta(t,x)$ is the solution of (2.1), then the conclusion of theorem 2.5 holds.*

Theorem 2.7. *If $\Delta(t,x)$ is the RF-integral of (1.4), $u(t,x) = c$ (c is a constant) is the first integral of (1.4), $\alpha(t,u)$ is an arbitrary continuously differentiable scalar odd function with respect to t, then (1.4) is equivalent to the system*

$$x' = X(t,x) + \alpha(t,u(t,x))\Delta(t,x). \tag{2.3}$$

Remark 2.8. *If in theorem 2.4 taking $\delta(t,x) = \beta(t)$, or in theorem 2.5 or theorem 2.7 taking $\alpha(t,u) = \beta(t)$, $\beta(t)$ is an arbitrary scalar odd continuous function, then from one of these three theorems can follow the result of theorem 1.5. That is said, our conclusions improve the result of Mironenko's[8].*

Example 2.9. *If $x\psi_x + y\psi_y = 2\psi$, then the differential system*

$$\begin{cases} x' = y + x\psi(x,y), \\ y' = -x + y\psi(x,y) \end{cases} \tag{2.4}$$

has a first integral $u = t - \arctan\frac{y}{x}$ and RF-integral $\Delta = (x(x^2+y^2), y(x^2+y^2))^T$. By theorem 2.7, this system is equivalent to the system

$$\begin{cases} x' = y + x\psi(x,y) + \alpha(t,\phi(t,t-\arctan\frac{y}{x}))x(x^2+y^2), \\ y' = -x + y\psi(x,y) + \alpha(t,\phi(t,t-\arctan\frac{y}{x}))y(x^2+y^2), \end{cases}$$

where $\alpha(t,u)$ is an arbitrary continuously differentiable odd function with respect to t, $\phi(t,u)$ is a continuously differential even function with respect to t.

If taking $\phi(t,u) = u$, $\alpha(t,u) = \beta(t)\cos^2 u = \beta(t)\frac{(x\sin t + y\cos t)^2}{x^2+y^2}$, then the system (2.4) is equivalent to the system

$$\begin{cases} x' = y + x\psi(x,y) + \beta(t)x(x\sin t + y\cos t)^2, \\ y' = -x + y\psi(x,y) + \beta(t)y(x\sin t + y\cos t)^2, \end{cases} \tag{2.5}$$

where $\beta(t)$ is an arbitrary differentiable odd scalar function. Furthermore, if $\beta(t)$ is 2π-periodic, then the qualitative behavior of the periodic solutions of the systems (2.4) and (2.5) are the same.

3. The Structure of the RF-Integrals of the Rational Equations

In this section, we will discuss the structure of the RF-integral of the following rational differential equation

$$x' = \frac{\sum_{i=0}^{m} b_i(t)x^i}{\sum_{i=0}^{n} a_i(t)x^i} = \frac{B}{A}. \tag{3.1}$$

It has a continuously differentiable right-hand side and the solutions of the initial value problems exist and unique. A and B are relatively prime polynomials with respect to x, $A \neq 0$ and $a_n \neq 0$ $(n \geq 1)$ near the point $x = 0$ for all $t \in R$.

In this section, the notation $(A,B) = 1$ means that A and B are relatively prime polynomials with respect to x, $A \mid B$ means that B is divisible by A, $A \nmid B$ denotes that B is not divisible by A.

By the definition 1.4, we know that if function $\Delta(t,x)$ is a solution of the following partial differential equation

$$\Delta_t(t,x) + \Delta_x(t,x)\frac{B}{A} - (\frac{B}{A})_x\Delta(t,x) = 0, \tag{3.2}$$

then $\Delta(t,x)$ is the RF-integral of (3,1).

Theorem 3.1. *If $\Delta(t,x)$ is the RF-integral of (3.1), then $\Delta(t,x)$ can not be a polynomial function with respect to x.*

Proof. On the contrary, if the present conclusion is not correct. Suppose $\Delta(t,x)$ is a polynomial of degree k with respect to x, by (3.2) we get

$$A(A\Delta_t + \Delta_x B - B_x\Delta) = -A_x B\Delta, \tag{3.3}$$

As $(A,B) = 1$, (3.3) implies that $A \mid A_x\Delta$.

Case 1 $(A,A_x) = 1$. From (3.3) follows that $A \mid \Delta$. So $\Delta = A^kW, k \geq 1, A \nmid W$, W is a polynomial function with respect to x. Substituting $\Delta = A^kW$ into (3.3) we get

$$A(kA_tW + AW_t + W_xB - B_xW) = -(k+1)A_xBW,$$

it implies that $A \mid W$. This is contradictory with the above hypothesis. Thus, Δ is not a polynomial function with respect to x.

Case 2. $(A,A_x) \neq 1$. Let $A = A_1^{k_1}A_2^{k_2}\cdots A_l^{k_l}$, $\sum_{i=1}^{l} k_i = n, k_i\,(i = 1,2,...,l)$ are positive integers, $(A_i,A_j) = 1, (A_i,A_{ix}) = 1\,(i \neq j,\ i,j = 1,2,...,l)$, the polynomials $A_i(i = 1,2,...l)$ can not be factored. As $(A,A_x) \neq 1$, so, there is at least one $k_i > 1 (1 \leq i \leq l)$. We might assume that $k_1 > 1$, thus

$$A_x = A_1^{k_1-1}G,\ G = k_1A_{1x}\Phi + A_1\Phi_x,\ \Phi := A_2^{k_2}A_3^{k_3}\cdots A_l^{k_l}.$$

Obviously, it implies that $(A_1,G) = 1$.

Substituting them into (3.3) and dividing by $A_1^{k_1-1}$, we get

$$A_1\Phi(A\Delta_t + \Delta_xB - B_x\delta) = -GB\Delta, \tag{3.3'}$$

As $(A_1,B) = 1$, $(A_1,G) = 1$, so $A_1 \mid \Delta$. Let $\Delta = A_1^kD$, $A_1 \nmid D$, $(k \geq 1)$. Substituting it into (3.3') and dividing by A_1^k, we obtain

$$A_1(\Phi(A_1^{k_1-1}\Phi(kA_{1t}D + A_1D_t) + D_xB - B_xD) + BD\Phi_x) = -BD(k_1+k)A_{1x}\Phi,$$

it implies that $A_1 \mid D$, this is contradictory with the above hypothesis. Thus, the present theorem is correct.

The proof is finished.

□

Theorem 3.2. *If R and S are relatively prime polynomials with respect to x, $\Delta = \frac{R(t,x)}{S(t,x)}$ is a rational fractional RF-integral of (3.1), then $S(t,x) = A(t,x)D(t,x)$ and $(A,D) = 1$, where D is a polynomial with respect to x.*

Proof. Suppose $\Delta = \frac{R}{S}$ is the RF-integral of (3.1), using (3.2) we get

$$A[A(R_tS - S_tR) + B(R_xS - S_xR) - B_xRS] = -A_xBRS. \tag{3.4}$$

As $(A,B) = 1$, so $A \mid A_xRS$.

Case 1. $(A,A_x) = 1$. From (3.4) follows $A \mid RS$.

1^0. If $(A,S) = 1$, then $A \mid R$ and $R = A^kG$, $A \nmid G$, where G is a polynomial with respect to x, k is a positive integer. Substituting it into (3.4) and dividing by A^k, we get

$$A[kA_tG + AG_tS - S_tAG + B(G_xS - S_xG) - B_xSG] = -(k+1)BGSA_x,$$

it implies that $A \mid BGSA_x$, as $(A, A_x) = 1, (A, B) = 1, (A, S) = 1$, so $A \mid G$. This is contradictory to the previous hypothesis. Thus $(A, S) \neq 1$.

2^0. If $(A, R) = 1$, then $A \mid S$. Because $(A, A_x) = 1$, so A can be expressed as $A = A_1 A_2 \cdots A_l$, $(A_i, A_j) = 1$, $(A_i, A_{ix}) = 1\,(i \neq j, i, j = 1, 2, ..., l)$. Let $S = A_1^{m_1} A_2^{m_2} \cdots A_l^{m_l} D$, $(A_i, D) = 1$, $m_i\,(i = 1, 2, ..., l)$ are positive integers. Substituting them into (3.4) and dividing by factor $A_1^{m_1} A_2^{m_2} \cdots A_l^{m_l}$ we get

$$A[R_t AD - RAD_t - AD(m_1 A_{1t} A_2 \cdots A_l + m_2 A_1 A_{2t} A_3 \cdots A_l + ... + m_l A_1 A_2 \cdots A_{lt}) +$$
$$+ B(R_x D - RD_x) - B_x RD] =$$
$$RDB((m_1 - 1)A_{1x} A_2 \cdots A_l + (m_2 - 1)A_1 A_{2x} A_3 \cdots A_l + ... + (m_l - 1)A_1 A_2 \cdots A_{lx}).$$

As $(A, A_x) = 1, (A, B) = 1, (A, R) = 1, (A, D) = 1$, from the above relation implies $m_i = 1\,(i = 1, 2, ..., l)$, i.e., $S = AD, (A, D) = 1$. Therefore, the conclusion of the present theorem is correct.

3^0. If $(A, S) \neq 1$ and $(A, R) \neq 1$. Let $A = A_1 A_2$, $(A_1, A_2) = 1, A_1 \mid S, A_2 \mid R$, $(A_1, A_{1x}) = 1, (A_2, A_{2x}) = 1$. Then $R = A_2^k G$, $A_2 \nmid G$, k is a positive integer. Substituting it into (3.4) and dividing by the factor A_2^k we get

$$A_2[A_1^2(kA_{2t} GB + A_2 SG_t - A_2 S_t G) + A_1 B(G_x S - S_x G) + SG(A_{1x} B - B_x A_1)] =$$
$$= -(k + 1)A_{2x} SBA_1 G,$$

it implies that $A_2 \mid A_{2x} SBGA_1$. As $(A_2, A_{2x}) = 1, (A_2, A_1) = 1$, $(A_2, B) = 1$, $(A_2, S) = 1$, so $A_2 \mid G$, this is contradictory to the previous hypothesis. Thus $A \mid S$, i.e., $S = AD, (A, D) = 1$.

Case 2. $(A, A_x) \neq 1$. Let $A = A_1^{k_1} A_2^{k_2} \cdots A_l^{k_l}$, $\sum_{i=1}^{l} k_i = n, k_i \geq 1$, $(A_i, A_j) = 1, (A_i, A_{ix}) = 1\,(i \neq j,\ i, j = 1, 2..., l)$, the polynomials $A_i (i = 1, 2, ...l)$ can not be factored. As $(A, A_x) \neq 1$, so, there is at least one $k_i > 1 (1 \leq i \leq l)$. Thus

$$A_x = A_1^{k_1 - 1} A_2^{k_2 - 1} \cdots A_l^{k_l - 1} G,$$
$$G = k_1 A_{1x} A_2 ... A_l + k_2 A_1 A_{2x} \cdots A_l + \cdots + k_l A_1 ... A_{l-1} A_{lx}.$$

Obviously, it implies that $(A_i, G) = 1$, $(i = 1, 2, ..., l)$. Substituting them into (3.4) and dividing by the factor $A_1^{k_1 - 1} A_2^{k_2 - 1} \cdots A_l^{k_l - 1}$, we obtain

$$\breve{A}[A(R_t S - S_t R) + B(R_x S - S_x R) - B_x RS] = -BRSG,$$

where

$$\breve{A} = A_1 A_2 A_3 \cdots A_l.$$

As $(A_i, B) = 1$, $(A_i, G) = 1$, so $\breve{A} \mid RS$.

1^0. If $(\breve{A}, S) = 1, \breve{A} \mid R$. Let $R = A_1^{m_1} A_2^{m_2} \cdots A_l^{m_l} \Psi, (A_i, \Psi) = 1, m_i\,(i = 1, 2, ..., l)$ are positive integers. In the similar way as case 2 of theorem 3.1, we can prove that $A_i \mid \Psi$, this is contradictory to the previous hypothesis .

2^0. If $(\breve{A}, R) = 1, \breve{A} \mid S$. Let $S = A_1^{m_1} A_2^{m_2} \cdots A_l^{m_l} D, (A_i, D) = 1, m_i\,(i = 1, 2, ..., l)$ are positive integers. Similar as case 1 of the above and theorem 3.1, we can prove that $m_i = k_i, (i = 1, 2, ..., l)$, thus, $S = AD, (A, D) = 1$ and the result of the present theorem is correct.

3^0. If $(\breve{A}, R) \neq 1, (\breve{A}, S) \neq 1$, similar as the case 1 of the above, we can prove $S = AD, (A, D) = 1$.

In summary, the proof is completed. □

4. On the RF-Integrals of $\frac{dx}{dt} = \frac{b_0(t) + b_1(t)x}{a_0(t) + a_1(t)x} x$

Consider the rational differential equation

$$\frac{dx}{dt} = \frac{b_0 + b_1 x}{a_0 + a_1 x} x = \frac{B}{A} x, \tag{4.1}$$

where $a_i = a_i(t), b_i = b_i(t)\,(i = 0, 1)$ are continuously differentiable functions and $a_0 b_1 - a_1 b_0 \neq 0, a_0 a_1 \neq 0$, $A = a_0 + a_1 x$, $B = b_0 + b_1 x$.

By section 3, we know the equation (4.1) has the RF-integral in the form of $\Delta = \frac{R}{AD} x$. Thus, in this section we will discuss under which conditions this rational fractional function is the RF-integral of (4.1).

Theorem 4.1. *Suppose that functions $v_k(k = 0, 1, 2, ..., n)$ are the solutions of the following linear differential systems*

$$\begin{cases} v_0' = 0, \\ v_k' = (-\frac{a_1}{a_0}v_{k-1}' + ((\frac{a_1}{a_0})' - (k-2)\frac{b_1}{a_0} + (k-1)\frac{a_1 b_0}{a_0^2})v_{k-1})e^{-\int \frac{b_0}{a_0}dt}, \ (k = 1, 2, ..., n) \end{cases} \tag{4.2}$$

$$(v_n \frac{a_0}{a_1} e^{\int (-n\frac{b_0}{a_0} + (n-1)\frac{b_1}{a_1})dt})' = 0. \tag{4.3}$$

Then

$$\Delta = \frac{\sum_{k=0}^{n} a_0 v_k e^{-k\int \frac{b_0}{a_0}dt} x^k}{a_0 + a_1 x} x \tag{4.4}$$

is the RF-integral of (4.1).

Proof. By (1.5), $\Delta = \frac{\sum_{k=0}^{n} r_k x^k}{a_0 + a_1 x} x = \frac{R}{A} x$ is the RF-integral of (4.1), if and only if

$$R_t A - A_t R = (RB_x - R_x B)x. \tag{4.5}$$

Equating the coefficients of the same power of x on the both sides of (4.5) we obtain

$$r_0' a_0 - a_0' r_0 = 0, \tag{4.6}$$

$$a_0 r_k' + a_1 r_{k-1}' - a_0' r_k - a_1' r_{k-1} = -kb_0 r_k - (k-2)b_1 r_{k-1}, \ (k = 1, 2, .., n) \tag{4.7}$$

$$r_n' a_1 - a_1' r_n = -(n-1)b_1 r_n. \tag{4.8}$$

Denoting $v_k = \frac{r_k}{a_0} e^{k\int \frac{b_0}{a_0}dt}$ and using (4.6) and (4.7) and simplify computing we get (4.2). From (4.8) follows $(\frac{r_n}{a_1} e^{(n-1)\int \frac{b_1}{a_1}dt})' = 0$, which implies the identity (4.3) holds. Therefore, the function (4.4) is the RF-integral of (4.1). □

Theorem 4.2. *If there are two different constants λ_0, λ_1 such that*

$$\lambda_0(a_0 b_1 + a_0 a_1' - a_1 a_0') = \lambda_1(a_1 b_0 + a_1' a_0 - a_1 a_0'), \tag{4.9}$$

then $\Delta = \frac{\lambda_0 a_0 + \lambda_1 a_1 x}{a_0 + a_1 x} x$ is the RF-integral of (4.1) and

1^0. *If $\lambda_0 \lambda_1 \neq 0$, $\lambda = \frac{\lambda_1}{\lambda_0} \neq 1$, then the general integral of (4.1) is*

$$u = (\frac{1}{\lambda} x^{\frac{\lambda}{1-\lambda}} + \frac{a_1}{a_0} x^{\frac{1}{1-\lambda}}) e^{\frac{\lambda}{\lambda-1}\int \frac{b_0}{a_0}dt}.$$

2^0. *If $\lambda_0 = 0, \lambda_1 \neq 0$, then the general integral of (4.1) is*

$$u = \ln|x| - \frac{a_0}{a_1 x} - \int \frac{b_1}{a_1} dt.$$

3^0. *If $\lambda_1 = 0, \lambda_0 \neq 0$, then the general integral of (4.1) is*

$$u = \ln|x| + \frac{a_1}{a_0} x - \int \frac{b_0}{a_0} dt.$$

And also, the equation (4.1) is equivalent to the equation

$$\frac{dx}{dt} = \frac{b_0 + b_1 x}{a_0 + a_1 x} x + \alpha(t, u) \frac{\lambda_0 a_0 + \lambda_1 a_1 x}{a_0 + a_1 x} x, \tag{4.10}$$

where $\alpha(t, u)$ is an arbitrary continuously differentiable odd function with respect to variable t, u is the general integral of (4.1).

Proof. By definition 1.4, we know the fractional function $\Delta = \frac{r_0+r_1x}{a_0+a_1x}x = \frac{R}{A}x$ is the RF-integral of (4.1), if and only if (4.5) holds. Equating the coefficients of the same power of x on the both sides of (4.5), we have

$$r_0'a_0 - a_0'r_0 = 0, \tag{4.6}$$

$$r_0'a_1 + r_1'a_0 - a_0'r_1 - a_1'r_0 = b_1r_0 - r_1b_0, \tag{4.11}$$

$$r_1'a_1 - a_1'r_1 = 0. \tag{4.12}$$

From (4.6) implies that $r_0 = \lambda_0 a_0$, λ_0 is a constant . From (4.12) yields that $r_1 = \lambda_1 a_1$, λ_1 is a constant. Substituting these relations into (4.11) we get (4.9) . As $a_0b_1 - a_1b_0 \neq 0$, so $\lambda_0 - \lambda_1 \neq 0$. Thus, under the hypothesis of the present theorem, the function $\Delta = \frac{\lambda_0a_0+\lambda_1a_1x}{a_0+a_1x}x$ is the RF-integral of (4.1).

1^0. If $\lambda_0\lambda_1 \neq 0$, $\lambda = \frac{\lambda_1}{\lambda_0} \neq 1$. From (4.9), we get

$$b_1 = \lambda\frac{a_1}{a_0}b_0 + (\lambda - 1)\frac{1}{a_0}(a_1'a_0 - a_0'a_1).$$

Substituting it into (4.1), we have

$$x' + (1-\lambda)(\frac{1}{1-\lambda}\frac{a_1}{a_0}xx' + (\frac{a_1}{a_0})'x^2) = \frac{b_0}{a_0}(x + \lambda\frac{a_1}{a_0}x^2).$$

Both sides of the above equation are multiplied by $\frac{1}{1-\lambda}x^{\frac{2\lambda-1}{1-\lambda}}$, we obtain

$$(\frac{1}{\lambda}x^{\frac{\lambda}{1-\lambda}} + \frac{a_1}{a_0}x^{\frac{1}{1-\lambda}})' = \frac{b_0\lambda}{(1-\lambda)a_0}(\frac{1}{\lambda}x^{\frac{\lambda}{1-\lambda}} + \frac{a_1}{a_0}x^{\frac{1}{1-\lambda}}).$$

Solving this linear differential equation we get

$$u = (\frac{1}{\lambda}x^{\frac{\lambda}{1-\lambda}} + \frac{a_1}{a_0}x^{\frac{1}{1-\lambda}})e^{-\int\frac{\lambda b_0}{(1-\lambda)a_0}dt} = c,$$

(where c is a constant) is the general solution of (4.1).

2^0. If $\lambda_0 = 0, \lambda_1 \neq 0$, by (4.9), we get $b_0 = \frac{1}{a_0}(a_0'a_1 - a_0a_1')$, substituting it into (4.1) we have

$$\frac{x'}{x} + \frac{a_0}{a_1}\frac{x'}{x^2} - (\frac{a_0}{a_1})'\frac{1}{x} = \frac{b_1}{a_1}.$$

Solving this linear equation we get its general integral is

$$u = \ln|x| - \frac{a_0}{a_1x} - \int\frac{b_1}{a_1}dt = c,$$

where c is constant.

3^0. If $\lambda_0 \neq 0, \lambda_1 = 0$, using (4.9), we get $b_1 = \frac{a_1a_0'-a_0a_1'}{a_0}$, substituting it into (4.1), we have

$$\frac{x'}{x} + (\frac{a_1}{a_0}x)' = \frac{b_0}{a_0}.$$

Solving this equation we get its general integral is

$$u = \ln|x| + \frac{a_1}{a_0}x - \int\frac{b_0}{a_0}dt = c,$$

where c is a constant.

According to theorem 2.7, the equation (4.1) is equivalent to the equation (4.10).

In summary, the proof is completed. □

By Theorem 3.2, we see that the equation (4.1) maybe has the RF-integral in the form of $\Delta = \frac{r_0+r_1x+r_2x^2}{(a_0+a_1x)(d_0+d_1x)}x = \frac{R}{AD}x$, $(A, D) = 1$.

Theorem 4.3. *Suppose there are two real numbers λ_0, λ_1 such that*

1^0. *If*

$$a_1\zeta_1 + a_0(\lambda_1 - \lambda_0) \neq 0$$

and

$$a_0\zeta_0 + a_1(\lambda_0 - \lambda_1) = z(a_1\zeta_1 + a_0(\lambda_1 - \lambda_0)), \tag{4.13}$$

where $z := \frac{d_1}{d_0}$ $(d_0d_1 \neq 0)$ is the solution of the linear differential equation

$$z'(-\zeta_0 + \frac{a_1}{a_0}(\lambda_1 - \lambda_0)) + z(\zeta_0' + (\lambda_1 - \lambda_0)(\frac{a_1}{a_0})' + 2\lambda_1\frac{a_1b_0}{a_0^2} - 2\lambda_0\frac{b_1}{a_0}) = \zeta_0(\frac{a_1}{a_0})' - \frac{a_1}{a_0}\zeta_0'. \tag{4.14}$$

ζ_0 and ζ_1 are respectively the solutions of the following linear differential equations

$$\zeta_0' = -\frac{b_0}{a_0}\zeta_0 + \lambda_0\frac{a_0b_1 - a_1b_0}{a_0^2}, \tag{4.15}$$

$$\zeta_1' = \frac{b_1}{a_1}\zeta_1 + \lambda_1\frac{a_0b_1 - a_1b_0}{a_1^2}. \tag{4.16}$$

Then

$$\Delta = \frac{\lambda_0a_0 + (a_0\zeta_0 + \lambda_0(a_1 + a_0z))x + \lambda_1a_1zx^2}{(a_0 + a_1x)(d_0 + d_1x)}x$$

is the RF-integral of (4.1).

2^0. *If $a_1\zeta_1 + a_0(\lambda_1 - \lambda_0) = 0$ and (4.16) holds, then $\Delta = \frac{\lambda_0a_0+\lambda_1a_1x}{a_0+a_1x}x$ is the RF-integral of (4.1).*

Proof. By (1.6), $\Delta = \frac{r_0+r_1x+r_2x^2}{(a_0+a_1x)(d_0+d_1x)}x = \frac{R}{AD}x$ is the RF-integral of (4.1), if and only if

$$R_tAD - (AD)_tR = x(B_xRD + D_xRB - R_xDB).$$

Equating coefficients of the same power of x on the both sides of above relation we obtain

$$r_0'e_0 - e_0'r_0 = 0, \tag{4.17}$$

$$r_0'e_1 + r_1'e_0 - e_0'r_1 - e_1'r_0 = (b_1d_0 + b_0d_1)r_0 - r_1b_0d_0, \tag{4.18}$$

$$e_2r_0' + e_1r_1' + e_0r_2' - r_2e_0' - r_1e_1' - r_0e_2' = 2(b_1d_1r_0 - b_0d_0r_2), \tag{4.19}$$

$$r_1'e_2 + r_2'e_1 - e_1'r_2 - e_2'r_1 = b_1d_1r_1 - (b_0d_1 + b_1d_0)r_2, \tag{4.20}$$

$$r_2'e_2 - e_2'r_2 = 0, \tag{4.21}$$

where $e_0 = a_0d_0$, $e_1 = a_0d_1 + a_1d_0$, $e_2 = a_1d_1$.

Solving equation (4.17) and (4.21), we get

$$r_0 = \lambda_0e_0 = \lambda_0a_0d_0, \quad r_2 = \lambda_1e_2 = \lambda_1a_1d_1.$$

Substituting these relations into (4.18)-(4.20), we obtain

$$r_1' = \frac{e_0' - b_0d_0}{e_0}r_1 + r_0((\frac{e_1}{e_0})' + \frac{b_1d_0 + b_0d_1}{e_0}), \tag{4.18'}$$

$$r_1' = \frac{e_2' - b_1 d_1}{e_2} r_1 + r_2((\frac{e_1}{e_2})' - \frac{b_0 d_1 + b_1 d_0}{e_2}), \tag{4.20'}$$

$$r_1' e_1 - e_1' r_1 = (\lambda_1 - \lambda_0)(e_2 e_0' - e_0 e_2') + 2(b_1 d_1 r_0 - b_0 d_0 r_2). \tag{4.19'}$$

Solving the equation (4.18′) and (4.20′) we get

$$r_1 = \lambda_0 e_1 + e_0 \zeta_0, \tag{4.18''}$$

$$r_1 = \lambda_1 e_1 + e_2 \zeta_1, \tag{4.20''}$$

where ζ_0 and ζ_1 are respectively the solutions of the equations (4.15) and (4.16).

Using (4.18″) and (4.20″) derive the relation (4.13). Substituting (4.18″) into (4.19′) implies (4.14) is held. Thus, under the conditions of the present theorem, the function $\Delta = \frac{\lambda_0 a_0 d_0 + r_1 x + \lambda_1 a_1 d_1 x^2}{(a_0 + a_1 x)(d_0 + d_1 x)} x$ is the RF-integral of (4.1).

If $a_1 \zeta_1 + a_0(\lambda_1 - \lambda_0) = 0$, using (4,16) follows the identity relation (4.9). According to theorem 4.2, the conclusion of the present theorem is correct.

The proof is completed. □

Similar, we can discuss under which conditions $\Delta = \frac{R}{AD} x$ is the RF-integral of (4.1). Where D, R are polynomials of degree n with respect to x. Here, we don't state it anymore.

Now, we apply the theorem 4.2 to discuss the center-focus problem and integrability problem of the quadratic systems and their equivalent systems.

Let's consider the quadratic system

$$\begin{cases} x' = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ y' = \ x + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{cases} \tag{4.21}$$

where a_{ij}, b_{ij} $(i, j = 0, 1, 2)$ are real numbers. Taking $x = r\cos\theta, y = r\sin\theta$, this system reduces to

$$\frac{dr}{d\theta} = \frac{b_1 r^2}{1 + a_1 r}, \tag{4.22}$$

where

$$a_1 = b_{20}\cos^3\theta + (b_{11} - a_{20})\cos^2\theta\sin\theta + (b_{02} - a_{11})\cos\theta\sin^2\theta - a_{02}\sin^3\theta,$$
$$b_1 = a_{20}\cos^3\theta + (a_{11} + b_{20})\cos^2\theta\sin\theta + (a_{02} + b_{11})\cos\theta\sin^2\theta + b_{02}\sin^3\theta.$$

The relation (4.9) is equivalent to

$$\lambda_0 b_1 = (\lambda_1 - \lambda_0) a_1'. \tag{4.9'}$$

By relation (4.9′) and theorem 4.2 we derive the following conclusions.

Theorem 4.4. *For nonzero constant λ , $\lambda \neq 1$. If*

$$a_{02} = \frac{1}{1-\lambda} a_{20}, b_{20} = -\frac{1}{\lambda} a_{11}, b_{11} = \frac{\lambda}{\lambda - 1} a_{20}, b_{02} = \frac{\lambda - 1}{\lambda} a_{11},$$

then system (4.21) has the first integral

$$u = (x^2 + y^2)^{\frac{\lambda}{2(1-\lambda)}} (1 - a_{11}x + \frac{\lambda a_{20}}{\lambda - 1} y)$$

and this system is equivalent to the system

$$\begin{cases} x' = -y + a_{20}x^2 + a_{11}xy + \frac{1}{1-\lambda} a_{20}y^2 + \alpha(\theta, u)x(1 - a_{11}x + \frac{\lambda}{\lambda-1} a_{20}y), \\ y' = x - \frac{1}{\lambda} a_{11}x^2 + \frac{\lambda}{\lambda-1} a_{20}xy + \frac{\lambda-1}{\lambda} a_{11}y^2 + \alpha(\theta, u)y(1 - a_{11}x + \frac{\lambda}{\lambda-1} a_{20}y). \end{cases} \tag{4.23}$$

where $\alpha(\theta, u)$ is an arbitrary continuously differentiable odd function with respect to θ, $\theta = arctan\frac{y}{x}$. And these equivalent systems have a center at (0,0).

Proof. Under the assumption of the present theorem, taking $\lambda = \frac{\lambda_1}{\lambda_0}$, it is not difficult to verify that (4.9′) holds. By theorem 4.2 we know

$$\Delta = \frac{\lambda_0 + \lambda_1 a_1 r}{1 + a_1 r} r,$$

in which

$$a_1 = -\frac{a_{11}}{\lambda}\cos\theta + \frac{1}{\lambda - 1} a_{20}\sin\theta,$$

is the RF-integral of (4.22) and its general integral is

$$u = r^{\frac{\lambda}{1-\lambda}}(1 + \lambda a_1 r) = r^{\frac{\lambda}{1-\lambda}}(1 - a_{11} r\cos\theta + \frac{\lambda}{\lambda - 1} a_{20} r\sin\theta).$$

Thus

$$u = (x^2 + y^2)^{\frac{\lambda}{2(1-\lambda)}}(1 - a_{11}x + \frac{\lambda a_{20}}{\lambda - 1} y)$$

is the first integral of (4.21).

By theorem 2.7, it implies that (4.22) is equivalent to the equation

$$\frac{dr}{d\theta} = \frac{b_1 r^2}{1 + a_1 r} + \alpha(\theta, u)\frac{\lambda_0 + \lambda_1 a_1 r}{1 + a_1 r} r. \tag{4.24}$$

As $u(\theta, r) = u(-\theta, F(\theta, r))$, in which $F(\theta, r)$ is the reflecting function of (4.22), thus $F(\theta + 2\pi, r) = F(\theta, r)$ and $r = 0$ is a center of (4.22) (see [6]) and the equivalent equation (4.24) has a center $r = 0$, too. On the other way, the equation (4.24) is equivalent to the system (4.23), therefore, the conclusion of the present theorem is true. □

Remark 4.5. *If in (4.23) taking $\alpha(\theta, u) = \sin\theta u^{1-\lambda}$, then the system (4.23) reduces to*

$$\begin{cases} x' = -y + a_{20}x^2 + a_{11}xy + \frac{1}{1-\lambda}a_{20}y^2 + xy(1 - a_{11}x + \frac{\lambda}{\lambda-1}a_{20}y)^{2-\lambda}, \\ y' = x - \frac{1}{\lambda}a_{11}x^2 + \frac{\lambda}{\lambda-1}a_{20}xy + \frac{\lambda-1}{\lambda}a_{11}y^2 + y^2(1 - a_{11}x + \frac{\lambda}{\lambda-1}a_{20}y)^{2-\lambda} \end{cases}$$

and this system has a center (0,0), too. That is said, if we know a quadratic system has a center at (0,0), meanwhile, we know its many equivalent complicated systems have a center at (0,0), too.

In (4.9′) taking $\lambda_1 = \frac{2}{3}\lambda_0$ and same discussing as theorem 4.4, we get the following theorem.

Theorem 4.6. *If*

$$b_{11} = -2a_{20}, b_{02} = -\frac{1}{2}a_{11},$$

then system (4.21) has a first integral

$$u(x, y) = \frac{3}{2}(x^2 + y^2) + b_{20}x^3 - 3a_{20}x^2y - \frac{3}{2}a_{11}xy^2 - a_{02}y^3$$

and this system is equivalent to system

$$\begin{cases} x' = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \alpha(\theta, u)u(x, y)\cos\theta, \\ y' = x + b_{20}x^2 - 2a_{20}xy - \frac{1}{2}a_{11}y^2 + \alpha(\theta, u)u(x, y)\sin\theta, \end{cases} \tag{4.25}$$

where $\alpha(\theta, u)$ is an arbitrary continuously differentiable odd function with respect to θ, $\theta = \arctan\frac{y}{x}$. And these equivalent systems have a center at (0,0).

If in (4.9′) taking $\lambda_0 = 0, \lambda_1 \neq 0$, then the relation (4.9) is equivalent to $a_1 = 0$ and the equation (4.22) is a polynomial equation, in the paper [14] we have discussed its RF-integral and its equivalence.

If in (4.9′) taking $\lambda_1 = 0, \lambda_0 \neq 0$. Applying theorem 4.2 and same discussing as theorem 4.4, we obtain

Theorem 4.7. *If $a_{11} = b_{11} = 0, a_{20} = a_{02}, b_{20} = b_{02}$, then the system (4.21) has the first integral*

$$u = \ln \sqrt{x^2 + y^2} + b_{20}x - a_{20}y$$

and it is equivalent to the system

$$\begin{cases} x' = -y + a_{20}(x^2 + y^2) + \alpha(\theta, u)x, \\ y' = x + b_{20}(x^2 + y^2) + \alpha(\theta, u)y, \end{cases} \tag{4.26}$$

where $\alpha(\theta, u)$ is an arbitrary continuously differentiable odd function with respect to θ, $\theta = arctan\frac{y}{x}$. And these equivalent systems have a center (0, 0).

If in (4.26) taking $\alpha = \lambda \sin\theta \cos\theta e^{2u}$ (λ is a constant), then this system becomes

$$\begin{cases} x' = -y + a_{20}(x^2 + y^2) + \lambda x^2 y e^{2(b_{20}x - a_{20}y)}, \\ y' = x + b_{20}(x^2 + y^2) + \lambda x y^2 e^{2(b_{20}x - a_{20}y)}. \end{cases}$$

From the above theorems, by the equivalence, we can know the qualitative behavior of a non-polynomial system and high-order polynomial systems by discuss a low-order polynomial system.

Applying theorem 4.3 to system (4.21) we derive the following theorem.

Theorem 4.8. *If there are two real numbers λ_0, λ_1 such that*

$$a_1\zeta_1 + \lambda_1 - \lambda_0 \neq 0,$$

$$(a_1(\lambda_1 - \lambda_0) - \zeta_0)^2 + (a_1\zeta_1 + \lambda_1 - \lambda_0)(a_1\zeta_0 - 2\lambda_0 \int a_1 b_1 d\theta) = 0,$$

$$\zeta_0 + a_1(\lambda_0 - \lambda_1) = z(a_1\zeta_1 + \lambda_1 - \lambda_0),$$

where

$$\zeta_0' = \lambda_0 b_1,\ \zeta_1' = \frac{b_1}{a_1}\zeta_1 + \lambda_1 \frac{b_1}{a_1^2},$$

then

$$\Delta = \frac{\lambda_0 + (\lambda_0(z + a_1) + \zeta_0)r + \lambda_1 a_1 z r^2}{(1 + a_1 r)(1 + zr)} r$$

is the RF-integral of (4.22), and (4.21) is equivalent to the system

$$\begin{cases} x' = (-y + a_{20}x^2 + a_{11}xy + a_{02}y^2)(1 + z\sqrt{x^2 + y^2}) + \alpha(\theta, u)Rx, \\ y' = (x + b_{20}x^2 + b_{11}xy + b_{02}y^2)(1 + z\sqrt{x^2 + y^2}) + \alpha(\theta, u)Ry, \end{cases} \tag{4.27}$$

where $\theta = arctan\frac{y}{x}$, $R = \lambda_0 + (\lambda_0(z + a_1) + \zeta_0)\sqrt{x^2 + y^2} + \lambda_1 a_1 z(x^2 + y^2)$, $\alpha(\theta, u)$ is an arbitrary continuously differentiable odd function with respect to θ, u is an even algebraic integral of (4.21).

In summary, to discuss the equivalence of the first-order differential equations is very important for researching the qualitative behavior of the planar differential systems. Applying our results to the study of center of planar differential systems, we generalize the conclusions made in existing literature for a specific polynomial system to a wide range of higher-order polynomial or non-polynomial systems.

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