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On Certain Operator Method for Solving Differential Equations

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Abstract. In this paper we study some properties of generalized-homogeneous operators. These properties are applied to find eigen functions and associated functions of some classes of differential operators, as well as to construct solutions of differential equations of fractional order.

1. Introduction and Problem Statement

Hereinafter, we need the notion of normalized systems of functions, considered in [6]. Now we give the definition. Let Ω be some domain in the space $R^n, n \geq 1$, X be a linear space of functions defined on the domain Ω. We denote elements of the space X by $f(x)$ and let $N_0 = N \cup \{0\}, N$ is the set of natural numbers. Assume, that the linear operators L_1 and L_2, mapping X into X, are given.

Definition 1.1. *A system of functions $\{f_k(x) : k \in N_0\}$ from X is called $f-$ normalized with respect to (L_1, L_2) in Ω with the basis $f_0(x)$, if everywhere of this domain the following equalities hold:*

$$L_1 f_0(x) = f(x),\ L_1 f_k(x) = L_2 f_{k-1}(x),\ k \in N_0, x \in \Omega.$$

If $L_2 = I$, where I is the unit operator, then in this case $f-$ normalized system of functions with respect to (L_1, I) has the property:

$$L_1 f_0(x) = f(x),\ L_1 f_k(x) = f_{k-1}(x),\ k \in N_0, x \in \Omega. \tag{1}$$

If in (1) , $f(x) = 0$, then in this case the system of functions $f_k(x) : k \in N_0$ is called simple normalized with respect to the operator L_1 in Ω.

Let $L = \frac{d}{dt}$ be a differentiation operator of the first order. Then the system of functions $f_k(t) = \frac{t^{k+s}}{(k+s)!}$, $k \in N_0, s \geq 1$ are $\frac{t^{s-1}}{(s-1)!}-$ normalized with respect to the operator $\frac{d}{dt}$ in the domain $\Omega = R$. If $s = 0$, then the system $f_k(t) = \frac{t^k}{k!}, k \in N_0$ will be simple normalized. Further, in [1],[2],[4] properties of generalized

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exponential functions associated with the Laguerre type derivatives were introduced and investigated. In particular, in [1] the properties of generalized exponential functions of the following form

$$\sum_{i=0}^{\infty} \frac{t^k}{(k!)^{n+1}}, n = 0, 1, ... \tag{2}$$

were studied. It was proved, that the functions of the form (2) are eigen functions of the operator $D_{nL} = Dt...DtDtD$, where $D = \frac{d}{dt}$, t is $n-$times multiplied, and the operator D is $n + 1-$ times used. In this paper we study properties of generalized - homogeneous operators of β order. Using the normalized systems for these operators, the results of [1],[2],[4] are generalized to the general class of differential operators. We prove that these solutions are eigen-functions and associated functions of the operator. Moreover, we build normalized systems for differential operators of fractional order, and discuss application of these systems to construct solutions of differential equations of fractional order.

2. Construction of Normalized Systems for Generalized Homogeneous Operators and their Applications

In this section we give a technique for constructing normalized systems for certain classes of operators in one dimensional case.

Definition 2.1. *Operator D_β is called generalized - homogeneous of the β order with respect to the variable t, if*

$$D_\beta t^\mu = C_{\beta,\mu} t^{\mu-\beta}, t \geq 0, \tag{3}$$

where $0 < \beta \leq \mu$ is a real number, $C_{\beta,\mu}$ is a constant.

Let D_β be a generalized homogeneous operator of the β order, and let for some $s = 0, 1, ...,$ the equality $D_\beta t^s = 0$ holds. We consider a monomial $t^{\beta k+s}, k = 0, 1, 2, ...$ Due to (3), we have

$$D_\beta t^{\beta k+s} = C_{\beta,k,s} t^{\beta k+s-\beta}. \tag{4}$$

Multiplying both sides of the equality (4) to the monomial $t^{-\beta k-s+\beta}$, we get

$$C_{\beta,k,s} = t^{-\beta k-s+\beta} D_\beta t^{\beta k+s}.$$

Let $s = 0, 1, ...$. Consider the coefficients:

$$C(\beta, s, i) = \prod_{k=1}^{i} C_{\beta,k,s} \equiv \prod_{k=1}^{i} \left(t^{-\beta k-s+\beta} D_\beta t^{\beta k+s}\right), i \geq 1, C(\beta, s, 0) = 1.$$

It is easy to show, that for the coefficients $C(\beta, s, i)$ the following equality holds:

$$\frac{1}{C(\beta, s, i)} = \frac{\left(t^{-\beta k-s} D_\beta t^{\beta k+\beta+s}\right)}{C(\beta, s, i+1)}. \tag{5}$$

Consider the system of functions:

$$f_{s,i}(t) = \frac{t^{\beta i+s}}{C(\beta, s, i)}. \tag{6}$$

Lemma 2.2. *Let the operator D_β be generalized homogeneous of β order with respect to the variable t, and for some $s = 0, 1, ...$ the equality $D_\beta t^s = 0$ holds. Then the system of functions (6) is $0-$ normalized with respect to the operator D_β.*

Proof. By the definition of the operator D_β we have:

$$D_\beta t^\mu = t^{-\mu+\beta} D_\beta t^\mu t^{\mu-\beta}.$$

Then $D_\beta f_{s,0}(t) = 0$, and for all $s = 0, 1, ...,$ and $i \geq 1$ the following equality holds:

$$D_\beta f_{s,i}(t) = \frac{D_\beta t^{\beta i+s}}{C(\beta, s, i)} = \frac{\left(t^{-\beta i+\beta-s} D_\beta t^{\beta i+s}\right)}{C(\beta, s, i)} t^{\beta i+s-\beta}.$$

Further, due to (5), we get:

$$\frac{\left(t^{-\beta i+\beta-s} D_\beta t^{\beta i+s}\right)}{C(\beta, s, i)} = \frac{1}{C(\beta, s, i-1)}, i \geq 1.$$

Consequently,

$$D_\beta f_{s,i}(t) = \frac{t^{\beta(i-1)+s}}{C(\beta, s, i-1)} = f_{s,i-1}(t).$$

□

Let the coefficients $C(\beta, s, i)$ be defined by (6). Consider the function:

$$y_s(t) = \sum_{i=0}^{\infty} \lambda^i \frac{t^{\beta i+s}}{C(\beta, s, i)}. \tag{7}$$

The following proposition is true.

Theorem 2.3. *Let the series* (7) *be convergent, and it be possible to use the operator* D_β *termwise to* (7). *If there exist values of the parameter* s *such that*

$$\left(t^{-\beta i-s+\beta} D_\beta t^{\beta i+s}\right)\Big|_{i=0} = 0,$$

then at these values of the parameter s *the functions* $y_s(x)$ *satisfy the following equation:*

$$D_\beta y(t) = \lambda y(t). \tag{8}$$

Proof. Due to Lemma 2.2, the system $f_{s,i}(t) = \frac{t^{\beta i+s}}{C(\beta,s,i)}$ is $0-$ normalized with respect to the operator D_β. Therefore, applying the operator D_β to the functions $y_s(x)$, and considering that the term, when $i = 0$, by the condition of Theorem, equals to zero, we have

$$D_\beta y_s(t) = \sum_{i=1}^{\infty} \lambda^i D_\beta f_{s,i}(t) = \sum_{i=1}^{\infty} \lambda^i f_{s,i-1}(t).$$

Further, replacing the summation index i to $j + 1$, we get

$$D_\beta y_s(t) = \sum_{j=0}^{\infty} \lambda^{j+1} f_{s,j}(t) = \lambda \sum_{j=0}^{\infty} \lambda^j f_{s,j}(t) = \lambda y_s(t).$$

□

Corollary 2.4. *Functions* $y_s(x)$ *at all values of* $s = 0, 1, ...$ *are eigen functions of the operator* D_β.

Now we consider the following generalization of the function (7):

$$y_{s,p}(t)=\sum_{i=p}^{\infty}\lambda^{i-p}\begin{pmatrix} i \\ p \end{pmatrix}\frac{t^{\beta i+s}}{C(\beta,s,i)}, \tag{9}$$

where $\begin{pmatrix} i \\ p \end{pmatrix}=\frac{i!}{p!(i-p)!}, p=0,1,2,...,.$

It is obvious that $y_{s,0}(t)=y_s(t)$. The following proposition is true.

Theorem 2.5. *Let the series* (9) *be convergent, and it be possible to use the operator $D_\beta-\lambda$ termwise to* (9). *If there exist values of the parameter s such that*

$$\left(t^{-\beta i-s+\beta}D_\beta t^{\beta i+s}\right)\Big|_{i=0}=0,$$

then the functions $y_{s,p}(x)$ at all values of s satisfy the following equations:

$$\left(D_\beta-\lambda\right)y_{s,p}(t)=y_{s,p-1}(t), p\geq 1, \tag{10}$$

$$\left(D_\beta-\lambda\right)y_{s,0}(t)=0, \tag{11}$$

i.e. it forms the normalized by the parameter p. system with respect to the operator $D_\beta-\lambda$.

Proof. The equality (11) has been proved in Theorem 2.3. Let's prove the equality (10). Applying the operator D_β to the functions $y_{s,p}(x)$, due to normability of the system $f_{s,i}(t)=\frac{t^{\beta i+s}}{C(\beta,s,i)}$, we obtain

$$D_\beta y_{s,p}(t)=\sum_{i=p}^{\infty}\lambda^{i-p}\begin{pmatrix} i \\ p \end{pmatrix}D_\beta f_{s,i}(t)=\sum_{i=p}^{\infty}\lambda^{i-p}\begin{pmatrix} i \\ p \end{pmatrix}f_{s,i-1}(t).$$

Changing the summation index i to $j+1$, we get

$$D_\beta y_{s,p}(t)=\sum_{j=p-1}^{\infty}\lambda^{j-(p-1)}\begin{pmatrix} j+1 \\ p \end{pmatrix}f_{s,j}(t).$$

Consider now the function $D_\beta y_{s,p}(t)-y_{s,p-1}(t)$. By the definition of the functions $y_{s,p-1}(t)$, we have

$$\begin{aligned}D_\beta y_{s,p}(t)-y_{s,p-1}(t)&=\sum_{j=p-1}^{\infty}\lambda^{j-(p-1)}\begin{pmatrix} j+1 \\ p \end{pmatrix}f_{s,j}(t)-\sum_{j=p-1}^{\infty}\lambda^{j-(p-1)}\begin{pmatrix} j \\ p-1 \end{pmatrix}f_{s,j}(t)\\&=\sum_{j=p-1}^{\infty}\lambda^{j-(p-1)}\left[\begin{pmatrix} j+1 \\ p \end{pmatrix}-\begin{pmatrix} j \\ p-1 \end{pmatrix}\right]f_{s,j}(t).\end{aligned}$$

Further, for $\begin{pmatrix} j+1 \\ p \end{pmatrix}-\begin{pmatrix} j \\ p-1 \end{pmatrix}$ we have

$$\begin{pmatrix} j+1 \\ p \end{pmatrix}-\begin{pmatrix} j \\ p-1 \end{pmatrix}=\begin{pmatrix} j \\ p \end{pmatrix}.$$

Moreover, if $j=p-1$, then

$$\left.\begin{pmatrix} j+1 \\ p \end{pmatrix}-\begin{pmatrix} j \\ p-1 \end{pmatrix}\right|_{j=p-1}=\begin{pmatrix} p \\ p \end{pmatrix}-\begin{pmatrix} p-1 \\ p-1 \end{pmatrix}=0.$$

Therefore,

$$D_\beta y_{s,p}(t) - y_{s,p-1}(t) = \sum_{j=p}^{\infty} \lambda^{j-(p-1)} \left(\begin{array}{c} j \\ p \end{array} \right) f_{s,j}(t) = \lambda \sum_{j=p}^{\infty} \lambda^{j-p} \left(\begin{array}{c} j \\ p \end{array} \right) f_{s,j}(t) = \lambda y_{s,p}(x),$$

i.e. the following equality holds:

$$\left(D_\beta - \lambda\right) y_{s,p}(t) = y_{s,p-1}(t).$$

□

Corollary 2.6. *By the conditions of Theorem 2.5, the functions $y_{s,p}(t)$ at all values of $s = 0, 1, ...$, are associated functions of the operator D_β.*

Corollary 2.7. *By the conditions of Theorem 2.5, the functions $y_{s,p}(t)$ at values of $p = 0, 1, ..., N-1$ are solutions of the equation:*

$$\left(D_\beta - \lambda\right)^N y(t) = 0.$$

Now we discuss some examples of generalized homogeneous operators.

Example 2.8. *Let $D = \frac{d}{dt}$ and $D_{nL} = Dt...DtDtD$, where t is $n-$times multiplied, and the operator D is $n+1-$times applied. In this case:*

$$D_{nL} t^m = m^{n+1} t^{m-1},$$

i.e. the given operator is generalized homogeneous of the order 1. It is obvious, that $D_{nL} t^0 = 0$ and

$$C(1, 0, i) = \prod_{k=1}^{i} \left(t^{-k+1} D_{nL} t^k\right) = \prod_{k=1}^{i} k^{n+1} = (i!)^{n+1}.$$

Then from Corollary 2.1 and Corollary 2.2 it follows that:

$$y_p(t) = \sum_{i=p}^{\infty} \lambda^{i-p} \left(\begin{array}{c} i \\ p \end{array} \right) \frac{t^i}{(i!)^{n+1}}$$

are eigen functions when $p = 0$, and associated functions when $p \geq 1$ of the operator D_{nL}. Proposition, that $y_0(t)$ is an eigen function for the operator D_{nL}, was proved in [1]. In this case the authors called the function $y_0(t)$ as a generalized exponential, and denote it by $e_n(\lambda t)$.

Example 2.9. *Let $D_2 = D^2 t^2 D^2$, where $D^2 = \frac{d^2}{dt^2}$. In this case $D^2 t^s = 0, s = 0, 1$, and $D^2 t^m = m(m-1)t^{m-2}, m \geq 2$. Therefore,*

$$D_2 t^m = D^2 t^2 D^2 t^m = m(m-1) D^2 t^m = m^2 (m-1)^2 t^{m-2}.$$

Thus, the operator $D_2 = D^2 t^2 D^2$ is generalized homogeneous of the order 2. For this case

$$C(2, s, i) = \prod_{k=1}^{i} \left(t^{-2k-s+2} D_2 t^{2k+s}\right) = \prod_{k=1}^{i} (2k+s)^2 (2k+s-1)^2.$$

If $s = 0$, then

$$C(2, 0, i) = \prod_{k=1}^{i} (2k+s)^2 (2k+s-1)^2 = 2^2 \cdot (2-1)^2 \cdot 4^2 \cdot (4-1)^2 \cdot ... \cdot (2i)^2 (2i-1)^2$$

$$= 2^2 \cdot 1^2 \cdot 4^2 \cdot 3^2 \cdot ... \cdot (2i)^2(2i-1)^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot ... \cdot (2i-1)^2(2i)^2 = [(2i)!]^2.$$

Analogously, if $s = 1$, then

$$C(2,1,i) = \prod_{k=1}^{i} (2k+s)^2(2k+s-1)^2 = (2+1)^2 \cdot (2)^2 \cdot ... \cdot (2i+1)^2(2i)^2$$

$$= 3^2 \cdot 2^2 \cdot 5^2 \cdot 4^2 \cdot ... \cdot (2i+1)^2(2i)^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot ... \cdot (2i)^2(2i+1)^2 = [(2i+1)!]^2.$$

Then the functions

$$y_0(x) = \sum_{i=0}^{\infty} (-\lambda)^i \frac{x^{2i}}{[(2i)!]^2}, y_1(x) = \sum_{i=0}^{\infty} (-\lambda)^i \frac{x^{2i+1}}{[(2i+1)!]^2}$$

are eigen functions, and

$$y_{0,p}(x) = \sum_{i=p}^{\infty} (-\lambda)^{i-p} \left(\begin{array}{c} i \\ p \end{array} \right) \frac{x^{2i}}{[(2i)!]^2},$$

$$y_{1,p}(x) = \sum_{i=p}^{\infty} (-\lambda)^{i-p} \left(\begin{array}{c} i \\ p \end{array} \right) \frac{x^{2i+1}}{[(2i+1)!]^2}, p \geq 1$$

are associated functions of the operator D_2.

Since

$$\sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} = \cos x, \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!} = \sin x,$$

then obtained functions generalize the trigonometrical functions $\sin x$ and $\cos x$. We denote them by $\sin_2 x, \cos_2 x$, i.e.

$$\cos_2(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{[(2i)!]^2}, \sin_2(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{[(2i+1)!]^2}.$$

These functions coincide with the generalized trigonometrical functions, obtained in [2].

Example 2.10. *Let $D_\alpha = D^\alpha t^\alpha D^\alpha$, where $m-1 < \alpha \leq m, m = 1, 2, ..., D^\alpha =_{RL} J^{m-\alpha} \frac{d^m}{dt^m}$,*

$$_{RL}J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0.$$

We apply the operator D^α to functions of the form $t^r, r \geq 0$. By the definition, $D^\alpha t^r = J^{m-\alpha} \frac{d^m}{dt^m} t^r$. It is clear that $D^\alpha t^r = 0, r = 0, 1, ..., m-1$.

If $r > m-1$, then

$$D^\alpha t^r = J^{m-\alpha} \frac{d^m}{dt^m} t^r = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} t^{r-\alpha}.$$

Further, we consider the operator of the form $D_{\alpha,n} = D^\alpha t^\alpha ... D^\alpha t^\alpha D^\alpha t^\alpha D^\alpha$, where t^α is $n-$times multiplied, and the operator D^α is $n+1-$times applied. Then

$$D_{\alpha,1} t^r = D^\alpha t^\alpha D^\alpha t^r = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} D^\alpha t^r = \gamma_{r-\alpha}^2 t^{r-\alpha},$$

$$D_{\alpha,n}t^m = D^\alpha t^\alpha ... D^\alpha t^\alpha D^\alpha t^\alpha D^\alpha t^m = \gamma_{m,\alpha}^{n+1} t^{m-\alpha},$$

where $\gamma_{r,\alpha}^k = \left[\frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}\right]^k$. *Thus, the operator* $D_{\alpha,n}$ *is generalized homogeneous of the* α *order. Now we construct the functions* $y_{s,p}(x)$. *To do this, we find the coefficients* $C(\alpha, s, i)$. *For this case we choose* $r = \alpha k + s, s = 0, 1, ..., p - 1$.

$$C(\alpha, s, i) = \prod_{k=1}^{i}\left(t^{-\alpha k - s + \alpha} D_{\alpha,n} t^{\alpha k + s}\right) = \prod_{k=1}^{i}\left[\frac{\Gamma(\alpha k + s + 1)}{\Gamma(\alpha k + s + 1 - \alpha)}\right]^{n+1}$$

$$= \left[\frac{\Gamma(\alpha + s + 1)}{\Gamma(s + 1)}\frac{\Gamma(2\alpha + s + 1)}{\Gamma(\alpha + s + 1)}\cdots\frac{\Gamma(\alpha i + s + 1)}{\Gamma(\alpha(i - 1) + s + 1)}\right]^{n+1}$$

$$= \left[\frac{\Gamma(\alpha i + s + 1)}{\Gamma(s + 1)}\right]^{n+1}.$$

For convenience, one can waive the requirement $C(\alpha, s, i) = 1$, *and consider the coefficients* $C(\alpha, s, i) = \Gamma^{n+1}(\alpha i + s + 1)$. *Then, functions of the following form:*

$$y_{s,p}(x) = \sum_{i=p}^{\infty} \lambda^{i-p} \left(\begin{array}{c} i \\ p \end{array}\right) \frac{t^{\alpha i + s}}{\Gamma^{n+1}(\alpha i + s + 1)}$$

in the case $p = 0$ *at all* $s = 0, 1, ..., m - 1$ *will be eigen functions, and in the case* $p \geq 1$ *associated functions of the operator* $D_{\alpha,n}$.

If $\alpha = m$ *is integer, then*

$$\frac{\Gamma(\alpha i + s + 1)}{\Gamma(s + 1)} = \frac{\Gamma(mi + s + 1)}{\Gamma(s + 1)} = \frac{(mi + s)!}{s!}$$

and therefore,

$$y_{s,p}(t) = \sum_{i=p}^{\infty} \lambda^{i-p} \left(\begin{array}{c} i \\ p \end{array}\right) \frac{(s!)^{n+1}}{[(mi + s)!]^{n+1}} t^{mi+s}.$$

In the case $m = 1$ *we get that* $s = 0$, *and obtain the function*

$$y_{0,0}(t) = \sum_{i=0}^{\infty} \lambda^i \frac{t^i}{[i!]^{n+1}} = e_n(\lambda t),$$

is considered in Example 2.8.

Example 2.11. *For any* $\alpha > 0$ *we consider the following integral operator of fractional order in Hadamard sense:*

$${}_HJ^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(ln\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}.$$

Let $m - 1 < \alpha \leq m, m = 1, 2, ..., \delta = t\frac{d}{d}, \delta^k = \delta\left(\delta^{k-1}\right)$ *and* ${}_{HC}D^\alpha = {}_HJ^{m-\alpha}\delta^m$ *be Hadamard - Caputo type operator of differentiation. It is easy to show, that the equalities* ${}_{HC}D^\alpha t^0 = 0$ *and* ${}_{HC}D^\alpha t^k = k^\alpha t^k, k \geq 1$. *hold.*

Consider the operator $D_{\alpha,1} = \frac{d}{dt} \cdot {}_{HC}D^\alpha$. *Then,*

$$D_{\alpha,1}t^0 = 0, D_{\alpha,1}t^k = k^{\alpha+1}t^{k-1}, k \geq 1.$$

This operator will be generalized homogeneous of the order 1, and

$$C(\alpha,0,i)=\prod_{k=1}^{i}k^{\alpha+1}=(i!)^{\alpha+1}.$$

Due to Corollary 2.4, eigen function of this operator has the form:

$$y_0(t)=\sum_{i=0}^{\infty}\lambda^i\frac{t^i}{(i!)^{\alpha+1}}, \tag{12}$$

Some properties of the functions (12) *have been studied in [4].*

Corollary 2.6 implies that in the case $p\geq 1$ the functions

$$y_{s,p}(t)=\sum_{i=p}^{\infty}\lambda^{i-p}\begin{pmatrix} i\\ p\end{pmatrix}\frac{t^i}{(k!)^{\alpha+1}}$$

are associated functions of the operator $D_{\alpha,1}$.

3. Application of Normalized Systems to Solve Differential Equations of Fractional Order

Let $m-1<\alpha\leq\gamma\leq m, m=1,2,\ldots$. Consider the operator:

$$D^{\alpha,\gamma}f(t)={}_{RL}J^{(\gamma-\alpha)}\cdot\frac{d^m}{dt^m}\cdot{}_{RL}J^{(m-\gamma)}f(t),t>0.$$

In the case $m=1$ this operator was introduced in [3], and it generalizes well known differentiation operators of fractional order. In particular, $D^{\alpha,\alpha}={}_{RL}D^{\alpha}$ is the operator of differentiation of fractional order in Riemann - Liouville sense, $D^{\alpha,m}={}_{C}D^{\alpha}$ is the operator of differentiation in Caputo sense, and if $\gamma=\beta(m-\alpha)+\alpha, 0<\beta\leq 1$, then $D^{\alpha,\gamma}=D^{\alpha,\beta}-$ is the Hilfer operator [5].

It is easy to show, that the system of functions

$$f_{s,i}(t)=\frac{t^{i\alpha+s}}{\Gamma(i\alpha+s+1)}, i\in N_0$$

at all values of $s=\gamma-1,\gamma-2,...,\gamma-m$ is 0-normalized with respect to the operator $D^{\alpha,\gamma}$. Then the following proposition is true.

Theorem 3.1. *Let $m-1<\alpha\leq\gamma\leq m, m=1,2,...,s=\gamma-1,...,\gamma-m$. Then the functions*

$$y_{s,p}(t)=\sum_{i=0}^{\infty}\lambda^{i-p}\frac{t^{i\alpha+s}}{\Gamma(i\alpha+s+1)}$$

for all values of $p=0,1,...,N-1$ are solutions of the following equation of fractional order:

$$(D^{\alpha,\gamma}-\lambda)^N y(t)=0,t>0. \tag{13}$$

The proof of the theorem follows from Corollary 2.7. Convergence of the corresponding series can be checked by using d'Alembert's test and the properties of the gamma function.

Now we give an example to $f-$ normalized with respect to the operator $D^{\alpha,\gamma}$ system and construct a solution of the inhomogeneous equation

$$(D^{\alpha,\gamma}-\lambda)^N y(t)=f(t),t>0. \tag{14}$$

Note, that the operator method of constructing the solution of the equation (14) in the case $\gamma=\beta(m-\alpha)+\alpha, 0<\beta\leq 1$ had previously been studied in [7],[8].

Let

$$E^p_{\alpha,\alpha}(\lambda,t)=\sum_{i=p}^{\infty}\lambda^{i-p}\begin{pmatrix} i \\ p \end{pmatrix}\frac{t^{i\alpha}}{\Gamma(i\alpha+\alpha)}, p=0,1,....$$

Consider the following function

$$y_p(f)(t)=\int\limits_0^t (t-\tau)^{\alpha-1}E^p_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha})f(\tau)d\tau. \tag{15}$$

Theorem 3.2. *Let $m-1<\alpha\le\gamma\le m, m=1,2,...$. Then functions $y_p(f)(t), p=0,1,...$ form $f-$ normalized system with respect to the operator $D^{\alpha,\gamma}-\lambda$, i.e. the following equality holds:*

$$\begin{cases} (D^{\alpha,\gamma}-\lambda)\,y_0(f)(t)=f(t), \\ (D^{\alpha,\gamma}-\lambda)\,y_p(f)(t)=f_{p-1}(t), \quad p\ge 1 \end{cases} \tag{16}$$

Proof. When $p=0$ the function $E^0_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha})$ coincides with the Mittag - Leffler type operator , i.e.

$$E^0_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha})=\sum_{i=0}^{\infty}(-\lambda)^i\frac{t^{i\alpha}}{\Gamma(\alpha i+\alpha)}.$$

Then the equality (16) for the case $p=0$ is proved, as in the case of Riemann - Liouville operator. Indeed,

$$J^{m-\alpha}y_0(f)(t)=\frac{1}{\Gamma(m-\alpha)}\int\limits_0^t (t-\tau)^{m-\alpha-1}y_0(f)(\tau)d\tau$$

$$=\frac{1}{\Gamma(m-\gamma)}\int\limits_0^t (t-\tau)^{m-\gamma-1}\int\limits_0^{\tau}(\tau-\xi)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau-\xi)^{\alpha})f(\xi)d\xi d\tau$$

$$=\frac{1}{\Gamma(m-\gamma)}\int\limits_0^t f(\xi)\int\limits_{\xi}^{t}(t-\tau)^{m-\gamma-1}(\tau-\xi)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau-\xi)^{\alpha})d\tau d\xi$$

We calculate the inner integral. Using representation of the function $E_{\alpha,\alpha}$, we have

$$\int\limits_{\xi}^{t}(t-\tau)^{m-\gamma-1}(\tau-\xi)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\tau-\xi)^{\alpha})d\tau$$

$$=\sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\alpha i+\alpha)}\int\limits_{\xi}^{t}(t-\tau)^{m-\gamma-1}(\tau-\xi)^{\alpha i+\alpha-1}d\tau$$

$$=\sum_{i=0}^{\infty}\frac{\lambda^{i-p}(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha)}\int\limits_0^1(1-\theta)^{m-\gamma-1}\theta^{\alpha i+\alpha-1}d\theta$$

$$=\Gamma(m-\gamma)\sum_{i=0}^{\infty}\frac{\lambda^i(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha+m-\gamma)}.$$

Therefore,

$$J^{m-\gamma}y_0(f)(t)=\int\limits_0^t\sum_{i=0}^{\infty}\frac{\lambda^i(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha+m-\gamma)}f(\xi)d\xi$$

$$=\int\limits_0^t(t-\xi)^{m-\gamma-1}E_{\alpha,\alpha}\left(\lambda(t-\xi)^{\alpha}\right)f(\xi)d\xi.$$

Further,

$$\frac{d^{m-1}}{dt^{m-1}}J^{m-\gamma}y_0(f)(t)=\frac{d^{m-1}}{dt^{m-1}}\int\limits_0^t\sum_{i=0}^{\infty}\frac{\lambda^i(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha+m-\gamma)}f(\xi)d\xi$$

$$=\int\limits_0^t f(\xi)\sum_{i=0}^{\infty}\lambda^i(\alpha i+\alpha+m-\gamma-1)...(\alpha i+\alpha+1-\gamma)\frac{(t-\xi)^{\alpha i+\alpha-\gamma}}{\Gamma(\alpha i+\alpha+m-\gamma)}d\xi$$

$$=\int\limits_0^t f(\xi)\sum_{i=0}^{\infty}\lambda^i\frac{(t-\xi)^{\alpha i+\alpha-\gamma}}{\Gamma(\alpha i+\alpha+1-\gamma)}d\xi.$$

Here we note, that, since $\alpha>m-1$, then $\alpha-\gamma>m-1-\gamma>-1$, i.e. integral from the functions $(t-\xi)^{\alpha i+\alpha-\gamma}$ converges. Further,

$$J^{\gamma-\alpha}\frac{d^m}{dt^m}J^{m-\gamma}y_0(f)(t)=\int\limits_0^t\frac{(t-\tau)^{\gamma-\alpha-1}}{\Gamma(\gamma-\alpha)}\frac{d}{d\tau}\left(\int\limits_0^{\tau}f(\xi)\sum_{i=0}^{\infty}\frac{\lambda^i(\tau-\xi)^{\alpha i+\alpha-\gamma}}{\Gamma(\alpha i+\alpha+1-\gamma)}d\xi\right)d\tau$$

We represent the last integral in the following form:

$$J^{\gamma-\alpha}\frac{d^m}{dt^m}J^{m-\gamma}y_0(f)(t)=$$

$$=\frac{1}{\Gamma(\gamma-\alpha)}\frac{d}{dt}\int\limits_0^t\frac{(t-\tau)^{\gamma-\alpha}}{\gamma-\alpha}\frac{d}{d\tau}\left(\int\limits_0^{\tau}f(\xi)\sum_{i=0}^{\infty}\lambda^i\frac{(\tau-\xi)^{\alpha i+\alpha-\gamma}}{\Gamma(\alpha i+\alpha+1-\gamma)}d\xi\right)d\tau$$

Further, taking integral by parts, we obtain

$$J^{\gamma-\alpha}\frac{d^m}{dt^m}J^{m-\gamma}y_0(f)(t)=\frac{d}{dt}\left[\frac{1}{\Gamma(\gamma-\alpha)}\int\limits_0^t(t-\tau)^{\gamma-\alpha-1}\left(\int\limits_0^{\tau}f(\xi)\sum_{i=0}^{\infty}\lambda^i\frac{(\tau-\xi)^{\alpha i+\alpha-\gamma}}{\Gamma(\alpha i+\alpha+1-\gamma)}d\xi\right)d\tau\right]$$

$$=\sum_{i=0}^{\infty}\frac{\lambda^i\Gamma(\gamma-\alpha)\Gamma(\alpha i+\alpha+1-\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha i+\alpha+1-\gamma)\Gamma(\alpha i+1)}\frac{d}{dt}\left[\int\limits_0^t(t-\xi)^{\alpha i}f(\xi)d\xi\right]=f(t)$$

The case $p\geq 1$ can be checked analogously. Indeed, let $p\geq 1$. Then, applying the operator $J^{m-\gamma}$ to the functions $y_p\left(f\right)(t)$, we have

$$J^{m-\alpha}y_p(f)(t)=\frac{1}{\Gamma(m-\alpha)}\int\limits_0^t(t-\tau)^{m-\alpha-1}y_p(f)(\tau)d\tau$$

$$= \frac{1}{\Gamma(m-\gamma)} \int_0^t f(\xi) \int_\xi^t (t-\tau)^{m-\gamma-1} (\tau-\xi)^{\alpha-1} E^p_{\alpha,\alpha}(\lambda(\tau-\xi)^\alpha) d\tau d\xi$$

We calculate the inner integral. Using the representation of the function $E^p_{\alpha,\alpha}$, we have

$$\int_\xi^t (t-\tau)^{m-\gamma-1} (\tau-\xi)^{\alpha-1} E^p_{\alpha,\alpha}(\lambda(\tau-\xi)^\alpha) d\tau$$

$$= \sum_{i=p}^\infty \frac{\lambda^{i-p}}{\Gamma(\alpha i+\alpha)} \left(\begin{array}{c} i \\ p \end{array} \right) \int_\xi^t (t-\tau)^{m-\gamma-1} (\tau-\xi)^{\alpha i+\alpha-1} d\tau$$

$$= \sum_{i=p}^\infty \left(\begin{array}{c} i \\ p \end{array} \right) \int_0^1 (1-\theta)^{m-\gamma-1} \theta^{\alpha i+\alpha-1} d\theta \frac{\lambda^{i-p}(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha)}$$

$$= \Gamma(m-\gamma) \sum_{i=p}^\infty \left(\begin{array}{c} i \\ p \end{array} \right) \frac{\lambda^{i-p}(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha+m-\gamma)}.$$

Hence,

$$J^{m-\gamma} y_p(f)(t) = \int_0^t f(\xi) \sum_{i=p}^\infty \left(\begin{array}{c} i \\ p \end{array} \right) \frac{\lambda^{i-p}(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha+m-\gamma)} d\xi$$

$$= \int_0^t (t-\xi)^{m-\gamma-1} E^p_{\alpha,\alpha}\left(\lambda(t-\xi)^\alpha\right) f(\xi) d\xi.$$

Further,

$$\frac{d^{m-1}}{dt^{m-1}} J^{m-\gamma} f_p(t) = \frac{d^{m-1}}{dt^{m-1}} \int_0^t f(\xi) \sum_{i=p}^\infty \left(\begin{array}{c} i \\ p \end{array} \right) \frac{\lambda^{i-p}(t-\xi)^{\alpha i+\alpha+m-\gamma-1}}{\Gamma(\alpha i+\alpha+m-\gamma)} d\xi$$

$$= \int_0^t f(\xi) \sum_{i=p}^\infty \lambda^{i-p} \left(\begin{array}{c} i \\ p \end{array} \right) \frac{(t-\xi)^{\alpha i+\alpha-\gamma}}{\Gamma(\alpha i+\alpha+1-\gamma)} d\xi.$$

Finally,

$$J^{\gamma-\alpha} \frac{d^m}{dt^m} J^{m-\gamma} y_p(f)(t) = \frac{1}{\Gamma(\gamma-\alpha)} \int_0^t (t-\tau)^{\gamma-\alpha-1} \frac{d^m}{d\tau^m} J^{m-\gamma} y_p(f)(\tau) d\tau$$

$$= \int_0^t \sum_{j=p-1}^\infty \lambda^{j-(p-1)} \left(\begin{array}{c} j+1 \\ p \end{array} \right) \frac{(t-\xi)^{\alpha j+\alpha-1}}{\Gamma(\alpha j+\alpha)} f(\xi) d\xi.$$

Thus,

$$D^{\alpha,\gamma}y_p(f)(t)=\int\limits_0^t\left[\sum_{i=p-1}^{\infty}\lambda^{i-(p-1)}\left(\begin{array}{c}i+1\\p\end{array}\right)\frac{(t-\xi)^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)}\right]f(\xi)d\xi.$$

Then

$$D^{\alpha,\gamma}y_p(f)(t)-y_{p-1}(f)(t)=\int\limits_0^t\sum_{i=p-1}^{\infty}\left(\begin{array}{c}i+1\\p\end{array}\right)\frac{\lambda^{i-(p-1)}(t-\xi)^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)}f(\xi)d\xi$$

$$=\int\limits_0^t\sum_{i=p}^{\infty}\left(\begin{array}{c}i\\p\end{array}\right)\frac{\lambda^{i-(p-1)}(t-\xi)^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)}f(\xi)d\xi==\lambda\int\limits_0^t\sum_{i=p}^{\infty}\left(\begin{array}{c}i\\p\end{array}\right)\frac{\lambda^{i-p}(t-\xi)^{\alpha i+\alpha-1}}{\Gamma(\alpha i+\alpha)}f(\xi)d\xi=\lambda f_p(t).$$

So,

$$D^{\alpha,\gamma}y_p(f)(t)-y_{p-1}(f)(t)=\lambda y_p(f)(t),$$

i.e., at all values of $p=1,2,...$ the following equalities hold:

$$\left(D^{\alpha,\gamma}-\lambda\right)y_p(f)(t)=y_{p-1}(f)(t).$$

$\square$

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