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$\mathcal{T}L$-Fuzzy Set Valued Homomorphisms and Generalized $(\mathcal{I},\mathcal{T})$-$L$-Fuzzy Rough Sets on Groups

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Abstract. In this paper we introduce the notion of $\mathcal{T}L$-fuzzy set valued homomorphisms of groups and investigate related properties. We also investigate the properties of the generalized $(\mathcal{I},\mathcal{T})$-$L$-fuzzy rough sets constructed by $\mathcal{T}L$-fuzzy set valued homomorphisms of groups.

1. Introduction

The rough set theory is proposed by Pawlak [18] as a new intelligent soft computing tool for dealing with vagueness or uncertainty. In Pawlak's rough set model the equivalence relations are employed to construct the lower and upper approximations as key notations. Biswas and Nanda [4] applied the notion of rough sets to algebra and introduced the concept of rough subgroups. Kuroki [14] investigated some properties of the lower and upper approximations on semigroups by use of the congruence relations. Using arbitrary relations for different universal sets, Yao [29] developed the generalized rough sets. Since the congruence relations are not suitable for working on an algebraic structure in Yao's model, Davvaz [5] introduced the set-valued homomorphisms of groups and used them to construct the approximation operators. Yamak et al. [27, 28] investigated some properties of the generalized rough sets with respect to set-valued homomorphisms of rings and modules. And also see [1, 2, 10, 26].

Rough set theory and fuzzy set theory (see [30]) are two distinct but complementary, powerful mathematical tools for studying incomplete and vague information. Fuzzy rough sets is the hybridization of the rough sets and fuzzy sets. By replacing crisp binary relations with fuzzy relations in the universe, Dubois and Prade [6] proposed the concept of fuzzy rough set. Radzikowska and Kerre [19] presented a more general approach to the fuzzification of rough sets and defined $(\mathcal{I},\mathcal{T})$-fuzzy rough set by using t-norms and implications via fuzzy similarity relation. Li et al. [16] studied $(\mathcal{I},\mathcal{T})$-fuzzy rough approximation operators on a ring as a universal set with respect to a $\mathcal{T}L$-fuzzy ideal of a ring. Recently, Li and Yin [15] investigated the properties of ν-lower and $\mathcal{T}$-upper fuzzy rough approximation operators with respect to a $\mathcal{T}$-congruence L-fuzzy relation on a semigroup as a universal set. Wu, Leung and Mi [22] expanded $(\mathcal{I},\mathcal{T})$-fuzzy rough set into two different universal sets. Since $\mathcal{T}$-congruence L-fuzzy relations are not suitable for

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generalized $(\mathcal{I},\mathcal{T})$-fuzzy rough set, Ekiz et al. [7] applied generalized $(\mathcal{I},\mathcal{T})$-fuzzy rough set to the theory of ring via $\mathcal{T}L$-fuzzy relational morphism introduced by Ignjatović et al. [11]. $\mathcal{T}L$-fuzzy relational morphisms and $\mathcal{T}L$-fuzzy set valued homomorphisms are related closely. Obviously, a $\mathcal{T}L$-fuzzy relational morphism can define a $\mathcal{T}L$-fuzzy set valued homomorphism and vice versa. $\mathcal{T}L$-fuzzy set valued homomorphisms are the fuzzification of the set valued homomorphisms in the sense of Davvaz's definition. This paper denotes some properties of generalized $(\mathcal{I},\mathcal{T})$-fuzzy rough sets constructed on two different groups using $\mathcal{T}L$-fuzzy set valued homomorphisms. This paper, in one respect, is an extension in the sense of fuzzy of the generalized rough sets constructed by set-valued homomorphisms [5, 27, 28].

The rest of the paper is organized as follows. Section 2, reviews some preliminary concepts. Section 3, introduces the $\mathcal{T}L$-fuzzy set valued homomorphisms of groups. Section 4, analyzes the generalized $(\mathcal{I},\mathcal{T})$-fuzzy rough sets with respect to $\mathcal{T}L$-fuzzy set valued homomorphisms of groups.

2. Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented brief. Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice with the least element 0 and the greatest element 1.

A triangular norm (see [13]), or t-norm in short, is an increasing, associative and commutative mapping $\mathcal{T} : L \times L \to L$ that satisfies the boundary condition: for all $\alpha \in L$, $\mathcal{T}(\alpha, 1) = \alpha$. The minimum t-norm $\mathcal{T}_M$ and drastic product t-norm $\mathcal{T}_D$ on L are defined as follows:

$$\mathcal{T}_M(\alpha,\beta) = \alpha \wedge \beta, \qquad \mathcal{T}_D(\alpha,\beta) = \begin{cases} \beta, & \text{if } \alpha = 1; \\ \alpha, & \text{if } \beta = 1; \\ 0, & \text{otherwise} \end{cases}, \qquad \forall \alpha,\beta \in L$$

A t-norm $\mathcal{T}$ on L is called $\vee$-distributive if $\mathcal{T}(\alpha, \beta_1 \vee \beta_2) = \mathcal{T}(\alpha,\beta_1) \vee \mathcal{T}(\alpha,\beta_2)$ for all $\alpha,\beta_1,\beta_2 \in L$. $\mathcal{T}$ is also called infinitely $\vee$-distributive if $\mathcal{T}(\alpha, \bigvee_{i\in J} \beta_i) = \bigvee_{i\in J} \mathcal{T}(\alpha,\beta_i)$ for all $\alpha,\beta_i \in L$, where J is an index set. Any $a \in L$ is called an idempotent element of L with respect to the t-norm T if $\mathcal{T}(a,a) = a$. All of the idempotent elements of L are denoted by the set $D_{\mathcal{T}} = \{a \in L \mid \mathcal{T}(a,a) = a\}$ and for any t-norm $\mathcal{T}$, the operation $\mathcal{T}$ is a binary operation on $D_{\mathcal{T}}$.

A function $\mathcal{I} : L \times L \to L$ is an implication if it fulfills the following, for all $x, y, z \in L$

1 $x \le z$ implies $\mathcal{I}(z, y) \le \mathcal{I}(x, y)$,

2 $y \le z$ implies $\mathcal{I}(x, y) \le \mathcal{I}(x, z)$,

3 $\mathcal{I}(0, 0) = \mathcal{I}(1, 1) = 1$ and $\mathcal{I}(1, 0) = 0$.

An implicator $\mathcal{I}$ defined as

$$\mathcal{I}(x,y) = \bigvee_{\mathcal{T}(x,\alpha)\le y} \alpha, \quad \alpha \in L$$

for all $x, y \in L$ is called an R-implication (residual implication) based on the t-norm $\mathcal{T}$ (see [3, 8]).

2.1. L-fuzzy subsets

Let X be a non-empty set called universe of discourse. An L-fuzzy subset of X is any function from X into L (see [9, 30]). The class of all subsets and L-fuzzy subsets of X will be denoted by $\mathcal{P}(X)$ and $F(X, L)$, respectively. In particular, if $L = [0, 1]$ (where $[0, 1]$ is the unit interval), then it is appropriate to replace fuzzy subset with L-fuzzy subset. In this case the set of all fuzzy subsets of X is denoted by $F(X)$. Let $\alpha \in L$. Then the sets $\mu_\alpha = \{x \in X \mid \mu(x) \ge \alpha\}$, $\mu_\alpha^{\mathcal{T}} = \{x \in X \mid \mu(x)\mathcal{T}\alpha \ge \alpha\}$ and $[\mu_\alpha^{\mathcal{T}}] = \{x \in X \mid \mu(x)\mathcal{T}\lambda \ge \alpha \quad \textit{for some} \quad \lambda \in L\}$ are called α-, $(\alpha,\mathcal{T})$- and $[\alpha,\mathcal{T}]$-cut (or level) subsets of μ, respectively. It is easy to see that $\mu_\alpha \subseteq \mu_\alpha^{\mathcal{T}} \subseteq [\mu_\alpha^{\mathcal{T}}]$ if $\alpha \in D_{\mathcal{T}}$. Let $\emptyset \ne B \subseteq X$ and $\alpha \in L$. Then $(\alpha,\beta)_B$ will denote an L-fuzzy subset of X with value β if $x \in B$ and α elsewhere. Then $(0, 1)_B$ is called the characteristic function of a set $B \subseteq X$ and it is denoted by 1_B.

Let X, Y be non-empty sets, $\mu \in F(X, L)$, $\nu \in F(Y, L)$ and $f : X \to Y$ be a function. Then $f(\mu) \in F(Y, L)$ and $f^{-1}(\nu) \in F(X, L)$ are defined by

$$f(\mu)(y) = \bigvee_{f(x)=y} \mu(x), \ (\forall y \in Y), \ f^{-1}(\nu) = \nu \circ f$$

Let μ and ν be any two L-fuzzy subsets of X and let $'' * ''$ be a binary relation on L. Then $\mu * \nu$ is a binary relation on $F(X, L)$ and $\mu * \nu$ is defined by $(\mu * \nu)(x) = \mu(x) * \nu(x)$ for all $x \in X$.

An L-fuzzy subset $R \in F(X \times Y, L)$ is called an L-fuzzy relation from X to Y. $R(x, y)$ is the degree of relation between x and y, where $(x, y) \in X \times Y$. If $X = Y$, then R is referred to as an L-fuzzy relation on X. An L-fuzzy relation R is called reflexive if $R(x, x) = 1$ for all $x \in X$, symmetric if $R(x, y) = R(y, x)$ for all $x, y \in X$ and $\mathcal{T}$-transitive if $R(x, z) \geq \bigvee_{y \in X}(R(x, y)\mathcal{T}R(y, z))$ for all $x, z \in X$. A reflexive, symmetric and $\mathcal{T}$-transitive L-fuzzy relation is called a $\mathcal{T}$-equivalence L-fuzzy relation. Let $R : X \to F(Y, L)$ be a function and $x \in X$. Then $R(x)$ denotes as an L-fuzzy set of Y. Let X, Y, X', Y' be non-empty sets and $R : X \to F(Y, L)$, $P : X' \to F(Y', L)$, $g : Y \to Y'$, $f : X \to X'$ be functions. Then $R^{-1} : Y \to F(X, L)$, $R_{(f,g)} : X' \to F(Y', L)$, $P^{-1}_{(f,g)} : X \to F(Y, L)$ and $P_f : X \to F(Y', L)$, respectively, are defined by $R^{-1}(y)(x) = R(x)(y)$, $R_{(f,g)}(x')(y') = \bigvee_{f(x)=x', g(y)=y'} R(x)(y)$, $P^{-1}_{(f,g)}(x)(y) = P(f(x))(g(y))$, $P_f(x)(y') = P(f(x))(y')$ for all $x \in X, x' \in X', y \in Y, y' \in Y$. The $\mathcal{T}$-compositions of the L-fuzzy set valued functions $R : X \to F(Y, L)$ and $P : Y \to F(Z, L)$ is a L-fuzzy set valued functions $P *_{\mathcal{T}} R : X \to F(Z, L)$ defined by $(P *_{\mathcal{T}} R)(x)(z) = \bigvee_{y \in Y} R(x)(y)\mathcal{T}P(y)(z)$ for all $x \in X, z \in Z$ (see [12, 21]).

2.2. Generalized rough approximation operators

Let X and Y be two non-empty sets. Let T be a set valued mapping given by $T : X \to \mathcal{P}(Y)$. Then the triple (X, Y, T) is referred to as a generalized approximation space or generalized rough set. Any set valued function from X to $\mathcal{P}(Y)$ defines a binary relation by setting $\varphi_T = \{(x, y) \mid y \in T(x)\}$. Obviously, if φ is an arbitrary binary relation from X to Y, then it can define a set valued mapping $T_\varphi : X \to \mathcal{P}(Y)$ by $T_\varphi(x) = \{y \in Y \mid (x, y) \in \varphi\}$ where $x \in X$. For any set $A \subseteq Y$ a pair of lower and upper approximations $\underline{T}(A)$ and $\overline{T}(A)$ are defined by $\underline{T}(A) = \{x \in X \mid T(x) \subseteq A\}$ and $\overline{T}(A) = \{x \in X \mid T(x) \cap A \neq \emptyset\}$. The pair $(\underline{T}(A), \overline{T}(A))$ is referred to as a generalized rough set, and $\underline{T}$ and $\overline{T}$ are referred to as lower and upper generalized approximation operators, respectively (see [1, 29]).

2.3. Generalized L-fuzzy rough approximation operators

Let X and Y be two non-empty sets and R be an L-fuzzy relation from X to Y. The triple (X, Y, R) is called a generalized L-fuzzy approximation space. If R is an L-fuzzy relation on X, then (X, R) is called an L-fuzzy approximation space. Let $\mathcal{T}$ be a t-norm and $\mathcal{I}$ be an implication on L. For any L-fuzzy subset μ of Y, the $\mathcal{T}$-upper and $\mathcal{I}$-lower L-fuzzy rough approximations of μ denoted by $\overline{R}^{\mathcal{T}}(\mu)$ and $\underline{R}_{\mathcal{I}}(\mu)$ respectively, are two L-fuzzy sets of X whose membership functions are defined by

$$\overline{R}^{\mathcal{T}}(\mu)(x) = \bigvee_{y \in Y} \mathcal{T}(R(x)(y), \mu(y)), \ \underline{R}_{\mathcal{I}}(\mu)(x) = \bigwedge_{y \in Y} \mathcal{I}(R(x)(y), \mu(y)), \ (\forall x \in X).$$

The operators $\overline{R}^{\mathcal{T}}$ and $\underline{R}_{\mathcal{I}}$ from $\mathcal{F}(Y, L)$ to $\mathcal{F}(X, L)$ are referred to as $\mathcal{T}$-upper and $\mathcal{I}$-lower fuzzy rough approximation operators of (X, Y, R) respectively, and the pair $(\underline{R}_{\mathcal{I}}(\mu), \overline{R}^{\mathcal{T}}(\mu))$ is called the $(\mathcal{I}, \mathcal{T})$-$L$-fuzzy rough set of μ with respect to (X, Y, R) (see [15, 22–25]).

2.4. Set valued homomorphisms

Let G, H be groups. A mapping $T : G \to \mathcal{P}(H)$ satisfying $T(a)T(b) \subseteq T(ab)$ and $(T(a))^{-1} \subseteq T(a^{-1})$ for all $a, b \in G$ is called set valued homomorphism of groups (see [2, 5, 10, 26–28]).

2.5. $\mathcal{T}L$-fuzzy subgroups

Let G be a group and $\mu \in \mathcal{F}(G, L)$. If, for all $x, y \in G$, $\mu(x)\mathcal{T}\mu(y) \leq \mu(xy)$ and $\mu(x) \leq \mu(x^{-1})$, then μ is called a $\mathcal{T}L$-fuzzy subgroup of G. If μ satisfies the supplementary condition $\mu(xy) = \mu(yx)$ for all $x, y \in G$, then it is called a normal $\mathcal{T}L$-fuzzy subgroup of G. μ is a normal $\mathcal{T}L$-fuzzy subgroup of G if and only if $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$. Let $\mu, \nu \in \mathcal{F}(G, L)$. Then $\mathcal{T}$-product of μ and ν denoted by $\mu \cdot_{\mathcal{T}} \nu$ is defined by $(\mu \cdot_{\mathcal{T}} \nu)(x) = \bigvee_{x=ab} \mu(a)\mathcal{T}\nu(b)$ and $\mu^{-1} \in F(G, L)$ is defined by $\mu^{-1}(x) = \mu(x^{-1})$ for all $x \in G$. Let μ be a

TL-fuzzy subgroup of G and a be any element in G. Then the L-fuzzy subset of G defined by $a\mu(x) = \mu(a^{-1}x)$ for all $x \in G$ is called a L-fuzzy left coset of μ in G. Similarly, the L-fuzzy right coset of μ in G is defined by $\mu a(x) = \mu(xa^{-1})$ for all $x \in G$. If μ be a normal TL-fuzzy subgroup of G, then $a\mu = \mu a$. The set of all the L-fuzzy left cosets of G with respect to μ is the set $G/\mu = \{a\mu \mid a \in G\}$. If μ is an normal TL-fuzzy subgroup of G, then G/μ is a group under the binary operation $(a\mu)(b\mu) = ab\mu$ for all $a, b \in G$. A $\mathcal{T}$-equivalence L-fuzzy relation R on G is called a $\mathcal{T}$-congruence L-fuzzy relation if it satisfies $R(x,y)\mathcal{T}R(a,b) \leq R(xa,yb)$ (see [12, 17, 20, 21, 31]).

Throughout this paper L will be denote a complete lattice and $\mathcal{T}$ will be referred as a t-norm on L, and $\mathcal{I}$ will be referred as an R-implication based on $\mathcal{T}$.

3. $\mathcal{T}L$-fuzzy Set Valued Homomorphism of Groups

In this section, the notion of $\mathcal{T}L$-fuzzy set valued homomorphism of groups is introduced and some related properties are investigated.

Definition 3.1. Let G and H be two groups. A mapping $R : G \to \mathcal{F}(H, L)$ satisfying for all $x, a \in G$,

(i) $R(x) \cdot_{\mathcal{T}} R(a) \leq R(xa)$,

(ii) $(R(x))^{-1} \leq R(x^{-1})$

is called $\mathcal{T}L$-fuzzy set valued homomorphism. The set of all of the $\mathcal{T}L$-fuzzy set valued homomorphism from G to H is denoted by $\mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$. If $\mathcal{T} = \wedge$, then R is called L-fuzzy set valued homomorphism and $\mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$ is replaced by $\mathrm{Hom}(G, \mathcal{F}(H, L))$. If $L = [0, 1]$, then R is called $\mathcal{T}$-fuzzy set valued homomorphism, and in this case $\mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$ is replaced by $\mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H))$. It is clear that if $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$, then $R^{-1} \in \mathrm{Hom}_{\mathcal{T}}(H, \mathcal{F}(G, L))$.

Example 3.2. Let $L = N_5$ whose Hasse diagram depicted in the figure 1.

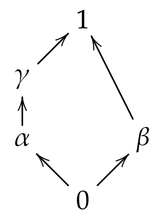

Figure 1: N_5

$\mathbb{R} \setminus \{0\}$ is a group under ordinary multiplication. For an $x \in \mathbb{R} \setminus \{0\}$, let $R(x) : \mathbb{R} \setminus \{0\} \to L$ be defined by

$$R(x)(y) = \begin{cases} 1, & \text{if } xy = 1; \\ \gamma, & \text{if } xy > 0, xy \neq 1; \\ \alpha, & \text{if } xy < 0. \end{cases}$$

for all $y \in \mathbb{R} \setminus \{0\}$. Then $R : \mathbb{R} \setminus \{0\} \to \mathcal{F}(\mathbb{R} \setminus \{0\}, L)$ is a $\mathcal{T}L$-fuzzy set valued homomorphism for any t-norm T on L.

Example 3.3. However, the set of all of the homomorphisms of group is $Hom(\mathbb{Z}_2, \mathbb{Z}_3) = \{\bar{0}\}$, while the set of all of the set valued homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}_3$ is $\{T_1, T_2, T_3, T_4\}$ where

$$T_1(x) = \begin{cases} \{\bar{0}\}, & \text{if } x = \bar{0}; \\ \emptyset, & \text{if } x = \bar{1}. \end{cases}, \quad T_2(x) = \begin{cases} \mathbb{Z}_3, & \text{if } x = \bar{0}; \\ \emptyset, & \text{if } x = \bar{1}. \end{cases}, \quad T_3(x) = \{\bar{0}\}, \quad T_4(x) = \mathbb{Z}_3.$$

Let $L = \{0, \alpha, \beta, 1\}$ be a lattice whose Hasse diagram depicted as follows:

$$\begin{tikzcd} & 1 & \\ \alpha \arrow[ur] & & \beta \arrow[ul] \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Figure 2: Lattice L

Then all of the L-fuzzy set valued homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}_3$ are;

x	$R_1(x)$	$R_2(x)$	$R_3(x)$	$R_4(x)$	$R_5(x)$	$R_6(x)$	$R_7(x)$	$R_8(x)$	$R_9(x)$	$R_{10}(x)$	$R_{11}(x)$	$R_{12}(x)$	$R_{13}(x)$
$\overline{0}$	μ_1	μ_1	μ_1	μ_1	μ_2	μ_2	μ_2	μ_2	μ_3	μ_3	μ_3	μ_3	μ_4
$\overline{1}$	μ_1	μ_5	μ_7	μ_9	μ_2	μ_5	μ_8	μ_9	μ_3	μ_6	μ_7	μ_9	μ_4

x	$R_{14}(x)$	$R_{15}(x)$	$R_{16}(x)$	$R_{17}(x)$	$R_{18}(x)$	$R_{19}(x)$	$R_{20}(x)$	$R_{21}(x)$	$R_{22}(x)$	$R_{23}(x)$	$R_{24}(x)$	$R_{25}(x)$
$\overline{0}$	μ_4	μ_4	μ_4	μ_5	μ_5	μ_6	μ_6	μ_7	μ_7	μ_8	μ_8	μ_9
$\overline{1}$	μ_6	μ_8	μ_9	μ_5	μ_9	μ_6	μ_9	μ_7	μ_9	μ_8	μ_9	μ_9

where $\mu_i \in \mathcal{F}(\mathbb{Z}_3, L), i = 1, 2, ..., 9$ as follows:

x	$\mu_1(x)$	$\mu_2(x)$	$\mu_3(x)$	$\mu_4(x)$	$\mu_5(x)$	$\mu_6(x)$	$\mu_7(x)$	$\mu_8(x)$	$\mu_9(x)$
$\overline{0}$	1	1	1	1	α	α	β	β	0
$\overline{1}$	1	α	β	0	α	0	β	0	0
$\overline{2}$	1	α	β	0	α	0	β	0	0

Example 3.4. Let G, H, X and Y be groups.

(1) Let $\Theta \in \mathcal{F}(G \times G, L)$ be a $\mathcal{T}$-congruence L-relation. For an $x \in G$, let $R(x) : G \to L$ be defined by $R(x)(y) = \Theta(x, y)$ for all $y \in G$. Then $R : G \to \mathcal{F}(G, L)$ is a $\mathcal{T}L$-fuzzy set valued homomorphism.

(2) Let $f : G \to H$ be homomorphism of groups. For an $x \in G$, let $R(f)_{(\alpha,\beta)} : G \to \mathcal{F}(H, L)$ be defined by
$$R(f)_{(\alpha,\beta)}(x)(y) = \begin{cases} \beta, & \text{if } f(x) = y; \\ \alpha, & \text{if } f(x) \neq y; \end{cases}$$
for all $y \in H$. If $\alpha \leq \beta$, then $R(f)_{(\alpha,\beta)}$ is a $\mathcal{T}L$-fuzzy set valued homomorphism.

(3) Let $T : G \to \mathcal{P}(H)$ be a set valued homomorphism, $\alpha, \beta \in L, \alpha \leq \beta$. Then $R : G \to \mathcal{F}(H, L)$ defined by $R(x) = (\alpha, \beta)_{T(x)}$ for all $x \in G$ is a $\mathcal{T}L$-fuzzy set valued homomorphism.

(4) Let $R : G \to \mathcal{F}(H, L)$ be a $\mathcal{T}L$-fuzzy set valued homomorphism, $x \in G$, $\alpha \in L$ and $\beta \in D_{\mathcal{T}}$. Then $T_1(x) = R(x)^{\mathcal{T}}_{\alpha}, T_2(x) = R(x)_{\beta}$ and $T_3(x) = [R(x)^{\mathcal{T}}_{\beta}]$ are set valued homomorphisms from G to $\mathcal{P}(H)$.

(5) Let $f : G \to H$ be homomorphism of groups and μ be a normal $\mathcal{T}L$-fuzzy subgroup of H. For an $x \in G$, let $R(x) : H \to L$ be defined by $R(x)(y) = \mu(f(x)y^{-1})$ for all $y \in H$. Then $R : G \to \mathcal{F}(H, L)$ is a $\mathcal{T}L$-fuzzy set valued homomorphism.

(6) Let $f : G \to H$ be homomorphism of groups and μ be a $\mathcal{T}L$-fuzzy subgroup of H. For an $x \in G$, let $R(x) : H \to L$ be defined by $R(x)(y) = \mu(f(x))\mathcal{T}\mu(y^{-1})$ for all $y \in H$. Then $R : G \to \mathcal{F}(H, L)$ is a $\mathcal{T}L$-fuzzy set valued homomorphism.

Proposition 3.5. Let G, H be groups and $R : G \to \mathcal{F}(H, L)$.

(i) Let $x, a \in G$. If $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$, then $R(x)(y)\mathcal{T}R(a)(b) \le R(xa^{-1})(yb^{-1})$ for all $y, b \in H$.

(ii) If $R(e_G)(e_H) = 1$ and $R(x)(y)\mathcal{T}R(a)(b) \le R(xa^{-1})(yb^{-1})$ for all $x, a \in G$ and $y, b \in H$, then $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$.

Proof.

(i) It is straightforward.

(ii) Let $x, a \in G$ and $y, b \in H$. Then $(R(a))^{-1}(b) = R(a)(b^{-1}) = R(e_G)(e_H)\mathcal{T}R(a)(b^{-1}) \le R(a^{-1})(b)$. Hence we have $(R(a))^{-1} \le R(a^{-1})$. Thus $(R(x) \cdot_{\mathcal{T}} R(a))(u) = \bigvee_{u=yb} R(x)(y)\mathcal{T}R(a)(b)$ $= \bigvee_{u=yb} R(x)(y)\mathcal{T}(R(a))^{-1}(b^{-1}) \le \bigvee_{u=yb} R(x)(y)\mathcal{T}R(a^{-1})(b^{-1}) \le \bigvee_{u=yb} R(xa)(yb) = R(xa)(u)$. We obtain that $R(x) \cdot_{\mathcal{T}} R(a) \le R(xa)$. Therefore $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$.

Theorem 3.6. Let G, H and K be groups and $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm on L. If $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$ and $P \in \mathrm{Hom}_{\mathcal{T}}(H, \mathcal{F}(K, L))$, then $P *_{\mathcal{T}} R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(K, L))$.
Proof. Let $x, a \in G$ and $k \in K$. Thus

$$\begin{aligned}
\Big((P *_{\mathcal{T}} R)(x) \cdot_{\mathcal{T}} (P *_{\mathcal{T}} R)(a)\Big)(k) &= \bigvee_{k=yb} (P *_{\mathcal{T}} R)(x)(y)\mathcal{T}(P *_{\mathcal{T}} R)(a)(b) \\
&= \bigvee_{k=yb} \Big(\bigvee_{h_1 \in H} R(x)(h_1)\mathcal{T}P(h_1)(y)\Big)\mathcal{T}\Big(\bigvee_{h_2 \in H} R(a)(h_2)\mathcal{T}P(h_2)(b)\Big) \\
&= \bigvee_{h_1, h_2 \in H} \Big(R(x)(h_1)\mathcal{T}R(a)(h_2)\mathcal{T}(\bigvee_{k=yb} P(h_1)(y)\mathcal{T}P(h_2)(b))\Big) \\
&= \bigvee_{h_1, h_2 \in H} \Big(R(x)(h_1)\mathcal{T}R(a)(h_2)\mathcal{T}(P(h_1) \cdot_{\mathcal{T}} P(h_2))(k)\Big) \\
&\le \bigvee_{h_1, h_2 \in H} \Big((R(x) \cdot_{\mathcal{T}} R(a))(h_1h_2)\mathcal{T}(P(h_1) \cdot_{\mathcal{T}} P(h_2))(k)\Big) \\
&\le \bigvee_{h_1h_2 \in H} R(xa)(h_1h_2)\mathcal{T}P(h_1h_2)(k) = \bigvee_{h \in H} R(xa)(h)\mathcal{T}(P(h)(k) = (P *_{\mathcal{T}} R)(xa)(k). \\
((P *_{\mathcal{T}} R)(x))^{-1}(k) &= (P *_{\mathcal{T}} R)(x)(k^{-1}) = \bigvee_{h \in H} R(x)(h)\mathcal{T}P(h)(k^{-1}) \\
&= \bigvee_{h \in H} (R(x))^{-1}(h^{-1})\mathcal{T}(P(h))^{-1}(k) \le \bigvee_{h \in H} R(x^{-1})(h^{-1})\mathcal{T}P(h^{-1})(k) \\
&= (P *_{\mathcal{T}} R)(x^{-1})(k).
\end{aligned}$$

So $P *_{\mathcal{T}} R$ is a $\mathcal{T}L$-set valued homomorphisms of groups.

Theorem 3.7. Let $f : G \to G'$ and $g : H \to H'$ be homomorphisms of groups.

(1) Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm and $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$. Then $R_{(f,g)} \in \mathrm{Hom}_{\mathcal{T}}(G', \mathcal{F}(H', L))$.

(2) If $P \in \mathrm{Hom}_{\mathcal{T}}(G', \mathcal{F}(H', L))$, then $P^{-1}_{(f,g)} \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$.

(3) If $S \in \mathrm{Hom}_{\mathcal{T}}(G', \mathcal{F}(H, L))$, then $S_f \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$.

Proof.

(1) Let $x', a' \in G'$ and $y' \in H'$. Thus

$$\begin{aligned}
(R_{(f,g)}(x') \cdot_{\mathcal{T}} R_{(f,g)}(a'))(h') &= \bigvee_{h'=y'b'} R_{(f,g)}(x')(y')\mathcal{T} R_{(f,g)}(a')(b') \\
&= \bigvee_{h'=y'b'} \Big(\bigvee_{f(xa)=x'a', g(yb)=y'b'} R(x)(y)\mathcal{T} R(a)(b)\Big) \\
&\leq \bigvee_{h'=y'b'} (\bigvee_{f(xa)=x'a', g(s)=y'b'} (R(x) \cdot_{\mathcal{T}} R(a))(s)) \\
&= (\bigvee_{f(xa)=x'a', g(s)=h'} (R(x) \cdot_{\mathcal{T}} R(a))(s)) \\
&\leq \bigvee_{f(xa)=x'a', g(s)=h'} R(xa)(s) \\
&= R_{(f,g)}(x'a')(h'). \\
(R_{(f,g)}(x'))^{-1}(y') &= (R_{(f,g)}(x'))((y')^{-1}) = \bigvee_{f(x)=x', g(y)=(y')^{-1}} R(x)(y) \\
&= \bigvee_{f(x)=x', g(y^{-1})=y'} R(x)(y) = \bigvee_{f(x)=x', g(y^{-1})=y'} (R(x))^{-1}(y^{-1}) \\
&\leq \bigvee_{f(x)=x', g(y^{-1})=y'} R(x^{-1})(y^{-1}) = \bigvee_{f(x^{-1})=(x')^{-1}, g(y^{-1})=y'} R(x^{-1})(y^{-1}) \\
&= R_{(f,g)}(x')^{-1}(y').
\end{aligned}$$

(2) Let $x, a \in G$ and $h \in H$. Thus

$$\begin{aligned}
(P^{-1}_{(f,g)}(x) \cdot_{\mathcal{T}} P^{-1}_{(f,g)}(a))(h) &= \bigvee_{h=yb} P^{-1}_{(f,g)}(x)(y)\mathcal{T} P^{-1}_{(f,g)}(a)(b) = \bigvee_{h=yb} P(f(x))(g(y))\mathcal{T} P(f(a))(g(b)) \\
&\leq \bigvee_{g(h)=g(y)g(b)} P(f(x))(g(y))\mathcal{T} P(f(a))(g(b)) = (P(f(x)) \cdot_{\mathcal{T}} P(f(a)))(g(h)) \\
&\leq (P(f(xa))(g(h)) = P^{-1}_{(f,g)}(xa)(h). \\
(P^{-1}_{(f,g)}(x))^{-1}(y) &= (P^{-1}_{(f,g)}(x))(y^{-1}) = P(f(x))(g(y^{-1})) = P(f(x))(g(y)^{-1}) \\
&= (P(f(x))^{-1}(g(y)) \leq P(f(x^{-1}))(g(y)) = P^{-1}_{(f,g)}(x^{-1})(y).
\end{aligned}$$

(3) Let $x, a \in G$ and $h \in H$. Thus

$$\begin{aligned}
(S_f(x) \cdot_{\mathcal{T}} S_f(a))(h) &= \bigvee_{h=yb} S_f(x)(y)\mathcal{T} S_f(a)(b) = \bigvee_{h=yb} S(f(x))(y)\mathcal{T} S(f(a))(b) \\
&= (S(f(x)) \cdot_{\mathcal{T}} S(f(a)))(h) \leq S(f(xa))(h) = S_f(xa)(h). \\
(S_f(x))^{-1}(y) &= (S_f(x))(y^{-1}) = S(f(x))(y^{-1}) = (S(f(x))^{-1}(y) \\
&\leq P(f(x^{-1}))(y) = S_f(x^{-1})(y).
\end{aligned}$$

Definition 3.8. Let $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$. Then the kernel and image of R denoted by $\mathrm{Ker}R$ and $\mathrm{Im}R$, respectively are L-fuzzy subsets of G and H, respectively that are defined as $\mathrm{Ker}R(x) = R(x)(e)$ and $\mathrm{Im}R(y) = \bigvee_{x\in G} R(x)(y)$ for all $x \in G$, $y \in H$.

Remark Let $f \in \mathrm{Hom}(G, H)$. Then $\mathrm{Ker}R(f)_{(\alpha,\beta)} = (\alpha, \beta)_{\mathrm{Ker}f}$ and $\mathrm{Im}R(f)_{(\alpha,\beta)} = (\alpha, \beta)_{\mathrm{Im}f}$, where $R(f)_{(\alpha,\beta)}$ is in the

Example 3.4 (2).

Proposition 3.9. Let $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L)), P \in \mathrm{Hom}_{\mathcal{T}}(G', \mathcal{F}(H', L)), f \in \mathrm{Hom}(G, G')$ and $g \in \mathrm{Hom}(H, H')$. Then

(1) $f(\mathrm{Ker}R) \leq \mathrm{Ker}R_{(f,g)}$,

(2) If g is a monomorphism, then $f(\mathrm{Ker}R) = \mathrm{Ker}R_{(f,g)}$,

(3) $\mathrm{Ker}P^{-1}_{(f,g)} = \mathrm{Ker}P_f$,

(4) $\mathrm{Im}R_{(f,g)} \leq g(\mathrm{Im}R)$,

(5) If f is an epimorphism, then $\mathrm{Im}R_{(f,g)} = g(\mathrm{Im}R)$,

(6) $\mathrm{Im}R^{-1}_{(f,g)} = \mathrm{Im}((R_f)^{-1}_g)^{-1}$.

Proof.

(1) Let $x' \in G'$. Then

$$f(\mathrm{Ker}R)(x') = \bigvee_{f(x)=x'} \mathrm{Ker}R(x) = \bigvee_{f(x)=x'} R(x)(e) \leq \bigvee_{f(x)=x', g(y)=e} R(x)(y) = R_{(f,g)}(x')(e) = \mathrm{Ker}R_{(f,g)}(x').$$

(2) Since g is a monomorphism, then it follows immediately from (1).

(3) Let $x \in G$. Then

$$\mathrm{Ker}P^{-1}_{(f,g)}(x) = P^{-1}_{(f,g)}(x)(e) = P(f(x))(g(e)) = P(f(x))(e) = P_f(x)(e) = \mathrm{Ker}P_f(x).$$

(4) Let $y' \in H'$. Then

$$\begin{aligned} \mathrm{Im}R_{(f,g)}(y') &= \bigvee_{x' \in G'} R_{(f,g)}(x')(y') = \bigvee_{x' \in G'} \Big(\bigvee_{f(x)=x', g(y)=y'} R(x)(y) \Big) \\ &\leq \bigvee_{g(y)=y'} \Big(\bigvee_{x \in G} R(x)(y) \Big) = \bigvee_{g(y)=y'} \mathrm{Im}R(y) = g(\mathrm{Im}R)(y'). \end{aligned}$$

(5) Since g is an epimorphism, then it follows immediately from (4).

(6) Let $y \in H$. Then

$$\begin{aligned} \mathrm{Im}R^{-1}_{(f,g)}(y) &= \bigvee_{x \in G} R^{-1}_{(f,g)}(x)(y) = \bigvee_{x \in G} R(f(x))(g(y)) = \bigvee_{x \in G} R_f(x)(g(y)) \\ &= \bigvee_{x \in G} (R_f)^{-1}(g(y))(x) = \bigvee_{x \in G} ((R_f)^{-1}_g)^{-1}(x)(y) = \mathrm{Im}((R_f)^{-1}_g)^{-1}(y). \end{aligned}$$

Proposition 3.10. Let $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$. Then

(1) $\mathrm{Ker}R$ is a $\mathcal{T}L$-fuzzy subgroup of G,

(2) Let $x \in G$. If $R(x) \in \mathcal{F}(H, L)$ and there exists an element $y \in H$ such that $R(x)(y) = 1$, then $\mathrm{Ker}R$ is a normal $\mathcal{T}L$-fuzzy subgroup of G,

(3) If $\mathcal{T}$ is an infinitely $\vee$-distributive t-norm, then $\mathrm{Im}R$ is a $\mathcal{T}L$-fuzzy subgroup of H,

(4) $R(e)$ is a $\mathcal{T}L$-fuzzy subgroup of H.

Proof.

(1) Let $x, a \in G$. Since

$$\begin{aligned}\mathrm{Ker}R(x)\mathcal{T}\mathrm{Ker}R(a) &= R(x)(e)\mathcal{T}R(a)(e) \le (R(x) \cdot_{\mathcal{T}} R(a))(e) \le R(xa)(e) = \mathrm{Ker}R(xa),\\ \mathrm{Ker}R(x) &= R(x)(e) = R(x)(e^{-1}) = (R(x))^{-1}(e) \le R(x^{-1})(e) = \mathrm{Ker}R(x^{-1}).\end{aligned}$$

Then $\mathrm{Ker}R$ is a $\mathcal{T}L$-fuzzy subgroup of G.

(2) Let $x, a \in G$. For each $x \in G$, there exists an element $y \in H$ such that $R(x)(y) = 1$. Hence

$$\begin{aligned}\mathrm{Ker}R(xa) &= R(xa)(e) = 1\mathcal{T}R(xa)(e)\mathcal{T}1 = R(x)(y)\mathcal{T}R(xa)(e)\mathcal{T}R(x)(y)\\ &= (R(x))^{-1}(y^{-1})\mathcal{T}R(xa)(e)\mathcal{T}R(x)(y) \le R(x^{-1})(y^{-1})\mathcal{T}R(xa)(e)\mathcal{T}R(x)(y)\\ &\le \Big((R(x^{-1}) \cdot_{\mathcal{T}} R(xa))(y^{-1})\Big)\mathcal{T}R(x)(y) \le R(a)(y^{-1})\mathcal{T}R(x)(y)\\ &\le (R(a) \cdot_{\mathcal{T}} R(x))(e) \le R(ax)(e) = \mathrm{Ker}R(ax).\end{aligned}$$

Thus $\mathrm{Ker}R$ is a normal $\mathcal{T}L$-fuzzy subgroup of G if for each $x \in G$, there exists an element $y \in H$ such that $R(x)(y) = 1$.

(3) Let $y, b \in H$. Since

$$\begin{aligned}\mathrm{Im}R(y)\mathcal{T}\mathrm{Im}R(b) &= \bigvee_{x\in G} R(x)(y)\mathcal{T}\bigvee_{a\in G} R(a)(b) = \bigvee_{x,a\in G} R(x)(y)\mathcal{T}R(a)(b)\\ &\le \bigvee_{x,a\in G} (R(x) \cdot_{\mathcal{T}} R(a))(yb) \le \bigvee_{t\in G} R(t)(yb) = \mathrm{Im}R(yb),\\ \mathrm{Im}R(x) &= \bigvee_{x\in G} R(x)(y) = \bigvee_{x\in G}(R(x))^{-1}(y^{-1}) \le \bigvee_{x\in G} R(x^{-1})(y^{-1}) = \mathrm{Im}R(y^{-1}).\end{aligned}$$

Then $\mathrm{Im}R$ is a $\mathcal{T}L$-fuzzy subgroup of H if $\mathcal{T}$ is an infinitely $\vee$-distributive t-norm,

(4) Let $y, b \in H$. Since

$$\begin{aligned}R(e)(y)\mathcal{T}R(e)(b) &\le (R(e) \cdot_{\mathcal{T}} R(e))(yb) \le R(e)(yb),\\ R(e)(y) &= R(e)^{-1}(y^{-1}) \le R(e)(y^{-1}).\end{aligned}$$

Then $R(e)$ is a $\mathcal{T}L$-fuzzy subgroup of H.

Theorem 3.11. Let μ be a normal TL-fuzzy subgroup of G and $R : G \to \mathcal{F}(G/\mu, L)$ be defined by $R(x)(y\mu) = \mu(xy^{-1})$ for all $x, y \in G$. Then $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(G/\mu, L))$ and $\mathrm{Ker}R = \mu$.
Proof. Let $x, a \in G$ and $y\mu, b\mu \in G/\mu$. Then

$$\begin{aligned}(R(a) \cdot_{\mathcal{T}} R(b))(u\mu) &= \bigvee_{u\mu=(y\mu)(b\mu)} R(x)(y\mu)\mathcal{T}R(a)(b\mu) = \bigvee_{u\mu=(y\mu)(b\mu)} \mu(xy^{-1})\mathcal{T}\mu(ab^{-1}) = \bigvee_{u\mu=(y\mu)(b\mu)} \mu(y^{-1}x)\mathcal{T}\mu(ab^{-1})\\ &\le \bigvee_{u\mu=(y\mu)(b\mu)} \mu(y^{-1}xab^{-1}) = \bigvee_{u\mu=(y\mu)(b\mu)} \mu(xab^{-1}y^{-1}) = \bigvee_{u\mu=(y\mu)(b\mu)} yb\mu(xa)\\ &= \bigvee_{u\mu=(y\mu)(b\mu)} R(xa)(yb\mu) = \bigvee_{u\mu=(y\mu)(b\mu)} R(xa)((y\mu)(b\mu)) = \bigvee_{u\mu=(y\mu)(b\mu)} R(xa)(u\mu)).\\ (R(x))^{-1}(y\mu) &= R(x)((y\mu)^{-1}) = \mu(x^{-1}y^{-1}) = R(x^{-1})(y\mu).\end{aligned}$$

Therefore $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(G/\mu, L))$. Since $\mathrm{Ker}R(x) = R(x)(\mu) = \mu(x)$ for all $x \in G$, then $\mathrm{Ker}R = \mu$.

4. $\mathcal{I}$-Lower and $\mathcal{T}$-Upper Fuzzy Rough Approximations with Respect to $\mathcal{T}L$-Fuzzy Set Valued Homomorphism of Groups

In this section, some properties of the $\mathcal{I}$-lower and $\mathcal{T}$-upper fuzzy rough approximations constructed on two different groups by using a $\mathcal{T}L$-fuzzy relational morphism are investigated.

Theorem 4.1. Let $\mu, \nu \in \mathcal{F}(H, L)$ and $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm on L. If $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$, then $\overline{R}^{\mathcal{T}}(\mu) \cdot_{\mathcal{T}} \overline{R}^{\mathcal{T}}(\nu) \leq \overline{R}^{\mathcal{T}}(\mu \cdot_{\mathcal{T}} \nu)$.

Proof. Let $x \in G$. Thus

$$
\begin{aligned}
\Big(\overline{R}^{\mathcal{T}}(\mu) \cdot_{\mathcal{T}} \overline{R}^{\mathcal{T}}(\nu)\Big)(x) &= \bigvee_{x=ab} \overline{R}^{\mathcal{T}}(\mu)(a)\mathcal{T}\overline{R}^{\mathcal{T}}(\nu)(b) \\
&= \bigvee_{x=ab} \Big(\bigvee_{k\in H} R(a)(k)\mathcal{T}\mu(k)\Big)\mathcal{T}\Big(\bigvee_{t\in H} R(b)(t)\mathcal{T}\nu(t)\Big) \\
&= \bigvee_{x=ab} \bigvee_{k,t\in H} (R(a)(k)\mathcal{T}R(b)(t))\mathcal{T}(\mu(k)\mathcal{T}\nu(t)) \\
&\leq \bigvee_{x=ab} \bigvee_{p\in H} (R(a) \cdot_{\mathcal{T}} R(b))(p)\mathcal{T}(\mu \cdot_{\mathcal{T}} \nu)(p) \\
&\leq \bigvee_{x=ab} \bigvee_{p\in H} R(ab)(p)\mathcal{T}(\mu \cdot_{\mathcal{T}} \nu)(p) \\
&= \bigvee_{p\in H} R(x)(p)\mathcal{T}(\mu \cdot_{\mathcal{T}} \nu)(p) \\
&= \overline{R}^{\mathcal{T}}(\mu \cdot_{\mathcal{T}} \nu)(x).
\end{aligned}
$$

So we have $\overline{R}^{\mathcal{T}}(\mu) \cdot_{\mathcal{T}} \overline{R}^{\mathcal{T}}(\nu) \leq \overline{R}^{\mathcal{T}}(\mu \cdot_{\mathcal{T}} \nu)$.

The following example shows that Theorem 4.1 may not be true for the I-lower fuzzy rough approximation of μ.

Example 4.2. Let L be the lattice which is given in Figure 2 and $\mathcal{T} = \wedge$. Let $\mu, \nu \in \mathcal{F}(\mathbb{Z}_3, L)$ and $R : \mathbb{Z}_4 \to \mathcal{F}(\mathbb{Z}_3, L)$ be defined by

$$
\mu(x) = \begin{cases} 1, & \text{if } x = \overline{0}; \\ \alpha, & \text{if } x = \overline{1}; \\ \alpha, & \text{if } x = \overline{2}. \end{cases} \qquad
\nu(x) = \begin{cases} \beta, & \text{if } x = \overline{0}; \\ 0, & \text{if } x = \overline{1}; \\ 0, & \text{if } x = \overline{2}. \end{cases} \qquad
R(x)(y) = \begin{cases} 1, & \text{if } x = \overline{0}; \\ 0, & \text{if } x = \overline{1}; \\ \beta, & \text{if } x = \overline{2}; \\ 0, & \text{if } x = \overline{3}. \end{cases}
$$

Then $R \in \mathrm{Hom}_{\mathcal{T}}(\mathbb{Z}_4, \mathcal{F}(\mathbb{Z}_3, L))$. Let $\mathcal{I}$ be the residual implication of $\mathcal{T}$. Then we obtain that

$$
(\underline{R}_{\mathcal{I}}(\mu) \cdot_{\mathcal{T}} \underline{R}_{\mathcal{I}}(\nu))(x) = \begin{cases} 1, & \text{if } x = \overline{0}; \\ \alpha, & \text{if } x = \overline{1}; \\ 1, & \text{if } x = \overline{2}; \\ \alpha, & \text{if } x = \overline{3}. \end{cases} \qquad
\underline{R}_{\mathcal{I}}(\mu \cdot_{\mathcal{T}} \nu)(x) = \begin{cases} 0, & \text{if } x = \overline{0}; \\ 1, & \text{if } x = \overline{1}; \\ \alpha, & \text{if } x = \overline{2}; \\ 1, & \text{if } x = \overline{3}. \end{cases}
$$

Therefore $\underline{R}_{\mathcal{I}}(\mu) \cdot_{\mathcal{T}} \underline{R}_{\mathcal{I}}(\nu) \nleq \underline{R}_{\mathcal{I}}(\mu \cdot_{\mathcal{T}} \nu)$ and $\underline{R}_{\mathcal{I}}(\mu \cdot_{\mathcal{T}} \nu) \nleq \underline{R}_{\mathcal{I}}(\mu) \cdot_{\mathcal{T}} \underline{R}_{\mathcal{I}}(\nu)$.

Theorem 4.3. Let μ be a $\mathcal{T}L$-fuzzy subgroup of H and $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm on L. If $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$, then $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy subgroup of G.

Proof. Let x, y any elements in G. Thus

$$\begin{aligned}
\overline{R}^{\mathcal{T}}(\mu)(x)\mathcal{T}\overline{R}^{\mathcal{T}}(\mu)(y) &= \Big(\bigvee_{k\in H} R(x)(k)\mathcal{T}\mu(k)\Big)\mathcal{T}\Big(\bigvee_{t\in H} R(y)(t)\mathcal{T}\mu(t)\Big)\\
&= \bigvee_{k\in H}\bigvee_{t\in H}(R(x)(k)\mathcal{T}\mu(k)\mathcal{T}R(y)(t)\mathcal{T}\mu(t))\\
&\le \bigvee_{k,t\in H}(R(x)\cdot_{\mathcal{T}}R(y))(kt)\mathcal{T}\mu(kt)) \le \bigvee_{p\in H} R(xy)(p)\mathcal{T}\mu(p)\\
&= \overline{R}^{\mathcal{T}}(\mu)(xy).\\
\overline{R}^{\mathcal{T}}(\mu)(x) &= \bigvee_{k\in H} R(x)(k)\mathcal{T}\mu(k) = \bigvee_{k\in H}(R(x))^{-1}(k^{-1})\mathcal{T}\mu(k)\\
&\le \bigvee_{k\in H} R(x^{-1})(k^{-1})\mathcal{T}\mu(k^{-1}) = \overline{R}^{\mathcal{T}}(\mu)(x^{-1}).
\end{aligned}$$

So $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy subgroup of G.

The following example shows that $\overline{R}^{\mathcal{T}}(\mu)$ in the Theorem 4.3 may not a normal $\mathcal{T}L$-fuzzy subgroup of G.

Example 4.4. The set of all of the 2×2 invertible matrices over the field of real numbers, $GL_2(\mathbb{R})$, forms a group under ordinary matrix multiplication. It is easy to see that $H = \{\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix} \mid x \neq 0\}$ is a subgroup of $GL_2(\mathbb{R})$. Let $L = \{0, \alpha, \beta, \gamma, \delta, 1\}$ be a lattice whose Hasse diagram depicted as follows:

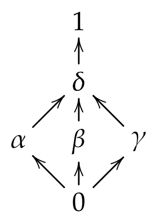

Figure 3: Lattice L

$\mathcal{T}_D$ is an infinitely $\vee$-distributive t-norm on L. Let $R : H \to \mathcal{F}(\mathbb{Z}, L)$ be defined by

$$R(\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix})(z) = \begin{cases} 1, & \text{if } y = 0;\\ \delta, & \text{if } y \neq 0;\end{cases} \quad , \qquad \forall \begin{pmatrix} x & y \\ 0 & 1\end{pmatrix} \in H, z \in \mathbb{Z}.$$

Then R is a $\mathcal{T}L$-fuzzy set valued homomorphisms. Let a normal $\mathcal{T}L$-fuzzy subgroup μ of $\mathbb{Z}$ be defined by

$$\mu(z) = \begin{cases} \delta, & \text{if } z \text{ is even};\\ \beta, & \text{if } z \text{ is odd}.\end{cases} \quad , \qquad \forall z \in \mathbb{Z}.$$

Thus we obtain that

$$\overline{R}^{\mathcal{T}}(\mu)(\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix}) = \begin{cases} \delta, & \text{if } y = 0;\\ 0, & \text{if } y \neq 0;\end{cases} \quad , \qquad \forall \begin{pmatrix} x & y \\ 0 & 1\end{pmatrix} \in H.$$

Since we have that

$$\overline{R}^{\mathcal{T}}(\mu)(\begin{pmatrix} 2 & -2 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 1\end{pmatrix}) = \overline{R}^{\mathcal{T}}(\mu)(\begin{pmatrix} 6 & 0 \\ 0 & 1\end{pmatrix}) = \delta$$

and

$$\overline{R}^{\mathcal{T}}(\mu)(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}) = \overline{R}^{\mathcal{T}}(\mu)(\begin{pmatrix} 6 & -5 \\ 0 & 1 \end{pmatrix}) = 0,$$

then $\overline{R}^{\mathcal{T}}(\mu)$ is not a normal $\mathcal{T}L$-fuzzy subgroup of G.

The following example shows that Theorem 4.3 may not be true for the I-lower fuzzy rough approximation of μ.

Example 4.5. Let $\mathcal{T} = \wedge$ and let $\mu \in \mathcal{F}(\mathbb{Z}, [0,1])$ and $R : \mathbb{Z} \to \mathcal{F}(\mathbb{Z}, [0,1])$ be defined by

$$\mu(x) = \begin{cases} \frac{1}{2}, & \text{if } x \text{ is even;} \\ \frac{1}{5}, & \text{if } x \text{ is odd,} \end{cases} \qquad R(x)(y) = \begin{cases} \frac{1}{3}, & \text{if } x \text{ is even;} \\ \frac{1}{7}, & \text{if } x \text{ is odd} \end{cases}$$

,respectively. Then $R \in \mathrm{Hom}_{\mathcal{T}}(\mathbb{Z}, \mathcal{F}(\mathbb{Z}, [0,1]))$. Let $\mathcal{I}$ be the residual implication of $\mathcal{T}$. Then we obtain that

$$\underline{R}_{\mathcal{I}}(\mu)(x) = \begin{cases} \frac{1}{5}, & \text{if } x \text{ is even;} \\ 1, & \text{if } x \text{ is odd.} \end{cases}$$

Since $\underline{R}_{\mathcal{I}}(\mu)(1)\mathcal{T}\underline{R}_{\mathcal{I}}(\mu)(1) = 1\mathcal{T}1 = 1 \nleq \frac{1}{5} = \underline{R}_{\mathcal{I}}(\mu)(2)$. Then $\underline{R}_{\mathcal{I}}(\mu)$ is not a $\mathcal{T}L$-fuzzy subgroup of $\mathbb{Z}$. **Theorem 4.6.** Let $R \in \mathrm{Hom}_{\mathcal{T}}(G, \mathcal{F}(H, L))$ and $P \in \mathrm{Hom}_{\mathcal{T}}(H, \mathcal{F}(K, L))$. Then

(1) $\overline{R}(R(x))(a) \leq \mathrm{Ker}R(ax^{-1})$ for all $x, a \in G$,

(2) $\overline{R}(\mathrm{Ker}P) = \mathrm{Ker}(P \cdot_{\mathcal{T}} R)$,

(3) $\underline{R}(\mathrm{Ker}P) = \mathrm{Ker}(P \cdot_{\mathcal{I}} R)$,

(4) $\overline{P^{-1}}(\mathrm{Ker}R^{-1}) = \mathrm{Ker}(P \cdot_{\mathcal{T}} R)^{-1}$,

(5) $\underline{P^{-1}}(\mathrm{Ker}R^{-1}) = \mathrm{Ker}(R^{-1} \cdot_{\mathcal{I}} P^{-1})$,

(6) If $\mathcal{T}$ is an infinitely $\vee$-distributive t-norm, then $\overline{R}(\mathrm{Im}P^{-1}) = \mathrm{Im}(P \cdot_{\mathcal{T}} R)^{-1}$,

(7) If $\mathcal{T}$ is an infinitely $\vee$-distributive t-norm, then $\overline{P^{-1}}(\mathrm{Im}R) = \mathrm{Im}P \cdot_{\mathcal{T}} R$,

(8) $\underline{R}(\mathrm{Im}R^{-1}) \geq \mathrm{Im}(P \cdot_{\mathcal{I}} R)^{-1}$,

(9) $\underline{P^{-1}}(\mathrm{Im}R) \geq \mathrm{Im}(P^{-1} \cdot_{\mathcal{I}} R^{-1})^{-1}$.

Proof.

(1) Let $x, a \in G$. Then

$$\begin{aligned} \overline{R}(R(x))(a) &= \bigvee_{b\in H} R(a)(b)\mathcal{T}R(x)(b) = \bigvee_{b\in H} R(a)(b)\mathcal{T}(R(x))^{-1}(b^{-1}) \leq \bigvee_{b\in H} R(a)(b)\mathcal{T}R(x^{-1})(b^{-1}) \\ &= (R(a) \cdot_{\mathcal{T}} R(x^{-1}))(e) \leq R(ax^{-1})(e) = \mathrm{Ker}R(ax^{-1}). \end{aligned}$$

(2) Let $x \in G$. Then

$$\overline{R}(\mathrm{Ker}P)(x) = \bigvee_{h\in H} R(x)(h)\mathcal{T}\mathrm{Ker}P(h) = \bigvee_{h\in H} R(x)(h)\mathcal{T}P(h)(e) = (P \cdot_{\mathcal{T}} R)(x)(e) = \mathrm{Ker}(P \cdot_{\mathcal{T}} R).$$

(3) Let $x \in G$. Then

$$\underline{R}(\mathrm{Ker}P)(x) = \bigwedge_{h\in H} R(x)(h)\mathcal{I}\mathrm{Ker}P(h) = \bigwedge_{h\in H} R(x)(h)\mathcal{I}P(h)(e) = (P \cdot_{\mathcal{I}} R)(x)(e) = \mathrm{Ker}(P \cdot_{\mathcal{I}} R)(x)$$

(4) Let $k \in K$. Then

$$\begin{aligned}\overline{P^{-1}}(\mathrm{Ker}R^{-1})(k) &= \bigvee_{h\in H} P^{-1}(k)(h)\mathcal{T}\mathrm{Ker}R^{-1}(h) = \bigvee_{h\in H} P(h)(k)\mathcal{T}R^{-1}(h)(e) \\ &= \bigvee_{h\in H} P(h)(k)\mathcal{T}R(e)(h) = (P\cdot_{\mathcal{T}} R)(e)(k) = \mathrm{Ker}(P\cdot_{\mathcal{T}} R)^{-1}(k)\end{aligned}$$

(5) Let $k \in K$. Then

$$\underline{P^{-1}}(\mathrm{Ker}R^{-1})(k) = \bigwedge_{h\in H} P^{-1}(k)(h)\mathcal{I}\mathrm{Ker}R^{-1}(h) = \bigwedge_{h\in H} P^{-1}(k)(h)\mathcal{I}R^{-1}(h)(e) = \mathrm{Ker}(R^{-1}\cdot_{\mathcal{I}} P^{-1})(k)$$

(6) Let $x \in G$. Then

$$\begin{aligned}\overline{R}(\mathrm{Im}P^{-1})(x) &= \bigvee_{h\in H} R(x)(h)\mathcal{T}\mathrm{Im}P^{-1}(h) = \bigvee_{h\in H} R(x)(h)\mathcal{T}(\bigvee_{k\in K} P^{-1}(k)(h)) = \bigvee_{h\in H,k\in K} R(x)(h)\mathcal{I}P(h)(k) \\ &= \bigvee_{k\in H}\bigvee_{h\in H} R(x)(h)\mathcal{I}P(h)(k) = \bigvee_{k\in K}(P\cdot_{\mathcal{T}} R)(x)(k) = \mathrm{Im}(P\cdot_{\mathcal{T}} R)^{-1}(k)\end{aligned}$$

(7) Let $k \in K$. Then

$$\begin{aligned}\overline{P^{-1}}(\mathrm{Im}R)(k) &= \bigvee_{h\in H} P^{-1}(k)(h)\mathcal{T}\mathrm{Im}R(h) = \bigvee_{h\in H} P(h)(k)\mathcal{T}(\bigvee_{g\in G} R(g)(h)) = \bigvee_{h\in H,g\in G} R(g)(h)\mathcal{I}P(h)(k) \\ &= \bigvee_{g\in G}\bigvee_{h\in H} R(x)(h)\mathcal{I}P(h)(k) = \bigvee_{g\in H}(P\cdot_{\mathcal{T}} R)(g)(k) = \mathrm{Im}(P\cdot_{\mathcal{T}} R)(k)\end{aligned}$$

(8) Let $x \in G$. Then

$$\begin{aligned}\underline{R}(\mathrm{Im}R^{-1})(x) &= \bigwedge_{h\in H} R(x)(h)\mathcal{I}\mathrm{Im}P^{-1}(h) = \bigwedge_{h\in H} R(x)(h)\mathcal{I}(\bigvee_{k\in K} P^{-1}(k)(h)) = \bigwedge_{h\in H} R(x)(h)\mathcal{I}(\bigvee_{k\in K} P(h)(k)) \\ &= \bigvee_{h\in H}(\bigvee_{k\in K} R(x)(h)\mathcal{I}P(h)(k)) \geq \bigvee_{k\in K}(\bigvee_{h\in H} R(x)(h)\mathcal{I}P(h)(k)) = \bigvee_{k\in K}(P\cdot_{\mathcal{T}} R)(x)(k) = \mathrm{Im}(P\cdot_{\mathcal{T}} R)^{-1}(k)\end{aligned}$$

(9) Let $k \in K$. Then

$$\begin{aligned}\underline{P^{-1}}(\mathrm{Im}R)(k) &= \bigwedge_{h\in H} P^{-1}(k)(h)\mathcal{I}\mathrm{Im}R(h) = \bigwedge_{h\in H} P^{-1}(k)(h)\mathcal{I}(\bigvee_{g\in G} R(g)(h)) \\ &\geq \bigwedge_{h\in H}(\bigvee_{g\in G} P^{-1}(k)(h)\mathcal{I}R^{-1}(h)(g)) \geq \bigvee_{g\in G}(\bigwedge_{h\in H} P^{-1}(k)(h)\mathcal{I}R^{-1}(h)(g)) \\ &= \bigvee_{g\in G}(R^{-1}\cdot_{\mathcal{I}} P^{-1})(k)(g) = \bigvee_{g\in G}(R^{-1}\cdot_{\mathcal{I}} P^{-1})^{-1}(g)(k) = \mathrm{Im}(R^{-1}\cdot_{\mathcal{I}} P^{-1})^{-1}(k)\end{aligned}$$

5. Conclusions

The generalized rough sets on algebraic sets such as group, ring, and module were mainly studied by set valued homomorphism [5, 10, 27, 28]. It is an interesting topic to research the generalized $(\mathcal{I},\mathcal{T})$-$L$-fuzzy rough sets which are constructed on two different groups instead of the universe of discourse. In this paper, we gave a definition for the fuzzification of set valued homomorphism to provide opportunity putting reasonable interpretations and explored the features of generalized $(\mathcal{I},\mathcal{T})$-$L$-fuzzy rough sets.

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