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A Note on Meir-Keeler Contractions on Dislocated Quasi-b-Metric

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Abstract. In this manuscript, we show that Meir-Keeler type contractions posses a fixed point in the setting of dislocated quasi-b-metric.

1. Introduction and Preliminaries

Throughout the paper, let $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and positive integers, respectively. In addition, let $\mathbb{R}_0^+ := [0,\infty)$ and $N_0 := \mathbb{N} \cup \{0\}$.

Definition 1.1. *[1] For a nonempty set M, a dislocated quasi-b-metric is a function $\rho : M \times M \to \mathbb{R}_0^+$ such that for all $u, v, w \in M$ and a fixed constant $s \geq 1$:*

(ρ_1) if $\rho(u,v) = 0$ then $u = v$.

(ρ_2) $\rho(u,v) \leq s[\rho(u,w) + \rho(w,v)]$.

Moreover, the pair (M,ρ) is called dislocated quasi-b-metric space (DqbMS).

Example 1.2. *The function $\rho_b : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined as*

$$\rho_b(x,y) = \max\{x,y\} + x$$

for all $x, y \in \mathbb{R}$ is a dislocated quasi-b-metric on $\mathbb{R}$.

Example 1.3. *The function $\rho_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined as*

$$\rho_b(x,y) = |x-y|^2 + 2|y|$$

for all $x, y \in \mathbb{R}$ is a dislocated quasi-b-metric on $\mathbb{R}$.

Example 1.4. *The function $\rho_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ defined as*

$$\rho_b(x,y) = |x-y| + 3|x|^2 + 2|y|^2$$

for all $x, y \in \mathbb{R}$ is a dislocated quasi-b-metric on $\mathbb{R}$.

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The basic topological notions and analog of standard topological tools have been derived in straight way, see e.g. [1]. Each dislocated quasi-b-metric ρ on a non-empty set M have a topology τ_ρ that was generated by the family of open balls

$$O_\varepsilon(u) = \{v \in M : |\rho(u,v) - \rho(u,u)| < \varepsilon, \} \text{ for all } u \in M \text{ and } \varepsilon > 0.$$

In the setting of the DqbMS (M,ρ), we say that a sequence $\{u_n\}$ converges to a point $u \in M$ if the following limit exists (and finite):

$$\lim_{n\to\infty} \rho(u_n,u) \underbrace{=}_{(R)} \rho(u,u) \underbrace{=}_{(L)} \lim_{n\to\infty} \rho(u,u_n). \tag{1}$$

If $\rho(u,u) = 0$, then it is called 0-convergence.

Moreover, a sequence $\{u_n\}$ is said to be Cauchy if the following limit

$$\lim_{n\to\infty} \rho(u_m,u_n) \underbrace{=}_{(R)} L_1 \underbrace{=}_{(L)} \lim_{n\to\infty} \rho(u_n,u_m), \tag{2}$$

exists and is finite, where $m \geq n$. Furthermore, if $L_1 = 0$ in (2), then we say that $\{u_n\}$ is a $0-$Cauchy sequence.

As it is expected, a pair (M,ρ) is called complete DqbMS if for each Cauchy sequence $\{u_n\}$, there is some $u \in M$ such that

$$L_2 = \lim_{n\to\infty} \rho(u_n,u) = \lim_{n\to\infty} \rho(u_m,u_n) \underbrace{=}_{(R)} \rho(u,u) \underbrace{=}_{(L)} \lim_{n\to\infty} \rho(u_n,u_m) = \lim_{n\to\infty} \rho(u,u_n). \tag{3}$$

Analogously, a pair (M,ρ) is called $0-$complete DqbMS if for each $0-$Cauchy sequence $\{u_n\}$, converges to a point $u \in M$ such that $L_2 = 0$ in (3).

If only the equality (R) holds in (1), (2), respectively, we say that $\{u_n\}$ has a *"left limit"*, is *"right Cauchy,"* respectively. Moreover, if (R) holds in (3), we say that X is right-complete. Analogously, the notions of left limit, left Cauchy and left complete can be defined. In these equations (1), (2),(3), if both (R) and (L) holds, then we can the corresponding standard definition.

Let (M,ρ) and (K,σ) be DqbMS's. A mapping $T : M \to K$ is called continuous if

$$\lim_{n\to\infty} \rho(u_n,u) = \rho(u,u) = \lim_{n,m\to\infty} \rho(u_n,u_m),$$

then we have

$$\lim_{n\to\infty} \sigma(Tu_n,Tu) = \sigma(Tu,Tu) = \lim_{n,m\to\infty} \sigma(Tu_n,Tu_m).$$

The proof of the following lemma is straightforward and hence we omit it.

Lemma 1.5. *(cf. [5]) For a DqbMS (M,ρ), we have the following observations:*

(A) If $\rho(u,v) = 0 = \rho(v,u)$ then $\rho(u,u) = \rho(v,v) = 0$.

(B) For a sequence $\{u_n\}$ with $\lim_{n\to\infty} \rho(u_n,u_{n+1}) = 0 = \lim_{n\to\infty} \rho(u_{n+1},u_n)$, we have

$$\lim_{n\to\infty} \rho(u_n,u_n) = \lim_{n\to\infty} \rho(u_{n+1},u_{n+1}) = 0.$$

(C) If $u \neq v$ then $\rho(u,v) > 0$ and $\rho(v,u) > 0$.

(D) Let V be a closed subset of M and $\{u_n\}$ be a sequence in V. If $u_n \to u$ as $n \to \infty$, then $u \in V$.

(E) For a sequence $\{u_n\}$ in M such that $u_n \to u$ as $n \to \infty$ with $\rho(u,u) = 0$, then $\lim_{n\to\infty} \rho(u_n,v) = \rho(u,v)$ for all $v \in M$.

Very recently, Popescu [6] propose the notion of triangular α-orbital admissible:

Definition 1.6. *[6] Suppose that $\alpha : M \times M \to \mathbb{R}_0^+$ is a function. A self-mapping $T : M \to M$ is said to be an α-orbital admissible if*

(T3) $\alpha(u, Tu) \geq 1 \Rightarrow \alpha(Tu, T^2u) \geq 1$

Furthermore, T is called triangular α-orbital admissible if T is α-orbital admissible and

(T4) $\alpha(u, v) \geq 1$ and $\alpha(v, Tv) \geq 1 \Rightarrow \alpha(u, Tv) \geq 1$

1.1. (b)-comparison functions.

For a fixed real number $s \geq 1$, let Ψ_b be all functions $\varphi_b : [0, +\infty) \to [0, +\infty)$ satisfying the conditions

(b_1) φ_b is increasing,
(b_2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} \nu_k$ such that $s^{k+1}\varphi_b^{k+1}(t) \leq as^k\varphi_b^k(t) + \nu_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

Any $\varphi_b \in \Psi_b$ is called (b)-comparison function [12]. For $s = 1$, in the definition above, φ_b is known as (c)-comparison functions.

We will need the following essential properties in our further discussion.

Lemma 1.7. *([11–13]) For a comparison function $\varphi_b : [0, +\infty) \to [0, +\infty)$ the following hold:*

(1) *the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in [0, +\infty)$;*
(2) *the function $b_s : [0, +\infty) \to [0, +\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t)$, $t \in [0, \infty)$ is increasing and continuous at 0.*
(3) *each iterate φ_b^k of φ $k \geq 1$, is also a comparison function;*
(4) *φ_b is continuous at 0;*
(5) *$\varphi_b(t) < t$, for any $t > 0$.*

For more details on comparison functions and examples we refer the reader to [11],[12].

2. Main Results

2.1. (α, ψ)-Meir-Keeler type contraction

We introduce the following notion which is an improved version of Meir-Keeler contraction.

Definition 2.1. *Let (M, ρ) be a DqbMS. We say that $T : M \to M$ is an (α, ψ)-*Meir-Keeler type contraction *if there exist two functions $\psi \in \Psi$ and $\alpha : M \times M \to \mathbb{R}_0^+$ satisfying the following condition:*
For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \psi(\rho(u, v)) < \varepsilon + \delta \quad \text{implies} \quad \alpha(u, v)\rho(Tu, Tv) < \varepsilon.$$

Notice that for an (α, ψ)-*Meir-Keeler type contraction* $T : M \to M$, we have

$$\alpha(u, v)\rho(Tu, Tv) \leq \psi(\rho(u, v)), \text{ for any } u, v \in M.$$

In what follows we shall state and prove the first main result of this section.

Theorem 2.2. *Suppose that (M, ρ) is a complete DqbMS and a self-mapping $T : M \to M$ is a (α, ψ)-Meir-Keeler type contraction. Assume also that*

(i) *T is α-orbital admissible;*
(ii) *there exists $u_0 \in M$ such that $\alpha(u_0, Tu_0) \geq 1$; and $\alpha(Tu_0, u_0) \geq 1$;*
(iii) *T is continuous.*

Then, there exists $u \in M$ such that $Tu = u$.

Proof. Due to assumption (*ii*) of theorem, there exists $u_0 \in M$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$. We shall setup an iterative sequence $\{u_n\}$ in M as

$$u_{n+1} = Tu_n, \text{ for all } n \in \mathbb{N}_0. \tag{4}$$

Notice that in case of existing a $k \in \mathbb{N}$ with $u_k = u_{k+1}$, the proof is over since $u = u_k = u_{k+1} = Tu_k = Tu$.

Consequently, we assume that $u_n \neq u_{n+1}$, for all n. Due to Lemma 1.5 part (C), for $u_n \neq u_{n+1}$, we have

$$\rho(u_n, u_{n+1}) > 0 \text{ and } \rho(u_{n+1}, u_n) > 0, \text{ for all } n \in \mathbb{N}_0. \tag{5}$$

Since T is α-orbital admissible, again by (*ii*), we get

$$\alpha(u_0, u_1) = \alpha(u_0, Tu_0) \geq 1 \Longrightarrow \alpha(Tu_0, Tu_1) = \alpha(u_1, u_2) \geq 1,$$

and

$$\alpha(u_1, u_0) = \alpha(Tu_0, u_0) \geq 1 \Longrightarrow \alpha(Tu_1, Tu_0) = \alpha(u_2, u_1) \geq 1.$$

Iteratively, we find that

$$\alpha(u_n, u_{n+1}) \geq 1 \text{ and } \alpha(u_{n+1}, u_n) \geq 1 \text{ for all } n \in \mathbb{N}_0. \tag{6}$$

Since T is an (α, ψ)-Meir-Keeler type contraction, we find

$$\begin{aligned} \rho(u_n, u_{n+1}) &= \rho(Tu_{n-1}, Tu_n) \\ &\leq \alpha(u_{n-1}, u_n)\rho(Tu_{n-1}, Tu_n) \\ &\leq \psi(\rho(u_{n-1}, u_n)), \end{aligned} \tag{7}$$

for each $n \in \mathbb{N}$ by taking (4) and (6) into account.

Combining (5) and the property of ψ, we derive that

$$\rho(u_n, u_{n+1}) \leq \psi(\rho(u_{n-1}, u_n)) < \rho(u_{n-1}, u_n), \tag{8}$$

for each $n \in \mathbb{N}$. Furthermore, by the repeating the same argument, we find that

$$\rho(u_{n+1}, u_n) \leq \psi(\rho(u_n, u_{n-1})) < \rho(u_n, u_{n-1}), \tag{9}$$

for each $n \in \mathbb{N}$.

Thus, we conclude that $\{\rho(u_{n-1}, u_n)\}$ is a non-increasing and bounded sequence. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n\to\infty} \rho(u_{n-1}, u_n) = t.$$

As a next step, we shall prove that $t = 0$. Suppose, on the contrary, that $t > 0$. Since T is an (α, ψ)-Meir-Keeler type contraction, for $\varepsilon = \psi(t) > 0$, there exists $\delta > 0$ and a natural number m such that

$$\varepsilon \leq \psi(\rho(u_{m-1}, u_m)) < \varepsilon + \delta \quad \text{implies} \quad \alpha(u_{m-1}, u_m)\rho(Tu_{m-1}, Tu_m) < \varepsilon.$$

On account of (6), we get that

$$\rho(u_m, u_{m+1}) = \rho(Tu_{m-1}, Tu_m) \leq \alpha(u_{m-1}, u_m)\rho(Tu_{m-1}, Tu_m) < \varepsilon = \psi(t) < t,$$

a contradiction, since $t = \inf\{\rho(u_n, u_{n+1}) : n \in \mathbb{N}\}$.

Eventually, we have

$$\lim_{n\to\infty} \rho(u_{n-1}, u_n) = 0. \tag{10}$$

In the same way, we conclude also that

$$\lim_{n\to\infty}\rho(u_n,u_{n-1})=0.$$

In what follows we shall prove that $\{u_n\}$ is a left-0-Cauchy sequence. That is,

$$\lim_{n\to\infty}\rho(u_n,u_{n+k})=0,\ \text{for all } k\in\mathbb{N}. \tag{11}$$

First, we observe from (8) that

$$\rho(u_n,u_{n+1})\le\psi(\rho(u_{n-1},u_n)) \text{ for all } n\in\mathbb{N}. \tag{12}$$

Since, ψ is nondecreasing, by iteration, we conclude that

$$\rho(u_n,u_{n+1})\le\psi^n(\rho(u_0,u_1)) \text{ for all } n\in\mathbb{N}. \tag{13}$$

Now, by using (ρ_2),(4),(6) and (13), we have the following

$$\begin{aligned}\rho(u_n,u_m) &\le s\rho(u_n,u_{n+1})+...+s^{m-n}\rho(u_{m-1},u_m)\\ &\le \textstyle\sum_{p=n}^{m-1} s^{p+1-n}\psi^p(\rho(u_0,u_1))\\ &= s\textstyle\sum_{p=n}^{m-1} s^p\psi^p(\rho(u_0,u_1))\end{aligned} \tag{14}$$

By Lemma 1.7, we know that the series $\sum_{k=0}^{\infty}s^k\varphi^k(t)$ converges for any $t\in[0,+\infty)$ Hence, letting $n\to\infty$ in the inequality above, we conclude that $\{u_n\}$ is left-Cauchy. Analogously, we derive that $\{u_n\}$ is right-0-Cauchy. Herewith, the iterative sequence $\{u_n\}$ is 0-Cauchy.

Since (M,ρ) is a 0-complete DqbMS, then there exists $u\in M$ such that

$$0=\lim_{n\to\infty}\rho(u_n,u)=\lim_{n\to\infty}\rho(u_m,u_n)=\rho(u,u)=\lim_{n\to\infty}\rho(u_n,u_m)=\lim_{n\to\infty}\rho(u,u_n) \tag{15}$$

As T is continuous, then we deduce that

$$\lim_{n\to\infty}\rho(u_{n+1},Tu)=\lim_{n\to\infty}\rho(Tu_n,Tu)=\rho(Tu,Tu)=\lim_{n\to\infty}\rho(Tu_n,Tu_{n+k})=0. \tag{16}$$

Since $u_{n+1}\to u$ as $n\to\infty$ and $\rho(u,u)=0$, then by using Lemma 1.5, we get

$$\lim_{n\to\infty}\rho(u_{n+1},Tu)=\rho(u,Tu). \tag{17}$$

From (16) and (17), we derive that $\rho(u,Tu)=0$. By (ρ_1), we conclude that $Tu=u$. □

Theorem 2.3. *Suppose that (M,ρ) is a complete DqbMS and a self-mapping $T:M\to M$ is a (α,ψ)-Meir-Keeler type contraction. Assume also that*

- (*i*) *T is α-orbital admissible;*
- (*ii*) *there exists $u_0\in M$ such that $\alpha(u_0,Tu_0)\ge1$ and $\alpha(Tu_0,u_0)\ge1$;*
- (*iii*) *if $\{u_n\}$ is a sequence in M such that $\alpha(u_n,u_{n+1})\ge1$ for all n and $u_n\to u\in M$ as $n\to\infty$, then $\alpha(u_n,u)\ge1$ for all n.*

Then, there exists $u\in M$ such that $Tu=u$.

Proof. By repeating the lines in the proof of Theorem 2.2, we conclude that there exists a Cauchy sequence $\{u_n\}$. Since M is complete, it converges to some $u\in M$. On the other hand, from (6) and (*iii*), we have

$$\alpha(u_n,u)\ge1,\ \text{for all } n. \tag{18}$$

By using (ρ_2) and (18) with the assumption of the Theorem that T is a (α,ψ)-Meir-Keeler type contraction, we obtain

$$\begin{aligned}\rho(Tu,u) &\leq s\rho(Tu,Tu_n)+s\rho(u_{n+1},u)\\ &\leq s\alpha(u_n,u)\rho(Tu_n,Tu)+s\rho(u_{n+1},u)\\ &\leq s\psi(\rho(u_n,u))+s\rho(u_{n+1},u).\end{aligned}$$

Since ψ is continuous at $t=0$, by letting $n\to\infty$ in the inequality above, we find

$$\rho(Tu,u)=0.$$

It yields that $Tu=u$ due to (ρ_1). □

Example 2.4. *Let $M=[0,\infty)$ endowed with $\rho(u,v)=|x-y|+|x|$ for all $u,v\in[0,\infty)$. It is clear that (M,ρ) is a complete DqbMS. Define $T:M\to M$ and $\alpha:M\times M\to\mathbb{R}_0^+$ by:*

$$Tu=\frac{u^2}{3},\text{ and }\alpha(u,v)=\begin{cases}1 & \text{if } u=v=0;\\ 0 & \text{otherwise .}\end{cases}$$

We can prove easily T is an(α,ψ)-Meir-Keeler type contraction and it is an α-orbital admissible. Moreover, there exists $u_0\in M$ such that $\alpha(u_0,Tu_0)\geq 1$. In fact, for $u_0=0$, we have

$$\alpha(0,T0)=1.$$

Now, we show that T is a continuous. Let $\lim_{n\to\infty}u_n=u$ in the context of DqbMS (M,ρ), that is,

$$\lim_{n\to\infty}\rho(u_n,u)=\rho(u,u)=\lim_{n,m\to\infty}\rho(u_n,u_m).$$

We shall show that T is continuous. Indeed,

$$\begin{aligned}\lim_{n\to\infty}\rho(Tu_n,Tu) &= \lim_{n\to\infty}|Tu_n-Tu|+|Tu_n|\\ &=\rho(Tu,Tu)=|Tu-Tu|+|Tu|=Tu=\tfrac{u^2}{3}\\ &=\lim_{n,m\to\infty}\rho(Tu_n,Tu_m)=\lim_{n,m\to\infty}|Tu_n-Tu_m|+|Tu_n|\\ &=\lim_{n,m\to\infty}\left|\tfrac{u_n^2}{3}-\tfrac{u_m^2}{3}\right|+|\tfrac{u_n^2}{3}|.\end{aligned}$$

So all hypotheses of Theorem 2.2 are satisfied. Consequently, T has a fixed point. Notice that $u=0$ is a fixed point of T.

In the following example, a self-mapping T is not continuous.

Example 2.5. *Let $M=[0,\infty)$ endowed with the dislocated metric $\rho(u,v)=\max\{u,v\}+|u|$ for all $u,v\in[0,\infty)$. Define $T:M\to M$ and $\alpha:M\times M\to\mathbb{R}_0^+$ by:*

$$Tu=\begin{cases}\frac{1}{2}u^3-1 & u>1,\\ 0 & 0\leq u\leq 1,\end{cases}\quad\text{and }\alpha(u,v)=\begin{cases}1 & \text{if } u,v\in[0,1],\\ 0 & \text{otherwise .}\end{cases}$$

Clearly T is not continuous at 1 which shows that Theorem 2.2 is not applicable in this case.

We shall prove that a self-mapping T is an $(\alpha-\psi)$-Meir-Keeler type contraction. Let $\varepsilon>0$ be given. Take $\delta>0$ and suppose that $\varepsilon\leq\psi(\rho(u,v))<\varepsilon+\delta$ we want to show that

$$\alpha(u,v)\rho(Tu,Tv)<\varepsilon.$$

Suppose that $\alpha(u,v)=1$, then $u,v\in[0,1]$ and so $Tu=0, Tv=0$. Hence

$$\begin{aligned}\rho(Tu,Tv) &= \rho(0,0)\\ &= \max\{0,0\}+|0|=0\\ &< \varepsilon.\end{aligned}$$

Also, T is an α-orbital-admissible. To see this, let $\alpha(u,v)\geq 1$, then both $u,v\in[0,1]$. Due to definition of T, we have $Tu=0\in[0,1]$ and $Tv=0\in[0,1]$. Thus, we get $\alpha(Tu,Tv)\geq 1$.

Moreover, there exists $u_0\in M$ such that $\alpha(u_0,Tu_0)\geq 1$. Indeed, for $u_0=0$ we have

$$\alpha(0,T0)=\alpha(0,0)=1=\alpha(0,0)=\alpha(T0,0).$$

Finally, let $\{u_n\}$ be a sequence in M such that $\alpha(u_n,u_{n+1})\geq 1$ for all n and $u_n\to u\in M$ as $n\to\infty$. Since $\alpha(u_n,u_{n+1})\geq 1$ for all n, by the definition of α, we have $u_n\in[0,1]$ for all n and $u\in[0,1]$, then $\alpha(u_n,u)=1$.

So, we conclude that all hypotheses of Theorem 2.3 are fulfilled. Hence, we proved that T has a fixed point.

2.2. Generalized (α,ψ)-Meir-Keeler type contraction

Definition 2.6. *Suppose that (M,ρ) is a DqbMS. A self-mapping $T: M\to M$ is said to be a generalized (α,ψ)-Meir-Keeler type contraction if there exist $\psi\in\Psi$ and $\alpha: M\times M\to\mathbb{R}_0^+$ such that for each $\varepsilon>0$, there exists $\delta>0$ such that*

$$\varepsilon\leq\psi(P(u,v))<\varepsilon+\delta \quad \textit{implies} \quad \alpha(u,v)\rho(Tu,Tv)<\varepsilon,$$

where

$$P(u,v)=\max\{\rho(u,v),\rho(u,Tu),\rho(v,Tv)\}.$$

If a self-mapping $T: M\to M$ is a generalized-(α,ψ)-Meir-Keeler type contraction, then we have

$$\alpha(u,v)\rho(Tu,Tv)\leq\psi(P(u,v)), \text{ for any } u,v\in M.$$

The following is the first main result of this section.

Theorem 2.7. *Suppose that (M,ρ) is a complete DqbMS, a self-mapping $T: M\to M$ is a generalized-(α,ψ)-Meir-Keeler type contraction and the following conditions are fulfilled:*

(*i*) *T is triangular α-orbital admissible mapping;*
(*ii*) *there exists $u_0\in M$ such that $\alpha(u_0,Tu_0)\geq 1$ and $\alpha(Tu_0,u_0)\geq 1$;*
(*iii*) *T is continuous.*

Then, T has a fixed point, that is, there exists $u\in M$ such that $Tu=u$.

Proof. Take $u_0\in M$ such that $\alpha(u_0,Tu_0)\geq 1$ and $\alpha(Tu_0,u_0)\geq 1$. As in the proof of Theorem 2.2, we construct an iterative sequence $\{u_n\}$ in M in the following way:

$$u_{n+1}=Tu_n, \text{ for each } n\in\mathbb{N}_0. \tag{19}$$

By the same reason in Theorem 2.2, we assume that $u_n\neq u_{n+1}$, for all n,

$$\rho(u_n,u_{n+1})>0, \text{ and } \rho(u_{n+1},u_n)>0, \text{ for all } n\in\mathbb{N}. \tag{20}$$

By assumption (*i*) of theorem in the mind, we find that

$$\alpha(u_n,u_m)\geq 1, \text{ and } \alpha(u_m,u_n)\geq 1, \text{ for all } m,n\in\mathbb{N} \text{ with } n<m. \tag{21}$$

Step 1: We shall prove that

$$\lim_{n\to\infty}\rho(u_n,u_{n+1})=0.$$

Taking (20) and (21) into account together with the fact that T is a generalized-(α, ψ)-Meir-Keeler type contraction mapping, for each $n \in \mathbb{N}$ we derive

$$\begin{aligned} \rho(u_n, u_{n+1}) &= \rho(Tu_{n-1}, Tu_n) \\ &\leq \alpha(u_{n-1}, u_n)\rho(Tu_{n-1}, Tu_n) \\ &\leq \psi(P(u_{n-1}, u_n)) \\ &< P(u_{n-1}, u_n), \end{aligned} \tag{22}$$

where

$$\begin{aligned} P(u_{n-1}, u_n) &= \max\{\rho(u_{n-1}, u_n), \rho(u_{n-1}, Tu_{n-1}), \rho(u_n, Tu_n)\} \\ &= \max\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1})\}. \end{aligned}$$

Let us analyze these cases. Let $P(u_{n-1}, u_n) = \rho(u_n, u_{n+1})$. Since ψ is nondecreasing, from (22), we derive

$$\rho(u_n, u_{n+1}) \leq \psi(\rho(u_n, u_{n+1})) < \rho(u_n, u_{n+1}), \tag{23}$$

a contradiction. Thus, $P(u_{n-1}, u_n) = \rho(u_{n-1}, u_n)$ and again by (22), we conclude

$$\rho(u_n, u_{n+1}) \leq \psi(\rho(u_{n-1}, u_n)) < \rho(u_{n-1}, u_n), \text{ for all } n \in \mathbb{N}. \tag{24}$$

Consequently, we find that the sequence $\{\rho(u_n, u_{n+1})\}$ is a non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n\to\infty} \rho(u_n, u_{n+1}) = t. \tag{25}$$

Recursively, we derive from (24) that

$$\rho(u_n, u_{n+1}) \leq \psi^n(\rho(u_0, u_1)), \text{ for all } n, \tag{26}$$

by keeping in the mind that ψ is nondecreasing.

On account of (26) and (Ψ_1), we obtain

$$\lim_{n\to\infty} \rho(u_n, u_{n+1}) = 0.$$

Analogously, one can derive that

$$\lim_{n\to\infty} \rho(u_{n+1}, u_n) = 0.$$

Step 2: We shall prove that $\{u_n\}$ is a Cauchy sequence by using the same arguments in the proof of Theorem 2.2

Now, by using (ρ_2),(4),(6) and (13), we have the following

$$\begin{aligned} \rho(u_n, u_m) &\leq s\rho(u_n, u_{n+1}) + ... + s^n\rho(u_{m-1}, u_m) \\ &\leq \sum_{p=n}^{m-1} s^{p+1-n}\psi^p(\rho(u_0, u_1)) \\ &= s\sum_{p=n}^{m-1} s^p\psi^p(\rho(u_0, u_1)) \end{aligned} \tag{27}$$

By Lemma (1.7), we know that the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in [0, +\infty)$ Hence, letting $n \to \infty$ in the inequality above, we conclude that $\{u_n\}$ is left-Cauchy. Analogously, we derive that $\{u_n\}$ is right-0-Cauchy. Herewith, the iterative sequence $\{u_n\}$ is 0-Cauchy.

Since (M, ρ) is a complete DqbMS, then there exists $u \in M$ such that

$$\lim_{n\to\infty} \rho(u_n, u) = \rho(u, u) = \lim_{n,m\to\infty} \rho(u_n, u_m) = 0. \tag{28}$$

On account of the continuity of the self-mapping T, we deduce that

$$\lim_{n\to\infty} \rho(u_{n+1}, Tu) = \lim_{n\to\infty} \rho(Tu_n, Tu) = \rho(Tu, Tu) = \lim_{n,m\to\infty} \rho(Tu_n, Tu_m) = 0. \tag{29}$$

Since $u_{n+1} \to u$ as $n \to \infty$ and $\rho(u, u) = 0$, then by using Lemma 1.5, we get

$$\lim_{n\to\infty} \rho(u_{n+1}, Tu) = \rho(u, Tu). \tag{30}$$

Regarding (29) and (30), we get $\rho(u, Tu) = 0$. Thus, by (ρ_1), we conclude that $Tu = u$. □

Theorem 2.8. *Suppose that (M, ρ) is a complete DqbMS, a self-mapping $T : M \to M$ is a generalized-(α, ψ)-Meir-Keeler type contraction, where $\alpha \in \psi \in$ with $\psi(t) < \frac{t}{s}$ for a constant $s \geq 1$.*
and the following conditions are fulfilled:

(i) T is triangular α-orbital admissible mapping;
(ii) there exists $u_0 \in M$ such that $\alpha(u_0, Tu_0) \geq 1$;
(iii) if $\{u_n\}$ is a sequence in M such that $\alpha(u_n, u_{n+1}) \geq 1$ for all n and $u_n \to u \in M$ as $n \to \infty$, then $\alpha(u_n, u) \geq 1$ for all n.

Then, T has a fixed point, that is, there exists $u \in M$ such that $Tu = u$.

Proof. Define an iterative sequence $\{u_n\}$ in M as in Theorem 2.7. Suppose that there is $k_0 \in \mathbb{N}$ such that $u_{k_0} = u_{k_0+1}$, then the proof is completed since $u = u_{k_0} = u_{k_0+1} = Tu_{k_0} = Tu$. So, it is interesting to assume that $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}_0$. Consequently, we have

$$\rho(u_n, u_{n+1}) > 0, \text{ for all } n \in \mathbb{N}_0,$$

From (21), we find that

$$\alpha(u_n, u_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{31}$$

Following the lines for the proofs of Step1 and Step2 in Theorem 2.7, we derive that $\{u_n\}$ is a Cauchy sequence and

$$\lim_{n,m\to\infty} \rho(u_n, u_m) = 0.$$

Since (M, ρ) is a complete DqbMS, then there exists $u \in M$ such that

$$\lim_{n\to\infty} \rho(u_n, u) = \rho(u, u) = \lim_{n,m\to\infty} \rho(u_n, u_m) = 0. \tag{32}$$

We shall prove that $u = Tu$. Suppose, on the contrary, that $\rho(u, Tu) > 0$.
Notice from (31), (32) and (iii) that

$$\alpha(u_n, u) \geq 1, \text{ for all } n. \tag{33}$$

By using (ρ_2) and (33) together with the assumption of the Theorem that T is a generalized-$(\alpha - \psi)$-Meir-Keeler type contraction, we obtain

$$\begin{aligned} \rho(Tu, u) &\leq s\rho(Tu, Tu_n) + s\rho(u_{n+1}, u) \\ &\leq s\alpha(u_n, u)\rho(Tu_n, Tu) + s\rho(u_{n+1}, u) \\ &\leq s\psi(P(u_n, u)) + s\rho(u_{n+1}, u), \end{aligned} \tag{34}$$

where

$$P(u_n, u) = \max\{\rho(u_n, u), \rho(u_n, u_{n+1}), \rho(u, Tu)\}.$$

Notice that as $\rho(u, Tu) > 0$, then we must have $P(u_n, u) > 0$.

Suppose that $P(u_n, u) = \rho(u_n, u)$, then from (34) we get

$$\begin{aligned} \rho(Tu, u) &\leq s\psi(\rho(u_n, u)) + s\rho(u_{n+1}, u), \\ &< s\rho(u_n, u) + s\rho(u_{n+1}, u). \end{aligned} \tag{35}$$

Taking $n \to \infty$ in (35), we have

$$\rho(Tu, u) < s\rho(u, u) + s\rho(u, u) = 0,$$

which is a contradiction.

Now, we suppose that $P(u_n, u) = \rho(u_n, u_{n+1})$, then by (34) we find that

$$\begin{aligned} \rho(Tu, u) &\leq s\psi(\rho(u_n, u_{n+1})) + s\rho(u_{n+1}, u), \\ &< \rho(u_n, u_{n+1}) + s\rho(u_{n+1}, u). \end{aligned}$$

Letting $n \to \infty$, this implies that

$$\rho(Tu, u) < \lim_{n\to\infty}[\rho(u_n, u_{n+1}) + s\rho(u_{n+1}, u)] = 0,$$

which is again a contradiction.

Finally, we suppose that $P(u_n, u) = \rho(u, Tu)$, then again from (34), we have

$$\rho(Tu, u) \leq s\psi(\rho(u, Tu)) + s\rho(u_{n+1}, u),$$

Letting $n \to \infty$, in the above inequality, we get

$$\begin{aligned} \rho(Tu, u) &\leq s\psi(\rho(u, Tu)) + s\rho(u, u), \\ &< \rho(u, Tu) + s\rho(u, u) = s\rho(Tu, u), \end{aligned} \tag{36}$$

also we have a contradiction. Thus we have $\rho(Tu, u) = 0$ and by (ρ_1), we have $Tu = u$. □

2.3. The uniqueness of the fixed point

We propose the following conditions for the uniqueness of the fixed points of the mappings discussed in sections 2.1 and 2.2. Let $Fix(T)$ denotes the set of fixed points of the mapping T.

We, first, recollect the following interesting condition for uniqueness of a fixed point of an $(\alpha - \psi)$-Meir-Keeler type contraction.

(H) For all $u, v \in Fix(T)$, then there exists $w \in M$ such that $\alpha(u, w) \geq 1$ and $\alpha(w, v) \geq 1$, where

Theorem 2.9. *Putting condition (H) to the statements of Theorem 2.2 (respectively, Theorem 2.3), we obtain that u is the unique fixed point of T.*

Proof. Let u and v be two distinct fixed points of T. From(H), there exists $w \in M$ such that

$$\alpha(u, w) \geq 1 \text{ and } \alpha(w, v) \geq 1.$$

Due to the fact that T is α-orbital admissible, we have

$$\alpha(u, Tw) \geq 1 \text{ and } \alpha(Tw, v) \geq 1.$$

Inductively, we obtain

$$\alpha(u, T^n w) \geq 1 \text{ and } \alpha(v, T^n w) \geq 1, \forall n \in \mathbb{N}.$$

From the above relation and since T is an (α, ψ)-MKC mapping, we have

$$\begin{aligned} \rho(u, T^n w) &\leq \alpha(u, T^{n-1} w)\rho(Tu, T^n w) \\ &\leq \psi(\rho(u, T^{n-1} w)). \end{aligned}$$

Iteratively, we get

$$\rho(u, T^n w) \leq \psi^n(\rho(u, w)).$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \rho(u, T^n w) = 0.$$

Similarly, we can prove that

$$\lim_{n\to\infty} \rho(v, T^n w) = 0.$$

Using (ρ_2), we have

$$\rho(u, v) \leq s[\rho(u, T^n w) + \rho(T^n w, v)].$$

Taking $n \to \infty$, we find that

$$\rho(u, v) = 0,$$

and so, by (ρ_1), $u = v$. □

The following is an alternative uniqueness condition: (U) For all $u, v \in Fix(T)$, then $\alpha(u, v) \geq 1$.

Theorem 2.10. *Putting condition (U) to the statements of Theorem 2.2 (resp. Theorem 2.3), we find that u is the unique fixed point of T.*

Proof. Let u, v be two distinct fixed point of T. Then by Lemma 1.5 part C, we have

$$\rho(u, v) > 0.$$

Due the property of ψ (Ψ_2), we get

$$\psi(\rho(u, v)) > 0.$$

Let $\varepsilon = \psi(\rho(u, v))$, then for any $\delta > 0$, we find that

$$\varepsilon = \psi(\rho(u, v)) < \varepsilon + \delta.$$

Regrading (U) and the assumption of the Theorem that T is an (α, ψ)-Meir-Keeler type contraction, we obtain

$$\begin{aligned} \rho(u, v) &\leq \alpha(u, v)\rho(Tu, Tv) \\ &< \psi(\rho(u, v)) \\ &< \rho(u, v), \end{aligned}$$

which is a contraction. Thus, $u = v$. □

In what follows, we propose the conditions for the uniqueness of a fixed point of a generalized $(\alpha-\psi)$-Meir-Keeler type contraction:
(H1) For all $u, v \in Fix(T)$, then there exists $w \in M$ such that $\alpha(u,w) \geq 1$, $\alpha(v,w) \geq 1$ and $\alpha(w,Tw) \geq 1$.
(H2) Let $u, v \in Fix(T)$. If there exists a sequence $\{w_n\}$ in M such that $\alpha(u,w_n) \geq 1$, $\alpha(v,w_n) \geq 1$ and $\alpha(w_n,w_{n+1}) \geq 1$, then

$$\rho(w_n,w_{n+1}) \leq inf\{\rho(u,w_n), \rho(v,w_n)\}.$$

(H3) For any $u \in Fix(T)$, then $\alpha(u,u) \geq 1$.

Theorem 2.11. *Putting conditions (H1), (H2)and (H3) to the statements of Theorem 2.7 (respectively, Theorem 2.8), we have that u is the unique fixed point of T.*

Proof. Let $u, v \in Fix(T)$ with $u \neq v$. By(H1), there exists $w \in M$ such that

$$\alpha(u,w) \geq 1, \alpha(\nu,w) \geq 1 \text{ and } \alpha(w,Tw) \geq 1.$$

On account of triangular α-orbital admissible property of T, we have

$$\alpha(Tw,T^2w) \geq 1 \text{ and hence } \alpha(T^{n-1}w,T^n w) \geq 1, \forall n \in \mathbb{N}..$$

By $(T4)$ $\alpha(u,w) \geq 1$ and $\alpha(w,Tw) \geq 1$ yields that

$$\alpha(u,Tz) \geq 1.$$

since $\alpha(u,Tw) \geq 1$ and $\alpha(Tw,T^2w) \geq 1$, again, by $(T4)$, we derive

$$\alpha(u,T^2w) \geq 1.$$

Recursively, we get

$$\alpha(u,T^n w) \geq 1, \forall n \in \mathbb{N}. \tag{37}$$

Analogously, one can get that

$$\alpha(\nu,T^n w) \geq 1, \forall n \in \mathbb{N}. \tag{38}$$

Set-up a new iterative sequence $\{w_n\}$ by $w_{n+1} = Tw_n$, for all $n \geq 0$ and $w_0 = w$.
Step1: We show that

$$\lim_{n\to\infty} \rho(u,w_n) = 0.$$

By (37) and the statement of the theorem that T is generalized-(α,ψ)-Meir-Keeler type contraction mapping, we have

$$\begin{aligned}\rho(u,w_{n+1}) &\leq \alpha(u,w_n)\rho(Tu,Tw_n)\\ &\leq \psi(P(u,w_n)).\end{aligned}$$

If $\psi(P(u,w_n)) = 0$, then

$$\lim_{n\to\infty} \rho(u,w_{n+1}) = 0.$$

Consequently, suppose that $\psi(P(u,w_n)) > 0$. Then, we have $P(u,w_n) > 0$. Since T is a generalized-(α,ψ)-Meir-Keeler type contraction mapping, we find

$$\begin{aligned}\rho(u,w_{n+1}) &\leq \alpha(u,w_n)\rho(Tu,Tw_n)\\ &\leq \psi(P(u,w_n)),\\ &< P(u,w_n).\end{aligned}$$

where

$$P(w_n, u) = \max\{\rho(w_n, u), \rho(w_n, w_{n+1}), \rho(u, Tu)\}.$$

Regarding (H2) and (ρ_2), we have

$$P(w_n, u) = \rho(w_n, u).$$

Thus, we have

$$\rho(u, w_{n+1}) < \rho(u, w_n).$$

Letting $n \to \infty$ in the inequality above , we obtain

$$\lim_{n\to\infty} \rho(u, w_{n+1}) < \lim_{n\to\infty} \rho(u, w_n),$$

which is a contradiction. Then

$$P(w_n, u) = \rho(w_n, u) = 0.$$

Hence, we get

$$\lim_{n\to\infty} \rho(u, w_n) = 0.$$

Step2: We shall prove that

$$\lim_{n\to\infty} \rho(\nu, w_n) = 0.$$

By (H3) and the assumption of the theorem that T is a generalized-(α, ψ)-Meir-Keeler type contraction mapping, we find that

$$\begin{aligned} \rho(\nu, \nu) &\leq \alpha(\nu, \nu)\rho(Tv, T\nu) \\ &\leq \psi(P(\nu, \nu)) \\ &= \psi(\rho(\nu, \nu)). \end{aligned}$$

Suppose, on the contrary, $\rho(\nu, \nu) > 0$. Then, from the above inequality, we obtain

$$\rho(\nu, \nu) \leq \psi(\rho(\nu, \nu)) < \rho(\nu, \nu),$$

which is a contradiction. Thus, $\rho(\nu, \nu) = \rho(\nu, T\nu) = 0$.

In analogous way of Step1, we can complete the proof of

$$\lim_{n\to\infty} \rho(\nu, w_n) = 0.$$

By (ρ_2), we have

$$\begin{aligned} \rho(u, \nu) &\leq \rho(u, T^{n-1}w) + \rho(T^{n-1}w, \nu) \\ &= \rho(u, w_n) + \rho(w_n, \nu). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we find that

$$\rho(u, \nu) = 0,$$

by (ρ_1), we have $u = \nu$. □

3. Consequences

If set $\alpha(u, v) = 1$ for all u, v in Theorem 2.2, we get the following result:

Theorem 3.1. *Suppose that (M, ρ) is a complete DqbMS and a self-mapping $T : M \to M$ is a (α, ψ)-Meir-Keeler type contraction. Then, there exists $u \in M$ such that $Tu = u$.*

Notice that (α, ψ)-Meir-Keeler type contraction $T : M \to M$ is non-expansive, $\rho(Tu, Tv) \leq \psi(\rho(u, v)) \leq \rho(u, v)$ and hence, it is continuous.

If set $\alpha(u, v) = 1$ for all u, v in Theorem 2.7 we find the following consequence:

Theorem 3.2. *Suppose that (M, ρ) is a complete DqbMS, a self-mapping $T : M \to M$ is a generalized-(α, ψ)-Meir-Keeler type contraction. If T is continuous T has a fixed point, that is, there exists $u \in M$ such that $Tu = u$.*

Conclusion

We first note that by adding a symmetry condition, "$\rho(x, y) = \rho(y, x)$" to the assumptions of dislocated quasi-b-metric, we obtain dislocated b-metric. Further, dislocated b-metric is both dislocated metric and b-metric (and hence standard metric). Thus, all result can be formulated in the setting of the mentioned abstract spaces.

It is also possible to list several existing results as a consequence of our main results by choosing both the auxiliary functions α and ψ in a proper like [2–4, 7, 8] and so on.

References

[1] C. Zhu, C. Chen, X. Zhang, Some results in quasi-b-metric-like spaces, Journal of Inequalities and Applications, (2014), 2014:437

[2] N. Bilgili, E. Karapınar, A note on "common fixed points for (ψ, α, β)-weakly contractive mappings in generalized metric spaces", Fixed Point Theory Appl. 2013, 2013:287.

[3] A. Latif, M. E. Gordji, E. Karapınar, W. Sintunavarat, Fixed point results for generalized (α, ψ)-Meir-Keeler contractive mappings and applications, J. Ineq. Appl. 2014, 2014:68.

[4] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. 75 (2012) 2154–2165.

[5] E. Karapınar, P.Salimi, Dislocated Metric Space to Metric Spaces with some fixed point theorems, Fixed Point Theory and Applications, (2013), 2013:222.

[6] O. Popescu, Some new fixed point theorems for α-Geraghty-contraction type maps in metric spaces, Fixed Point Theory Appl. 2014, 2014:190

[7] E. Karapınar, B.Samet, Generalized $\alpha - \psi$-Contractive Type Mappings and Related Fixed Point Theorems with Applications Abstract and Applied Analysis Volume 2012, Article ID 793486, 17 pages

[8] H. Aydi, E. Karapınar, B.Samet, Fixed points for generalized $(\alpha - \psi)$-contractions on generalized metric spaces, Journal of Inequalities and Applications 2014, 2014:229

[9] M. U. Ali, T. Kamran, E. Karapinar, On (α, ψ, u)-contractive multi-valued mappings, Fixed Point Theory Appl., 2014, 2014:7 doi:10.1186/1687-1812-2014-7.

[10] I. A. Rus, *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, 10(2009), No. 2, 305-320.

[11] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, Romania, 2001.

[12] V. Berinde, *Contracţii Generalizate şi Aplicaţii*, vol. 2, Editura Cub Press, Baia Mare, Romania, 1997.

[13] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in b-metric spaces, Cent. Eur. J. Math., 8(2) (2010), 367-377.