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Scalarized Solutions of Set-Valued Optimization Problems and Generalized Variational-like Inequalities

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Abstract. In this paper by using the scalariation method we introduced the concept of relaxed K-preinvex set-valued maps and obtain some equivalence results of them in terms of normal subdifferential. Also, we consider generalized Minty variational-like inequalities and show that the set of solutions is equal to scalarized set-valued optimization problems's solutions under generalized relaxed convexity assumptions.

1. Introduction

Variational inequalities are identified either in the form presented by Stampacchia [18] or in the form by Minty [11]. The concept of vector variational inequality, which was first introduced by Giannessi [8] for differentiable functions in finite dimensional spaces, has many applications in problems such that economics, finance, optimization and operational research. In recent years, various kinds of variational inequalities and optimization problems have been studied in a general setting by many authors; see, e.g. [1–3, 16, 17, 19]. By using the Clarke's generalized directional derivative, Santos et al. [19] considered scalarized variational-like inequalities and showed that the set of their solution is equal to weak efficient solution set. Afterward, Alshahrani et al. [2] extended results in [19] and obtained some existence results for solutions of nonsmooth variational-like inequalities under densely pseudomonotonicity. Very recently, Oveisiha and Zafarani [16] extended results in [2] to set-valued optimization problems and prove some characterization of the solution sets of pseudoinvex extremum problems.
In this paper, we introduce a relaxed K-preinvex set-valued map which extends and unifies the concepts of (strong) K-preinvexity for set-valued maps and (strong) preinvexity for vector-valued functions as well as classical strong convexity for real-valued functions in the literature. Because there are many examples of set-valued optimization problems that their solutions are not a solution of standard Minty variational-like inequality (e.g. Example 4.3), by a modification, we obtain generalized Minty variational-like inequality, that its solution set is larger than the solution set of Minty variational-like inequality. The paper is organized as follows: Section 1 prepares the notions and preliminary results used in the sequel. In section 2, some properties of relaxed K-preinvex maps in terms of normal subdifferential are established. Section 3 is devoted to obtain an equivalence relation between set-valued optimization problems and a generalization concept of Minty variational inequalities. Finally, in section 4, some conclusions are presented, which summarize this work.

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2. Preliminaries

Let X be a Banach space and X^* be its topological dual space. The norm in X and X^* will be denoted by $\|.\|$. We denote by $\langle .,.\rangle, [x,y]$ and $]x,y[$ the dual pair between X and X^*, the line segment for $x,y\in X$, and the interior of $[x,y]$, respectively. Also, suppose that B_X and S_X to be the closed unit ball and unit sphere of X, respectively. Now, we recall some concepts of subdifferentials and coderivatives that we need in next sections.

Definition 2.1. *[13] Let X be a Banach space, Ω be a nonempty subset of X, $x\in\Omega$ and $\varepsilon\geq 0$. The set of ε-normals to Ω at x is*

$$\widehat{N}_\varepsilon(x;\Omega) := \{x^*\in X^* | \limsup_{u\xrightarrow{\Omega}x} \frac{\langle x^*, u-x\rangle}{\|u-x\|}\leq\varepsilon\}.$$

If $\varepsilon=0$, the above set is denoted by $\widehat{N}(x;\Omega)$ and called regular normal cone to Ω at x. Let $\bar{x}\in\Omega$, the basic normal cone to Ω at $\bar{x}$ is

$$N(\bar{x};\Omega) := \mathrm{Limsup}_{x\to\bar{x},\varepsilon\downarrow 0}\widehat{N}_\varepsilon(x;\Omega).$$

Definition 2.2. *[13] Let X be a Banach space and $\varphi : X\to\bar{\mathbb{R}}$ be finite at $\bar{x}\in X$. The basic (limiting, Mordukhovich) subdifferential due to [13] of φ at $\bar{x}$ is defined by*

$$\partial\varphi(\bar{x}) := \{x^*\in X^* | (x^*,-1)\in N((\bar{x},\varphi(\bar{x})); \mathrm{epi}\varphi)\}.$$

Mean-value Theorems are important and useful tools in nonsmooth analysis. We here present a mean value theorem for limiting subdifferential.

Theorem 2.3. [13] *Let X be an Asplund space (i.e., every continuous convex function defined on X is Fréchet differentiable on a dense set of points) and φ be Lipschitz continuous on an open set containing $[a,b]$ in X. Then one has*

$$\langle x^*, b-a\rangle\geq\varphi(b)-\varphi(a),\quad \text{for some } x^*\in\partial\varphi(c);\ c\in[a,b[.$$

Given a set-valued mapping $F : X\rightrightarrows Y$ between Banach spaces with the range space Y partially ordered by a nonempty, closed and convex cone K. Denoting the ordering relation on Y by "$\leq$", we have

$$y_1\leq y_2 \quad\text{if and only if}\quad y_2-y_1\in K.$$

Now, we present some definitions and results about coderivatives and subdifferentials of set-valued mappings.

Definition 2.4. *[13] Let $F : X\rightrightarrows Y$ be a set-valued mapping between Banach spaces and $(\bar{x},\bar{y})\in\mathrm{gr}F$. Then, the normal coderivative of F at $(\bar{x},\bar{y})$ is the set-valued mapping $D^*_N F(\bar{x},\bar{y}) : Y^*\rightrightarrows X^*$ given by*

$$D^*_N F(\bar{x},\bar{y})(y^*) := \{x^*\in X^* | (x^*,-y^*)\in N((\bar{x},\bar{y}); \mathrm{gr}F)\}.$$

Definition 2.5. *[4] Let $F : X\rightrightarrows Y$ be a set-valued mapping. Then, the epigraphical multifunction $\mathcal{E}_F : X\rightrightarrows Y$ is defined by*

$$\mathcal{E}_F(x) := \{y\in Y | y\in F(x)+K\}.$$

The normal subdifferentials of F at the point $(\bar{x},\bar{y})\in\mathrm{epi}F$ in the direction $y^*\in Y^*$ is defined by $\partial F(\bar{x},\bar{y})(y^*) := D^*_N\mathcal{E}_F(\bar{x},\bar{y})(y^*)$.

Definition 2.6. *[13] Let $F : \Omega\subset X\rightrightarrows Y$ with $\mathrm{dom}F\neq\emptyset$.*

(i) F is said to be Lipschitz around $\bar{x}\in\mathrm{dom}F$ iff there are a neighborhood U of $\bar{x}$ and $\ell\geq 0$ such that

$$F(x)\subset F(u)+\ell\|x-u\|B_Y,\quad \text{for all } x,u\in\Omega\cap U.$$

(ii) F is said to be epi-Lipschitz around $\bar{x} \in \mathrm{dom}F$ iff $\mathcal{E}_F$ is Lipschitz around this point.

Let K be a closed, convex and pointed cone in Y and denote the positive polar cone of K by

$$K^+ := \{y^* \in Y^* | \langle y^*, k\rangle \geq 0,\ \forall k \in K\}.$$

The next object is the marginal function associated with a set-valued mapping. Given $F : X \rightrightarrows Y$ and $y^* \in Y^*$. We associate to F and y^* a marginal function $f_{y^*} : X \to \mathbb{R} \cup \{\pm\infty\}$

$$f_{y^*}(x) := \inf\{y^*(y) | y \in F(x)\},$$

and the minimum set

$$M_{y^*}(x) := \{y \in F(x) | f_{y^*}(x) = y^*(y)\}.$$

Throughout this paper, we suppose that $\mathrm{gr}F$ is closed, and for all $x \in \mathrm{dom}F$ and $y^* \in K^+$, $M_{y^*}(x)$ is nonempty.

Lemma 2.7. [15] *Suppose that $F : \Omega \subset X \rightrightarrows Y$ is a set-valued map and $\bar{x} \in \mathrm{dom}F$. If F is epi-Lipschitz around $\bar{x}$ and $y^* \in K^+$, then the scalar-function f_{y^*} is locally Lipschitz at $\bar{x}$.*

The next theorem gives some relations between normal subdifferential and normal coderivative of F and limiting subdifferential of its marginal functions. (see also Theorem 3.4 and Corollary 3.5 in [15])

Theorem 2.8. [15] *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ and $y^* \in K^+$. Suppose that $\bar{x} \in \mathrm{dom}F$ and $\bar{y} \in M_{y^*}(\bar{x})$.*

(i) If F is Lipschitz around $\bar{x}$, then $\partial f_{y^*}(\bar{x}) \subseteq D^*_N F(\bar{x}, \bar{y})(y^*)$.

(ii) If F is epi-Lipschitz around $\bar{x}$, then $\partial f_{y^*}(\bar{x}) \subseteq \partial F(\bar{x}, \bar{y})(y^*)$.

Definition 2.9. *[21] Let $\eta : X \times X \to X$. A subset Ω of X is said to be invex with respect to η if for any $x, y \in \Omega$ and $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in \Omega$.*

The following conditions are useful in the sequel.
Condition *A*.[10] A mapping $F : \Omega \subset X \rightrightarrows Y$ from an invex set Ω with respect to η to an ordered Banach space is said to enjoy Condition A if

$$F(x_1) \subset F(x_2 + \eta(x_1, x_2)) + K, \qquad \text{for all } x_1, x_2 \in \Omega.$$

Condition *C*.[12] Let $\eta{:}X \times X \to X$. Then for any $x, y \in X, \lambda \in [0, 1]$

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y); \qquad \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Remark 2.10. *By some computation, we can see that if Condition C holds, then for any $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$*

$$\eta(x_1 + \lambda_1\eta(x_2, x_1), x_1 + \lambda_2\eta(x_2, x_1)) = (\lambda_1 - \lambda_2)\eta(x_2, x_1).$$

Let Ω be a convex subset of a vector space X. Then a mapping $F : \Omega \rightrightarrows \Omega$ is called a KKM mapping iff for each nonempty finite subset A of Ω, conv(A) $\subseteq F(A)$, where conv(A) denotes the convex hull of A, and $F(A) = \bigcup\{F(x) | x \in A\}$.

Lemma 2.11. *(see e.g. [7]) Let Ω be a nonempty and convex subset of a Hausdorff topological vector space X. Suppose that $\Gamma, \widehat{\Gamma} : \Omega \rightrightarrows \Omega$ are two set-valued mappings such that the following conditions are satisfied:*

(A1) $\widehat{\Gamma}(x) \subseteq \Gamma(x), \quad \forall x \in \Omega,$
(A2) *$\widehat{\Gamma}$ is a KKM map,*
(A3) *Γ is closed-valued,*
(A4) *there is a nonempty compact convex set $B \subseteq \Omega$, such that $\mathrm{cl}_\Omega(\bigcap_{x \in B}\Gamma(x))$ is compact.*

Then, $\bigcap_{x \in \Omega}\Gamma(x) \neq \emptyset$.

3. Relaxed K-Preinvex Set-Valued Maps

In this section, we study the concept of relaxed K-preinvex maps and obtain some equivalence results about them.

Definition 3.1. *Let $\Omega \subset X$ be an invex set with respect to η and $F : \Omega \subset X \rightrightarrows Y$.*

(i) F is said to be relaxed K-preinvex with respect to η on Ω, if there exists a constant α and $e \in intK$ such that for any $x_1, x_2 \in \Omega$ and $\lambda \in [0,1]$, one has

$$\lambda F(x_1) + (1-\lambda)F(x_2) - \alpha\lambda(1-\lambda) \parallel \eta(x_1,x_2) \parallel^2 e \subset F(x_2 + \lambda\eta(x_1,x_2)) + K,$$

(ii) F is said to be relaxed K-invex with respect to η on Ω, if there exists a constant α such that for any $y^* \in K^+ \cap S_{Y^*}, x_i \in \Omega, y_i \in M_{y^*}(x_i), (i = 1,2)$ and $\xi \in \partial F(x_1,y_1)(y^*)$, one has

$$\langle \xi, \eta(x_2,x_1)\rangle \leq y^*(y_2) - y^*(y_1) - \alpha \parallel \eta(x_2,x_1) \parallel^2,$$

(iii) F is said to be weakly relaxed K-invex with respect to η on Ω, if there exists a constant α such that for any $y^* \in K^+ \cap S_{Y^*}, x_i \in \Omega, y_i \in M_{y^*}(x_i), (i = 1,2)$ there exists $\xi \in \partial F(x_1,y_1)(y^*)$, such that

$$\langle \xi, \eta(x_2,x_1)\rangle \leq y^*(y_2) - y^*(y_1) - \alpha \parallel \eta(x_2,x_1) \parallel^2,$$

(iv) The set-valued map $\partial F : X \times Y \times Y^* \rightrightarrows X^*$ is said to be invariant relaxed K-monotone on Ω with respect to η, if there exists a constant α such that for any $y^* \in K^+ \cap S_{Y^*}, x_i \in \Omega, y_i \in M_{y^*}(x_i)$ and $\xi_i \in \partial F(x_i,y_i)(y^*), (i = 1,2)$, one has

$$\langle \xi_1, \eta(x_2,x_1)\rangle + \langle \xi_2, \eta(x_1,x_2)\rangle \leq -\alpha(\parallel \eta(x_2,x_1) \parallel^2 + \parallel \eta(x_1,x_2) \parallel^2).$$

Remark 3.2. *(i) If $\alpha = 0$, then the above definition reduces to Definition 3.1 in [15], of K-preinvexity, K-invexity, weak K-invexity and invariant K-monotonicity, respectively, for set-valued maps.*

(ii) If $F = f : X \to \mathbb{R}$ is a real-valued function, then we get definition of relaxed preinveity, relaxed invexity, weak relaxed invexity and invariant relaxed monotonicity, respectively, for real-valued functions, that has been studied in [9, 20], when $\alpha \geq 0$.

Lemma 3.3. *Let $F : X \rightrightarrows Y$ be relaxed K-preinvex with respect to η. Then, for every $y^* \in K^+ \cap S_{Y^*}$, f_{y^*} is relaxed preinvex.*

Proof. The proof deduces easily from Definition 3.1. □

Lemma 3.4. *Let $F : X \rightrightarrows Y$ be relaxed K-invex with respect to η and constant α. Then ∂F is invariant relaxed K-monotone with the same constant.*

Proof. By using the Definition 3.1, we can obtain the proof. □

Theorem 3.5. *Suppose that X, Y are Asplund spaces and $F : X \rightrightarrows Y$ is a locally epi-Lipschitz map satisfying Condition A. If η satisfies Condition C and ∂F is invariant relaxed K-monotone with respect to η, then F is relaxed K-invex.*

Proof. Let ∂F be invariant relaxed K-monotone with respect to η and $x_1, x_2 \in X$. Let $z = x_2 + \frac{1}{2}\eta(x_1,x_2)$ and fix $y^* \in K^+ \cap S_{Y^*}$ be arbitrary. By Lemma 2.7, f_{y^*} is locally Lipschitz. Now, Theorem 2.3 implies that there exist λ_1, λ_2 such that $0 < \lambda_2 \leq \frac{1}{2} < \lambda_1 \leq 1$, $\xi_1 \in \partial f_{y^*}(u_1)$ and $\xi_2 \in \partial f_{y^*}(u_2)$ such that

$$f_{y^*}(x_2 + \eta(x_1,x_2)) - f_{y^*}(z) \geq \frac{1}{2}\langle \xi_1, \eta(x_1,x_2)\rangle, \tag{1}$$

and

$$f_{y^*}(z) - f_{y^*}(x_2) \geq \frac{1}{2}\langle \xi_2, \eta(x_1, x_2)\rangle$$

where $u_1 = x_2 + \lambda_1\eta(x_1, x_2)$ and $u_2 = x_2 + \lambda_2\eta(x_1, x_2)$. By using Corollary 2.8, $\xi_i \in \partial f_{y^*}(u_i) \subseteq \partial F(u_i, z_i)(y^*)$ that $z_i \in M_{y^*}(u_i), (i = 1, 2)$. Since ∂F is invariant relaxed K-monotone, we obtain

$$\langle \xi_1, \eta(x_2, u_1)\rangle + \langle w, \eta(u_1, x_2)\rangle \leq -\alpha(\| \eta(x_2, u_1) \|^2 + \| \eta(u_1, x_2) \|^2) \tag{2}$$

for any $y_2 \in M_{y^*}(x_2)$ and $w \in \partial F(x_2, y_2)(y^*)$. Now, Condition C implies that

$$\eta(u_1, x_2) = \lambda_1\eta(x_1, x_2), \quad \eta(x_2, u_1) = -\lambda_1\eta(x_1, x_2)$$

If we replace these reletions in inequality (2), we get

$$\langle \xi_1, \eta(x_1, x_2)\rangle \geq \langle w, \eta(x_1, x_2)\rangle + 2\alpha\lambda_1 \| \eta(x_1, x_2) \|^2 .$$

Now, by (1), we have

$$f_{y^*}(x_2 + \eta(x_1, x_2)) - f_{y^*}(z) \geq \frac{1}{2}\langle w, \eta(x_1, x_2)\rangle + \alpha\lambda_1 \| \eta(x_1, x_2) \|^2 .$$

In a similar way, we can obtain

$$f_{y^*}(z) - f_{y^*}(x_2) \geq \frac{1}{2}\langle w, \eta(x_1, x_2)\rangle + \alpha\lambda_2 \| \eta(x_1, x_2) \|^2 .$$

By adding the latter two relations, we have

$$f_{y^*}(x_2 + \eta(x_1, x_2)) - f_{y^*}(x_2) \geq \langle w, \eta(x_1, x_2)\rangle + \alpha(\lambda_1 + \lambda_2) \| \eta(x_1, x_2) \|^2 .$$

Since F satisfies Condition A, we deduce that f_{y^*} also satisfies Condition A for real single-valued functions. Hence

$$f_{y^*}(x_1) - f_{y^*}(x_2) \geq \langle w, \eta(x_1, x_2)\rangle + \frac{\alpha}{2} \| \eta(x_1, x_2) \|^2,$$

for any $y_i \in M_{y^*}(x_i), (i = 1, 2)$ and $w \in \partial F(x_2, y_2)(y^*)$. Therefore,

$$y^*(y_1) - y^*(y_2) \geq \langle w, \eta(x_1, x_2)\rangle + \frac{\alpha}{2} \| \eta(x_1, x_2) \|^2,$$

which implies that F is relaxed K-invex. □

Theorem 3.6. *Suppose that $F : X \rightrightarrows Y$ is a locally epi-Lipschitz set-valued map that satisfies Condition A, η satisfies Condition C and $\mathcal{E}_F$ is closed convex-valued. If F is relaxed K-invex with respect to η, then F is relaxed K-preinvex.*

Proof. Suppose that F is relaxed K-invex with constant α_0. By Corollary 2.8, we can easily see that f_{y^*} is relaxed invex for all $y^* \in K^+ \cap S_{Y^*}$ with the same constant. In a similar way of lemma 3.2 in [9], we deduce that f_{y^*} is relaxed preinvex with constant α_0. Now, we suppose to the contrary that F is not relaxed K-preinvex. Hence, for any $\alpha \in \mathbb{R}$ and $e \in intK$ there exist $x_1, x_2 \in \Omega$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ and $\lambda \in [0, 1]$ such that

$$\lambda y_1 + (1 - \lambda)y_2 - \alpha\lambda(1 - \lambda)e \| \eta(x_1, x_2) \|^2 \notin F(x_2 + \lambda\eta(x_1, x_2)) + K. \tag{3}$$

By applying the separating theorem for separating the nonempty disjoint convex sets: $\{\lambda y_1 + (1 - \lambda)y_2 - \alpha_0\lambda(1 - \lambda)e \| \eta(x_1, x_2) \|^2\}$ (which is compact) and $F(x_2 + \lambda\eta(x_1, x_2)) + K$ (which is closed), we deduce the existence of a functional $\tilde{y}^* \in Y^* \setminus \{0\}$ such that

$$\begin{aligned}\tilde{y}^*(\lambda y_1 + (1 - \lambda)y_2 - \alpha_0\lambda(1 - \lambda)e \| \eta(x_1, x_2) \|^2) &< \inf \tilde{y}^*(F(x_2 + \lambda\eta(x_1, x_2)) + K) \\ &= \inf \tilde{y}^*(F(x_2 + \lambda\eta(x_1, x_2))) + \inf \tilde{y}^*(K),\end{aligned}$$

then it can be easily see that $\tilde{y}^* \in K^+ \setminus \{0\}$ and therefore $\inf \tilde{y}^*(K) = 0$. Moreover, without loss of generality, we can suppose that $\tilde{y}^*(e) = 1$. Hence

$$\lambda \tilde{y}^*(y_1) + (1-\lambda)\tilde{y}^*(y_2) - \alpha_0 \lambda(1-\lambda) \parallel \eta(x_1, x_2) \parallel^2) < f_{\tilde{y}^*}(x_2 + \lambda \eta(x_1, x_2)). \tag{4}$$

Since $f_{\tilde{y}^*}$ is relaxed preinvex with constant α_0, one has

$$f_{\tilde{y}^*}(x_2 + \lambda\eta(x_1, x_2)) \le \lambda f_{\tilde{y}^*}(x_1) + (1-\lambda) f_{\tilde{y}^*}(x_2) - \alpha_0 \lambda(1-\lambda) \parallel \eta(x_1, x_2) \parallel^2 .$$

Because $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$, by using the definition of marginal functions, we deduce that

$$f_{\tilde{y}^*}(x_2 + \lambda\eta(x_1, x_2)) \le \lambda \tilde{y}^*(y_1) + (1-\lambda)\tilde{y}^*(y_2) - \alpha_0 \lambda(1-\lambda) \parallel \eta(x_1, x_2) \parallel^2),$$

which is a contradiction with (4). □

Theorem 3.7. *Suppose that X, Y are Asplund space and $F : X \rightrightarrows Y$ is a locally epi-Lipschitz map. If F is relaxed K-preinvex with respect to η, then F is weakly relaxed K-invex.*

Proof. By Lemmas 2.7 and 3.3 for any $y^* \in K^+ \cap S_{Y^*}$, f_{y^*} is a locally Lipschitz relaxed preinvex function. Now, we suppose that $x_1, x_2 \in \Omega$ and $y^* \in K^+ \cap S_{Y^*}$ are fixed. By relaxed preinvexity of f_{y^*}, we have

$$f_{y^*}(x_1 + \lambda\eta(x_2, x_1)) \le \lambda f_{y^*}(x_2) + (1-\lambda) f_{y^*}(x_1) - \alpha\lambda(1-\lambda) \parallel \eta(x_2, x_1) \parallel^2 .$$

Hence,

$$\frac{f_{y^*}(x_1 + \lambda\eta(x_2, x_1)) - f_{y^*}(x_1)}{\lambda} \le f_{y^*}(x_2) - f_{y^*}(x_1) - \alpha(1-\lambda) \parallel \eta(x_2, x_1) \parallel^2, \tag{5}$$

for any $\lambda \in (0, 1)$. Since f_{y^*} is Lipschitz around x_1, there exists a $\theta \in (0, 1)$ suth that f_{y^*} is Lipschitz on an open set containing $[x_1, x_1 + \lambda\eta(x_2, x_1)]$ for any $\lambda \in [0, \theta)$. Thus by using Theorem 2.3, there exists a $c_\lambda \in [x_1, x_1 + \lambda\eta(x_2, x_1))$ and a $x^*_\lambda \in \partial f_{y^*}(c_\lambda)$ such that

$$f_{y^*}(x_1 + \lambda\eta(x_2, x_1)) - f_{y^*}(x_1) \ge \lambda\langle x^*_\lambda, \lambda\eta(x_2, x_1)\rangle.$$

Now, by using (5), we can obtain

$$f_{y^*}(x_2) - f_{y^*}(x_1) - \alpha(1-\lambda) \parallel \eta(x_2, x_1) \parallel^2 \ge \langle x^*_\lambda, \eta(x_2, x_1)\rangle.$$

Since ∂f_{y^*} is locally bounded (Corollary 1.81 in [13]), there exists a neighborhood of x_1 and a constant $\ell > 0$ such that for each z in this neighborhood and $\xi \in \partial f_(y^*)(z)$, we have $\|\xi\| \le \ell$. Since, $c_\lambda \to x_1$ when $\lambda \to 0$, for λ be sufficiently small $\|x^*_\lambda\| \le \ell$, therefore, without loss of generality we may assume that $x^*_\lambda \to x^*$ in weak*-topology. Since the set-valued mapping $\partial f(.)$ has closed graph, we have $x^* \in \partial f_{y^*}(x_1)$ and

$$f_{y^*}(x_2) - f_{y^*}(x_1) - \alpha \parallel \eta(x_2, x_1) \parallel^2 \ge \langle x^*, \eta(x_2, x_1)\rangle.$$

Hence, f_{y^*} is weakly relaxed invex for some $x^* \in \partial f_{y^*}(x_1)$. Now, by using Corollary 2.8, F is weakly relaxed K-invex. □

4. (GMVLI) and Set-Valued Optimization Problems

In this section, we obtain relations between generalized Minty variational-like inequalities and scalarized optimization problems. Suppose that $F : X \rightrightarrows Y$ is a set-valued map between Banach spaces. We consider the following set-valued optimization problem

$$\min \quad F(x), \qquad \text{s.t.} \quad x \in \Omega \subset X. \tag{6}$$

By using the scalarization method, we consider the concept of scalarized solution of problem (6).

Definition 4.1. *(i) [6] A point $\bar{x}$ is said to be a weakly efficient solution of problem (6) iff there exists $\bar{y} \in F(\bar{x})$ such that*

$$(F(\Omega) - \bar{y}) \cap -\mathrm{int}K = \emptyset.$$

(ii) $\bar{x}$ is said to be a scalaraized solution of problem (6) ($\bar{x}$ is a solution of (SOP)) iff, for any $y^* \in K^+ \setminus \{0\}$, there exists $\bar{y} \in F(\bar{x})$ such that

$$y^*(\bar{y}) \leq y^*(y) \quad \text{for all} \quad y \in F(\Omega).$$

Generalized Minty variational-like inequality (GMVLI) consists of finding a vector $\bar{x}$ and $\alpha \in \mathbb{R}$ such that, for any $x \in \Omega$ and $y^* \in K^+ \cap S_{Y^*}$, there exist $y \in M_{y^*}(x)$ and $x^* \in \partial F(x, y)(y^*)$ such that

$$\langle x^*, \eta(\bar{x}, x)\rangle + \alpha \parallel \eta(\bar{x}, x) \parallel^2 \leq 0$$

Remark 4.2. *(i) If $\alpha = 0$, then it reduces to Minty variational-like inequality (MVLI) that has been studied in [2, 16, 19].*

(ii) Notice that, if $\bar{x}$ is a solution of (GMVLI) with constant α, then $\bar{x}$ is also a solution for all parameters $\alpha' \leq \alpha$.

The role of term $\alpha\|\eta(\bar{x}, x)\|^2$ in (GMVLI) is similar to a kind of perturbation in Minty variational inequalities. Because α is choosed in $\mathbb{R}$, the solution set of generalized Minty variational-like inequalities is larger than the solution set of Minty variational-like inequalities.

Example 4.3. *Let $X = Y = \mathbb{R}$, $\Omega = [-1, 1], K = [0, +\infty[$ and $F : \Omega \subset X \rightrightarrows Y$ such that $F(x) = [-x^2 + x, 2]$ for $x \geq 0$ and $F(x) = [x^2, 2]$ for $x < 0$. Let $\eta : X \times X \to X$ be defined as*

$$\eta(x, y) = \begin{cases} x - y & \text{if } x > 0, y > 0 \text{ or } x < 0, y < 0, \\ 1 - y & \text{otherwise.} \end{cases}$$

Then, the normal subdifferential of F is

$$\begin{cases} \partial F(x, -x^2 + x)(1) = -2x + 1 & \text{if } x > 0, \\ \partial F(0, 0)(1) = [0, 1] & \text{if } x = 0, \\ \partial F(x, x^2)(1) = 2x & \text{if } x < 0. \end{cases}$$

Then, by some computation we can see that $x = 0$ is a solution of (SOP) and (GMVLI) with constant $\alpha = -2$, but is not a solution for positive constants.

Lemma 4.4. *[16] Every solution of* (SOP) *is a weakly efficient solution of problem* (6).

Theorem 4.5. *Let $F : \Omega \subseteq X \rightrightarrows Y$ be weakly relaxed K-invex with respect to η. If $\bar{x}$ is a solution of (SOP), then it is a solution of (GMVLI).*

Proof. Suppose that $\bar{x}$ is a solution of (SOP), but not a solution of (GMVLI). Then, for any $\alpha \in \mathbb{R}$ there exists $x \in \Omega$ and $y^* \in K^+ \cap S_{Y^*}$ such that for all $y \in M_{y^*}(x)$ and $x^* \in \partial F(x, y)(y^*)$, we have

$$\langle x^*, \eta(\bar{x}, x)\rangle + \alpha \parallel \eta(\bar{x}, x) \parallel^2 > 0 \tag{7}$$

Since F is weakly relaxed K-invex, then there exists a constant $\alpha \in \mathbb{R}$ such that for any $y^* \in K^+ \cap S_{Y^*}, x \in \Omega, \bar{y} \in M_{y^*}(\bar{x})$ and $y \in M_{y^*}(x)$ there exists $x^* \in \partial F(x, y)(y^*)$, one has

$$\langle x^*, \eta(\bar{x}, x)\rangle + \alpha \parallel \eta(\bar{x}, x) \parallel^2 \leq y^*(\bar{y}) - y^*(y). \tag{8}$$

By using (7) and (8), we get

$$y^*(\bar{y}) - y^*(y) > 0,$$

which is a contradiction with $\bar{x}$ is a solution of (SOP). Hence, $\bar{x}$ is a solution of (GMVLI). □

Theorem 4.6. *Let $F : \Omega \subseteq X \rightrightarrows Y$ be a epi-Lipschitz set-valued map between Asplund spaces and relaxed K-invex with respect to η and constant $\alpha > 0$. Suppose that η satisfies Condition C and F satisfies Condition A. If $\bar{x}$ is a solution of (GMVLI), then it is a solution of (SOP) and hence, a weakly efficient solution of problem (6).*

Proof. Suppose that $\bar{x}$ is a solution of (GMVLI), but not a solution of (SOP). Then there exist $y^* \in K^+ \cap S_{Y^*}$ such that for any $\bar{y} \in F(\bar{x})$, there exist $x \in \Omega$ and $y \in F(x)$ such that

$$y^*(y) < y^*(\bar{y}).$$

Hence, we have

$$f_{y^*}(x) < f_{y^*}(\bar{x}). \tag{9}$$

Let $x(t) = \bar{x} + t\eta(x, \bar{x})$ for $t \in [0, 1]$. Since Ω is invex then $x(t) \in \Omega$. By lemma 2.7, f_{y^*} is a real-valued locally Lipschitz function. Choose $t' \in (0, 1)$ arbitrary. Now, by using Theorem 2.3, there exists $t_1 \in (0, t']$ and $\xi \in \partial f_{y^*}(\bar{x} + t_1\eta(x, \bar{x}))$, such that

$$t'\langle \xi, \eta(x, \bar{x})\rangle \le f_{y^*}(\bar{x} + t'\eta(x, \bar{x})) - f_{y^*}(\bar{x}). \tag{10}$$

Because F is relaxed K-invex, by Theorem 2.8, we have f_{y^*} is relaxed invex for any $y^* \in K^+ \cap S_{Y^*}$. Now, By a similar way of lemma 3.2 in [9], relaxed invexity of f_{y^*} implies that f_{y^*} is relaxed preinvex. Hence, we obtain

$$f_{y^*}(\bar{x} + t'\eta(x, \bar{x})) - f_{y^*}(\bar{x}) \le t'(f_{y^*}(x) - f_{y^*}(\bar{x}) - \alpha(1 - t') \parallel \eta(x, \bar{x}) \parallel^2).$$

From relation (9), we deduce that

$$f_{y^*}(\bar{x} + t'\eta(x, \bar{x})) - f_{y^*}(\bar{x}) < -\alpha t'(1 - t') \parallel \eta(x, \bar{x}) \parallel^2 .$$

Now, by using (10), we have

$$\langle \xi, \eta(x, \bar{x})\rangle < -\alpha(1 - t') \parallel \eta(x, \bar{x}) \parallel^2 . \tag{11}$$

Because $t_1 \in (0, 1)$, we can choose $t^* \in (0, 1)$ such that $t^* < t_1$ and to be sufficiently small. Now, by using Condition C, we can obtain

$$\eta(x(t^*), x(t_1)) = (t^* - t_1)\eta(x, \bar{x}), \quad \eta(x(t_1), x(t^*)) = (t_1 - t^*)\eta(x, \bar{x}). \tag{12}$$

From relations (11) and (12), we have

$$\langle \xi, \eta(x(t^*), x(t_1))\rangle > \alpha(t_1 - t^*)(1 - t') \parallel \eta(x, \bar{x}) \parallel^2 . \tag{13}$$

Since f_{y^*} is relaxed preinvex, then similar to the Theorem 3.2 in [9], we can deduce that ∂f_{y^*} is invariant relaxed monotone. Therefore

$$\langle \xi, \eta(x(t^*), x(t_1))\rangle + \langle \zeta, \eta(x(t_1), x(t^*))\rangle \le -\alpha(\parallel x(t^*), x(t_1) \parallel^2 + \parallel x(t_1), x(t^*) \parallel^2),$$

that $\zeta \in \partial f_{y^*}(x(t^*))$. If we use relations in (12), we obtain

$$\langle \xi, \eta(x(t^*), x(t_1))\rangle + \langle \zeta, \eta(x(t_1), x(t^*))\rangle \le -2\alpha(t_1 - t^*)^2 \parallel \eta(x, \bar{x}) \parallel^2 . \tag{14}$$

Now, by relations (12), (13) and (14), we deduce that

$$\begin{aligned}(t_1 - t^*))\langle \zeta, \eta(x, \bar{x})\rangle &= \langle \zeta, \eta(x(t_1), x(t^*))\rangle \\ &< -\alpha(t_1 - t^*) \parallel \eta(x, \bar{x}) \parallel^2 (2(t_1 - t^*) + (1 - t')).\end{aligned}$$

Since $\eta(\bar{x}, x(t^*)) = -t^*\eta(x, \bar{x})$, we obtain

$$\langle \zeta, \eta(\bar{x}, x(t^*)) \rangle > \frac{\alpha}{t^*}[2(t_1 - t^*) + (1 - t')] \parallel \eta(\bar{x}, x(t^*)) \parallel^2 .$$

Hence, we get

$$\langle \zeta, \eta(\bar{x}, x(t^*)) \rangle + \alpha' \parallel \eta(\bar{x}, x(t^*)) \parallel^2 > 0,$$

where $\alpha' = -\frac{\alpha}{t^*}[2(t_1 - t^*) + (1 - t')] < 0$. Notice that if $t^* \to 0^+$ then $\alpha' \to -\infty$. Hence, we can suppose that $\alpha' \leq c$, where c is a constant in $\mathbb{R}$ such that $\bar{x}$ is a solution of (GMVLI) with it. By using Theorem 2.8 we have $\zeta \in \partial f_{y^*}(x(t^*)) \subseteq \partial F(x(t^*), y(t^*))$, where $y(t^*) \in M_{y^*}(x(t^*))$. Therefore $\bar{x}$ is not a solution of (GMVLI) with constant α'. Now, by Remark 4.2 (ii), this contradicts with $\bar{x}$ is a solution of (GMVLI) with constant c. □

Remark 4.7. *Theorems 4.5 and 4.6 generalize Theorem 3.1 in [1], Theorem 3.1 in [5] and Theorem 2 in [14] to set-valued maps.*

Example 4.8. *Let $X = Y = \mathbb{R}$, $\Omega = [-1, 1]$, $K = [0, +\infty[$ and $F : \Omega \subset X \rightrightarrows Y$ such that $F(x) = [x^2, 3]$ for $x \geq 0$ and $F(x) = [x^2 - 2x, 3]$ for $x < 0$. Let $\eta : X \times X \to X$ be defined as*

$$\eta(x, y) = \begin{cases} x - y & \text{if } x \geq 0, y \geq 0 \text{ or } x \leq 0, y \leq 0, \\ -y & \text{otherwise.} \end{cases}$$

Then, the normal subdifferential of F is

$$\begin{cases} \partial F(x, x^2)(1) = 2x & \text{if } x > 0, \\ \partial F(0, 0)(1) = [-2, 0] & \text{if } x = 0, \\ \partial F(x, x^2 - 2x)(1) = 2x - 2 & \text{if } x < 0. \end{cases}$$

Then, η satisfies Condition C, *F satisfies Condition* A *and is relaxed K-invex with constant $\alpha = 1$. Hence, by some computation we can see that all assumptions of Theorem 4.5 are fulfilled and $x = 0$ is a solution of (GMVLI) and therefore is a solution of (SOP).*

Here, we obtain an existence theorem for the solution of (GMVLI) and therefore a weak efficient solution of problem (6). For normal subdifferential, we need the following condition to get an existence result for them.

Condition D: Let $F : X \rightrightarrows Y$ and $y^* \in K^* \cap S_{Y^*}$. Then, for any $\bar{x} \in \mathrm{dom}F$ and $\bar{y}_1, \bar{y}_2 \in M_{y^*}(\bar{x})$, we have

$$\partial F(\bar{x}, \bar{y}_1)(y^*) = \partial F(\bar{x}, \bar{y}_2)(y^*).$$

Theorem 4.9. *Let $F : X \rightrightarrows Y$ be relaxed K-invex with constant α and satisfy Condition D. Assume that the following conditions are satisfied:*

1. *η is affine and continuous in the first argument and skew.*
2. *There are a nonempty compact set $M \subset X$ and a nonempty compact convex set $B \subset X$ such that for each $x' \in X \backslash M$, there exists $x \in B$ and $y^* \in K^+ \cap S_{Y^*}$ such that for any $y \in M_{y^*}(x)$ and $x^* \in \partial F(x, y)(y^*)$, we have $\langle x^*, \eta(x', x) \rangle + 2\alpha \|\eta(x', x)\|^2 > 0$.*

Then, (GMVLI) has a solution. Also, the set of (GMVLI) solutions is compact.

Proof. Define two set-valued mappings $\Gamma, \widehat{\Gamma} : X \rightrightarrows X$ by

$$\Gamma(x) := \{x' \in X : \forall y^* \in K^+ \cap S_{Y^*}, \exists y \in M_{y^*}(x) \text{ and } x^* \in \partial F(x, y)(y^*); \langle x^*, \eta(x', x) \rangle + 2\alpha \|\eta(x', x)\|^2 \leq 0\},$$

$$\widehat{\Gamma}(x) := \{x' \in X : \forall y^* \in K^+ \cap S_{Y^*}, \exists y' \in M_{y^*}(x') \text{ and } x^* \in \partial F(x', y')(y^*); \langle x^*, \eta(x, x') \rangle \geq 0\},$$

for each $x \in X$. $\Gamma(x)$ and $\widehat{\Gamma}(x)$ are nonempty because they contain x. The proof is divided in the following steps.

(i) $\widehat{\Gamma}$ is a KKM mapping on X. Suppose that $\widehat{\Gamma}$ is not a KKM mapping. Then, there exist $\{x_1, x_2, \ldots, x_m\}$ and $\lambda_i \geq 0$, $i = 1, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$ such that $x_0 = \sum_{i=1}^{m} \lambda_i x_i \notin \bigcup_{i=1}^{m} \widehat{\Gamma}(x_i)$. Hence, it follows that $x_0 \notin \widehat{\Gamma}(x_i)$ for all $i = 1, \ldots, m$, i.e.

$$\exists y^* \in K^+ \cap S_{Y^*}; \forall y_0 \in M_{y^*}(x_0), x^* \in \partial F(x_0, y_0)(y^*) : \quad \langle x^*, \eta(x_i, x_0) \rangle < 0; \tag{15}$$

for each $i = 1, \ldots, m$. Therefore, for any $y_0 \in M_{y^*}(x_0)$ and $x^* \in \partial F(x_0, y_0)(y^*)$, one has

$$0 = \langle x^*, \eta(x_0, x_0) \rangle = \sum_{i=1}^{m} \lambda_i \langle x^*, \eta(x_i, x_0) \rangle < 0,$$

which yields a contradiction. Hence, $\widehat{\Gamma}$ is a KKM mapping.
(ii) Because relaxed K-invexity of F implies invariant relaxed K-monotonicity of ∂F (Lemma 3.4), hence we obtain $\widehat{\Gamma}(x) \subseteq \Gamma(x)$ and therefore, Γ is also a KKM mapping.
(iii) Γ is closed valued: Let $\{x_n\}$ be a sequence in $\Gamma(x)$ which converges to a x_0. Therefore, for any $y^* \in K^+ \cap S_{Y^*}$, there exist $y \in M_{y^*}(x)$ and $x_n^* \in \partial F(x, y)(y^*)$ such that

$$\langle x_n^*, \eta(x_n, x) \rangle + 2\alpha \|\eta(x_n, x)\|^2 \leq 0.$$

Since F satisfies Condition D and is epi-Lipschitz, x_n^* has a convergent subsequence x_m^*, that its limit x_0^* should be in $\partial F(x, y)(y^*)$ for a $y \in M_{y^*}(x)$. Since η is continuous in the first argument, $\{\eta(x_n, x)\}$ is a convergent sequence. Hence, we obtain

$$\langle x_0^*, \eta(x_0, x) \rangle + 2\alpha \|\eta(x_0, x)\|^2 \leq 0.$$

Thus, $x_0 \in \Gamma(x)$, this means that Γ is closed valued.
(iv) From condition 2, there exists a nonempty compact convex set B, such that $\mathrm{cl}(\bigcap_{x \in B} \Gamma(x))$ is compact.
(v) Thus, all of the conditions of Lemma 2.11 are fulfilled by mapping Γ. Therefore,

$$\bigcap_{x \in X} \Gamma(x) \neq \emptyset.$$

Hence, there exists $\bar{x}$ such that for any $x \in X$ and $y^* \in K^+ \cap S_{Y^*}$ there exist $y \in M_{y^*}(x)$ and $x^* \in \partial F(x, y)(y^*)$ such that

$$\langle x^*, \eta(\bar{x}, x) \rangle + 2\alpha \|\eta(\bar{x}, x)\|^2 \leq 0.$$

Thus, (GMVLI) has a solution. From (iii), Γ is closed valued and therefore, the set of solutions of (GMVLI), i.e. $\bigcap_{x} \Gamma(x)$ is closed. Now, from (2), the set of solutions must be contained in the compact set M, hence it is compact. $\square$

5. Conclusions

In this work, some new notions of relaxed preinvexity based on normal subdifferential for set-valued maps has been presented, which extends strong convexity concept for real-valued and vector-valued functions. Also, we have considered a generalization of Minty variational inequalities that the set of its solution is larger than Minty variational inequalities and investigated the relations between their solutions and set-valued optimization problem's solutions under generalized convexity. An existence result for them is also given.

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