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Real Hypersurfaces with Pseudo-parallel Normal Jacobi Operator in Complex Two-Plane Grassmannians

Avik De[a], Tee-How Loo[b]

[a]*Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, 43000 Cheras, Malaysia.*
[b]*Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia.*

Abstract. The objective of the present paper is to prove the non-existence of real hypersurface with pseudo-parallel normal Jacobi operator in complex two-plane Grassmannians. As a corollary, we show that there does not exist any real hypersurface with semi-parallel or recurrent normal Jacobi operator in complex two-plane Grassmannians. This answers a question considered in [Monatsh Math, 172 (2013), 167-178] in negative.

1. Introduction

The complex two-plane Grassmannian $G_2(\mathbb{C}_{m+2})$ is the set of all complex two-dimensional linear subspaces of $\mathbb{C}_{m+2}$. It is the unique compact, irreducible Riemannian symmetric space with positive scalar curvature, equipped with both a Kaehler structure J and a quaternionic Kaehler structure $\mathfrak{J}$ not containing J [3].

Typical examples of real hypersurfaces M in $G_2(\mathbb{C}_{m+2})$ are tubes around $G_2(\mathbb{C}_{m+1})$ and tubes around $\mathbb{H}P^n$ in $G_2(\mathbb{C}_{2n+2})$. These two classes of real hypersurfaces possess a number of interesting geometric properties. Characterizing them or subclasses of them under certain nice geometric conditions has been one of the main focus of researchers in the theory of real hypersurfaces in $G_2(\mathbb{C}_{m+2})$. One of the foremost results along this line was obtained by Berndt and Suh [4], they characterized these two classes of real hypersurfaces under the invariance of vector bundles $JT^\perp M$ and $\mathfrak{J}T^\perp M$ over the real hypersurfaces M under the shape operator A of M, where $T^\perp M$ is the normal bundle of M.

On the other hand, these structures J and $\mathfrak{J}$ of $G_2(\mathbb{C}_{m+2})$ significantly impose restrictions on the geometry of its real hypersurfaces. For instance, there does not exist any parallel real hypersurface [18]. Determining the existence (or non-existence) of real hypersurfaces in $G_2(\mathbb{C}_{m+2})$ satisfying certain geometric properties has also became another main research topic in this theory. The main objective of this paper is to prove the non-existence of real hypersurfaces in $G_2(\mathbb{C}_{m+2})$ with pseudo-parallel normal Jacobi operator.

Recall that the *normal Jacobi operator* $\hat{R}_N$, for a hypersurface M in a Riemannian manifold, is defined as $\hat{R}_N(X) = \hat{R}(X, N)N$, for any vector X tangent to M, where $\hat{R}$ is the curvature tensor of the ambient space and N is a unit vector normal to M [2].

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Email addresses: de.math@gmail.com (Avik De), looth@um.edu.my (Tee-How Loo)

Let M be an orientable real hypersurface isometrically immersed in $G_2(\mathbb{C}_{m+2})$. Denote by (ϕ, ξ, η) the almost contact structure on M induced by J, (ϕ_a, ξ_a, η_a), $a \in \{1, 2, 3\}$, the local almost contact 3-structure on M induced by $\mathfrak{J}$ and $\mathfrak{D}^\perp = \mathfrak{J}T^\perp M$. The real hypersurface M is said to be *Hopf* if $AJT^\perp M \subset JT^\perp M$, or equivalently, the Reeb vector field ξ is principal with principal curvature α.

Pérez et al. [17] studied the real hypersurfaces in $G_2(\mathbb{C}_{m+2})$, in which the normal Jacobi operator commutes with both the shape operator and the structure tensor ϕ. In [7] Jeong et al. proved the following:

Theorem 1.1 ([7]). *There does not exist any connected Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}_{m+2})$, $m \geq 3$, with parallel normal Jacobi operator.*

Machado et al. [14] proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}_{m+2})$ with Codazzi type $\hat{R}_N$ under certain conditions on the $\mathfrak{D}$- and $\mathfrak{D}^\perp$-component of ξ . Later, the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}_{m+2})$ whose normal Jacobi operator is $(\mathbb{R}\xi \cup \mathfrak{D}^\perp)$-parallel was proved [11]. In [10], Suh and Jeong investigated real hypersurfaces in $G_2(\mathbb{C}_{m+2})$ with $L_\xi \hat{R}_N = 0$, and proved the non-existence of such real hypersurfaces under the condition either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$. They also proved the non-existence of Hopf hypersurfaces with Lie parallel normal Jacobi operator, that is, $L_X \hat{R}_N = 0$ in $G_2(\mathbb{C}_{m+2})$ [8].

A real hypersurface M is said to have *recurrent* normal Jacobi operator if $(\hat{\nabla}_X \hat{R}_N)Y = \omega(X)\hat{R}_N Y$, for some 1-form ω. In [9], Jeong et al. generalized Theorem 1.1 and proved the following:

Theorem 1.2 ([9]). *There does not exist any connected Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}_{m+2})$, $m \geq 3$, with recurrent normal Jacobi operator.*

Deprez [6] first studied a submanifold M in a Riemannian manifold whose second fundamental form h satisfies $\bar{R} \cdot h = 0$, where $\bar{R}$ is the curvature tensor corresponding to the van der Waerden-Bortolotti connection. Such submanifolds are said to be *semi-parallel*. In [15], Ortega proved that there does not exist any semi-parallel real hypersurface in a non-flat complex space form.

A $(1, p)$-tensor T, in a Riemannian manifold M with Riemannian curvature tensor R is said to be *pseudo-parallel*, if it satisfies $R(X, Y)T = f\{(X \wedge Y)T\}$, for some function f, where

$$(X \wedge Y)Z := \langle Y, Z\rangle X - \langle X, Z\rangle Y,$$

and

$$\{(X \wedge Y)T\}(X_1, \cdots, X_p) := (X \wedge Y)T(X_1, \cdots, X_p) - \sum_{j=1}^{p} T(X_1, \cdots (X \wedge Y)X_j, \cdots, X_p),$$

for any $X, Y, Z, X_1, \cdots, X_p \in TM$.

The notion of pseudo-parallel submanifolds, that is, submanifolds with pseudo-parallel second fundamental forms, can be considered as a generalization of semi-parallel submanifolds. Asperti et al. [1] classified all pseudo-parallel hypersurfaces in space forms as quasi-umbilic hypersurfaces or cyclids of duplin. The classification of pseudo-parallel real hypersurfaces in a non-flat complex space form was obtained in [12].

Recently, Panagiotidou and Tripathi [16] studied Hopf hypersurfaces with semi-parallel normal Jacobi operator in $G_2(\mathbb{C}_{m+2})$ and proved the following:

Theorem 1.3 ([16]). *There does not exist any connected Hopf hypersurface M in $G_2(\mathbb{C}_{m+2})$, $m \geq 3$, equipped with semi-parallel normal Jacobi operator, if $\alpha \neq 0$ and $\mathfrak{D}$- or $\mathfrak{D}^\perp$-component of the Reeb vector field ξ is invariant by the shape operator A.*

One of the challenges in the theory of real hypersurfaces M in $G_2(\mathbb{C}_{m+2})$ is handling lengthy and complicated expressions resulting from the complexity of the geometric structures on M, induced by the Kaehler and the quaternionic Kaehler structure of $G_2(\mathbb{C}_{m+2})$. For technical reasons, certain additional restrictions such as M being Hopf, having non-vanishing geodesic Reeb flow, etc have often been imposed while dealing with real hypersurfaces in $G_2(\mathbb{C}_{m+2})$. It would be interesting to see whether any nice results on real hypersurfaces of $G_2(\mathbb{C}_{m+2})$ can be obtained without these restrictions.

Motivated by Theorem 1.1, Theorem 1.2 and Theorem 1.3, a question arises naturally:

Problem 1.4. *Does there exist real hypersurface in $G_2(\mathbb{C}_{m+2})$ with parallel, recurrent, semi-parallel or pseudo-parallel normal Jacobi operator?*

We first prove the following:

Theorem 1.5. *There does not exist any real hypersurface with pseudo-parallel normal Jacobi operator in $G_2(\mathbb{C}_{m+2})$, $m \geq 3$.*

Remark 1.6. *It is worthwhile to note that no additional condition has been imposed in the above theorem.*

Since a semi-parallel tensor is always pseudo-parallel with the associated function $f = 0$, we state that

Corollary 1.7. *There does not exist any real hypersurface in $G_2(\mathbb{C}_{m+2})$, $m \geq 3$, equipped with semi-parallel normal Jacobi operator.*

In the last section we prove the non-existence of real hypersurfaces with recurrent or parallel normal Jacobi operator in $G_2(\mathbb{C}_{m+2})$, $m \geq 3$ (see Corollary 4.2). Thus Problem 1.4 has been solved completely.

2. Real Hypersurfaces in $G_2(\mathbb{C}_{m+2})$

In this section we state some structural equations as well as some known results in the theory of real hypersurfaces in $G_2(\mathbb{C}_{m+2})$. A thorough study on the Riemannian geometry on $G_2(\mathbb{C}_{m+2})$ can be found in [3]. Denote by $\langle , \rangle$ the Riemannian metric, J the Kaehler structure and $\mathfrak{J}$ the quaternionic Kaehler structure on $G_2(\mathbb{C}_{m+2})$. For each $x \in G_2(\mathbb{C}_{m+2})$, we denote by $\{J_1, J_2, J_3\}$ a canonical local basis of $\mathfrak{J}$ on a neighborhood $\mathcal{U}$ of x in $G_2(\mathbb{C}_{m+2})$, that is, each J_a is a local almost Hermitian structure such that

$$J_a J_{a+1} = J_{a+2} = -J_{a+1} J_a, \quad a \in \{1, 2, 3\}. \tag{1}$$

Here, the index is taken modulo three. Denote by $\hat{\nabla}$ the Levi-Civita connection of $G_2(\mathbb{C}_{m+2})$. There exist local 1-forms q_1, q_2 and q_3 such that

$$\hat{\nabla}_X J_a = q_{a+2}(X) J_{a+1} - q_{a+1}(X) J_{a+2}$$

for any $X \in T_x G_2(\mathbb{C}_{m+2})$, that is, $\mathfrak{J}$ is parallel with respect to $\hat{\nabla}$. The Kaehler structure J and quaternionic Kaehler structure $\mathfrak{J}$ is related by

$$J J_a = J_a J; \quad \mathrm{Trace}(J J_a) = 0, \quad a \in \{1, 2, 3\}. \tag{2}$$

The Riemannian curvature tensor $\hat{R}$ of $G_2(\mathbb{C}_{m+2})$ is locally given by

$$\begin{aligned}\hat{R}(X, Y)Z =& \langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY - 2\langle JX, Y\rangle JZ \\ &+ \sum_{a=1}^{3} \{\langle J_a Y, Z\rangle J_a X - \langle J_a X, Z\rangle J_a Y - 2\langle J_a X, Y\rangle J_a Z + \langle J J_a Y, Z\rangle J J_a X - \langle J J_a X, Z\rangle J J_a Y\}.\end{aligned} \tag{3}$$

for all X, Y and $Z \in T_x G_2(\mathbb{C}_{m+2})$.

For a nonzero vector $X \in T_x G_2(\mathbb{C}_{m+2})$, we denote by $\mathbb{C}X = \mathrm{Span}\{X, JX\}$, $\mathfrak{J}X = \{J'X | J' \in \mathfrak{J}_x\}$, $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$, and $\mathbb{HC}X$ the subspace spanned by $\mathbb{H}X$ and $\mathbb{H}JX$. If $JX \in \mathfrak{J}X$, we denote by $\mathbb{C}^{\perp}X$ the orthogonal complement of $\mathbb{C}X$ in $\mathbb{H}X$.

Let M be an oriented real hypersurface isometrically immersed in $G_2(\mathbb{C}_{m+2})$, $m \geq 3$, N a unit normal vector field on M. The Riemannian metric on M is denoted by the same $\langle , \rangle$. A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ on $G_2(\mathbb{C}_{m+2})$ induces a local almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, \langle , \rangle)$ on M by

$$J_a X = \phi_a X + \eta_a(X) N, \quad J_a N = -\xi_a, \quad \eta_a(X) = \langle \xi_a, X\rangle$$

for any $X \in TM$. It follows from (1) that

$$\begin{aligned}&\phi_a\phi_{a+1} - \xi_a \otimes \eta_{a+1} = \phi_{a+2} = -\phi_{a+1}\phi_a + \xi_{a+1} \otimes \eta_a\\&\phi_a\xi_{a+1} = \xi_{a+2} = -\phi_{a+1}\xi_a.\end{aligned}$$

Denote by $(\phi, \xi, \eta, \langle,\rangle)$ the almost contact metric structure on M induced by J, that is,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = \langle \xi, X\rangle.$$

The vector field ξ is known as the *Reeb vector field*.

It follows from (2) that the two structures $(\phi, \xi, \eta, \langle,\rangle)$ and $(\phi_a, \xi_a, \eta_a, \langle,\rangle)$ can be related as follows

$$\phi_a\phi - \xi_a \otimes \eta = \phi\phi_a - \xi \otimes \eta_a; \quad \phi\xi_a = \phi_a\xi.$$

Denote by ∇ the Levi-Civita connection and A the shape operator on M. Then

$$\begin{aligned}&(\nabla_X\phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi, \quad \nabla_X\xi = \phi AX\\&(\nabla_X\phi_a)Y = \eta_a(Y)AX - \langle AX, Y\rangle\xi_a + q_{a+2}(X)\phi_{a+1}Y - q_{a+1}(X)\phi_{a+2}Y\\&\nabla_X\xi_a = \phi_a AX + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2}\end{aligned}$$

for any $X, Y \in TM$.

Corresponding to each canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$, we define a local endomorphism θ_a on TM by

$$\theta_a X := \tan(J_a J X) = \phi_a\phi X - \eta(X)\xi_a.$$

Some properties of θ_a are given in the following:

Lemma 2.1 ([13]).

(a) *θ_a is symmetric,*
(b) *Trace $\theta_a = \eta(\xi_a)$,*
(c) *$\theta_a^2 X = X + \eta_a(\phi X)\phi\xi_a$, for all $X \in TM$,*
(d) *$\theta_a\xi = -\xi_a$; $\theta_a\xi_a = -\xi$; $\theta_a\phi\xi_a = \eta(\xi_a)\phi\xi_a$,*
(e) *$\theta_a\xi_{a+1} = \phi\xi_{a+2} = -\theta_{a+1}\xi_a$,*
(f) *$\theta_a\phi\xi_{a+1} = -\xi_{a+2} + \eta(\xi_{a+1})\phi\xi_a$,*
(g) *$\theta_{a+1}\phi\xi_a = \xi_{a+2} + \eta(\xi_a)\phi\xi_{a+1}$.*

For each $x \in M$, we define a subspace $\mathcal{H}^\perp$ of T_xM by

$$\mathcal{H}^\perp := \mathrm{Span}\{\xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3\}.$$

Let $\mathcal{H}$ be the orthogonal complement of $\mathbb{HC}\xi$ in $T_xG_2(\mathbb{C}_{m+2})$. Then $T_xM = \mathcal{H} \oplus \mathcal{H}^\perp$.

Lemma 2.2 ([13]). *Let $\mathcal{H}_a(\varepsilon)$ be the eigenspace corresponding to eigenvalue ε of θ_a. Then*

(a) *$\theta_{a|\mathcal{H}}$ has two eigenvalues $\varepsilon = \pm 1$,*
(b) *$\phi\mathcal{H}_a(\varepsilon) = \phi_a\mathcal{H}_a(\varepsilon) = \mathcal{H}_a(\varepsilon)$,*
(c) *$\theta_b\mathcal{H}_a(\varepsilon) = \mathcal{H}_a(-\varepsilon)$, for $a \neq b$.*
(d) *$\phi_b\mathcal{H}_a(\varepsilon) = \mathcal{H}_a(-\varepsilon)$, for $a \neq b$.*

Let R be the curvature tensor of M. It follows from (3) that the equation of Gauss is given by

$$\begin{aligned} R(X,Y)Z &= \langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \phi Y,Z\rangle\phi X - \langle \phi X,Z\rangle \phi Y - 2\langle \phi X,Y\rangle \phi Z \\ &\quad + \sum_{a=1}^{3}\{\langle \phi_a Y,Z\rangle \phi_a X - \langle \phi_a X,Z\rangle\phi_a Y - 2\langle \phi_a X,Y\rangle \phi_a Z \\ &\quad + \langle \theta_a Y,Z\rangle\theta_a X - \langle \theta_a X,Z\rangle \theta_a Y\} + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY, \end{aligned} \tag{4}$$

for any $X,Y,Z \in TM$.

The normal Jacobi operator $\hat{R}_N$ is given by

$$\hat{R}_N X = X + 3\eta(X)\xi + \sum_{a=1}^{3}\{3\eta_a(X)\xi_a - \eta(\xi_a)\theta_a + \eta_a(\phi X)\phi\xi_a\}. \tag{5}$$

for any $X \in TM$.

Finally, we state two known results which we use in the next section:

Lemma 2.3 ([13]). *Let M be a real hypersurface in $G_2(\mathbb{C}_{m+2})$. If ξ is tangent to $\mathfrak{D}$ then $A\phi\xi_a = 0$, for $a \in \{1,2,3\}$.*

Lemma 2.4 ([13]). *Let M be a real hypersurface in $G_2(\mathbb{C}_{m+2})$. If ξ is tangent to $\mathfrak{D}$ then $\xi, \xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3$ are orthonormal.*

3. Proof of Theorem 1.5

Let the normal Jacobi operator $\hat{R}_N$ for a real hypersurface M in $G_2(\mathbb{C}_{m+2})$, be pseudo-parallel. Then we have

$$\langle (R(X,Y)\hat{R}_N)Z, W\rangle = f\langle [(X\wedge Y)\hat{R}_N]Z, W\rangle, \tag{6}$$

for any $X,Y,Z,W \in TM$, where f is a real-valued function on M, This implies that

$$\begin{aligned} \langle R(X,Y)\hat{R}_N Z, W\rangle - \langle \hat{R}_N R(X,Y)Z, W\rangle = \quad & f\{\langle Y,\hat{R}_N Z\rangle\langle X,W\rangle - \langle X,\hat{R}_N Z\rangle\langle Y,W\rangle \\ & -\langle Y,Z\rangle\langle \hat{R}_N X,W\rangle + \langle X,Z\rangle\langle \hat{R}_N Y,W\rangle\}. \end{aligned} \tag{7}$$

We consider two cases: $\xi \notin \mathfrak{D}$ at a point $x \in M$; and $\xi \in \mathfrak{D}$ on M.

Case 1: $\xi \notin \mathfrak{D}$ at a point $x \in M$.

Without loss of generality, we assume $0 < \eta(\xi_1) \le 1$, $\eta(\xi_2) = \eta(\xi_3) = 0$. Let β, $\mu \in \mathbb{R}$ and $U \in \mathcal{H}_1(1)$, $V \in \mathcal{H}_1(-1)$ be unit vectors such that the $\mathcal{H}_1(1)$-component $(A\xi)^+$ and $\mathcal{H}_1(-1)$-component $(A\xi)^-$ of $A\xi$ are given by

$$(A\xi)^+ = \beta U, \qquad (A\xi)^- = \mu V.$$

Since, $\eta(\xi_2) = 0 = \eta(\xi_3)$, for $Z \in \mathcal{H}_1(1)$ and $W \in \mathcal{H}_1(-1)$, from (5) we have

$$\hat{R}_N Z = (1-\eta(\xi_1))Z, \quad \hat{R}_N W = (1+\eta(\xi_1))W.$$

Since $\eta(\xi_1) \neq 0$, by putting $Z \in \mathcal{H}_1(1)$ and $W \in \mathcal{H}_1(-1)$ in (7), we obtain

$$\langle R(X,Y)Z,W\rangle = f\{\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\}, \tag{8}$$

for any X, $Y \in T_xM$, $Z \in \mathcal{H}_1(1)$ and $W \in \mathcal{H}_1(-1)$. In particular, for $X = \xi$ and $Y \perp \xi$, using the Gauss equation (4), the above equation becomes

$$\begin{aligned} -2\textstyle\sum_{a=1}^{3}\langle \phi\xi_a, Y\rangle\langle \phi_a Z, W\rangle + \sum_{a=1}^{3}\{\langle \theta_a Y,Z\rangle\langle \theta_a\xi, W\rangle - \langle \theta_a Y, W\rangle\langle \theta_a \xi, Z\rangle\} \\ +\langle A\xi,W\rangle\langle AY,Z\rangle - \langle A\xi,Z\rangle\langle AY,W\rangle = 0, \end{aligned}$$

for any $Y \perp \xi$, $Z \in \mathcal{H}_1(1)$ and $W \in \mathcal{H}_1(-1)$. Using Lemma 2.1(d) and Lemma 2.2(b), we obtain

$$\mu\langle V, W\rangle\langle AY, Z\rangle - \beta\langle U, Z\rangle\langle AY, W\rangle - 2\sum_{a=2}^{3}\langle\phi\xi_a, Y\rangle\langle\phi_a Z, W\rangle = 0, \tag{9}$$

for any $Y \perp \xi$, $Z \in \mathcal{H}_1(1)$ and $W \in \mathcal{H}_1(-1)$. If $Z \perp U$ and $W \perp V$, then $\sum_{a=2}^{3}\langle\phi\xi_a, Y\rangle\langle\phi_a Z, W\rangle = 0$, for any $Y \perp \xi$. Since $\phi\xi_2$ and $\phi\xi_3$ are linearly independent,

$$\langle\phi_a Z, W\rangle = 0, \quad a \in \{2, 3\}, \tag{10}$$

for any $Z \in \mathcal{H}_1(1)$ $(\perp U)$ and $W \in \mathcal{H}_1(-1)$ $(\perp V)$.

If $\dim \mathcal{H}_1(1) \geq 4$ the above equation implies that ϕ_a is not a monomorphism on $\mathcal{H}_1(1)$, which contradicts Lemma 2.2(d). Hence, we conclude that $\dim \mathcal{H}_1(1) = \dim \mathcal{H}_1(-1) = 2$, and $\mathcal{H}_1(1) = \mathbb{C}U$ and $\mathcal{H}_1(-1) = \mathbb{C}V$.

The equation (10) directly implies $\langle\phi_a\phi U, \phi V\rangle = 0$, for $a \in \{2, 3\}$. Hence, by Lemma 2.2(d), $\phi_a\phi U = \pm V$, for $a \in \{2, 3\}$. We can express

$$\phi_a\phi U = \epsilon_a V, \quad \epsilon_a \in \{1, -1\}. \tag{11}$$

Next, by putting $Z = \phi U$ and $W = V$ in (9), and using (11) we obtain

$$\mu A\phi U = 2\sum_{a=2}^{3}\epsilon_a\phi\xi_a,$$

which implies that $A\phi U \perp \mathcal{H}$.

Putting $Y = \phi U, Z = U, W = V$ in (8) and using Lemma 2.2(b), we get $\langle R(X, \phi U)U, V)\rangle = 0$. Since X is arbitrary, we have

$$R(V, U)\phi U = 0.$$

Using Lemma 2.2(b), Lemma 2.2(d) and the Gauss equation (4), we deduce from the above equation that

$$\phi V - \phi_1 V + \sum_{a=2}^{3}\langle V, \phi_a\phi U\rangle\phi_a U = 0. \tag{12}$$

Since $V \in \mathcal{H}_1(-1)$, we have $\theta_1 V = -V$, which implies $\phi V = \phi_1 V$. Hence, from (12) we conclude that $\sum_{a=2}^{3}\langle V, \phi_a\phi U\rangle\phi_a U = 0$. This contradicts (11) and the orthogonolity of $\phi_2 U$ and $\phi_3 U$. Accordingly, this case cannot occur.

Case 2: $\xi \in \mathfrak{D}$ **on** M**.**

In this case, we have each $\eta(\xi_a) = 0$ for all a, everywhere. It follows from (5) that the normal Jacobi operator has three constant eigenvalues 0, 4 and 1 at each point of M with eigenspaces

$$T_0 = \text{Span}\{\phi\xi_a : a = 1, 2, 3\}, \quad T_4 = \text{Span}\{\xi, \xi_a : a = 1, 2, 3\}, \quad T_1 = \mathcal{H}$$

respectively.

If we put $X, Y \perp \xi$ and $Z = \xi$ in (7) then $\hat{R}_N R(X, Y)\xi = 4R(X, Y)\xi$ and so $R(X, Y)\xi \in T_4 \ominus \mathbb{R}\xi = \text{Span}\{\xi_a : a = 1, 2, 3\}$, for any $X, Y \perp \xi$. Hence, it follows from Lemma 2.4 that $\langle R(X, Y)\xi, \phi\xi_a\rangle = 0$, for any $X, Y \perp \xi$. Furthermore, using Lemma 2.1(d) and the Gauss equation (4), we obtain

$$\eta(AY)\langle AX, \phi\xi_a\rangle - \eta(AX)\langle AY, \phi\xi_a\rangle - 2\langle\phi_a X, Y\rangle = 0,$$

for any $X, Y \in \mathcal{H}$ and $a \in \{1, 2, 3\}$. This equation, together with Lemma 2.3, yields $\langle\phi_a X, Y\rangle = 0$, for any $X, Y \in \mathcal{H}$. This is a contradiction and the proof is completed.

4. Real Hypersurfaces with Recurrent $\hat{R}_N$ in $G_2(\mathbb{C}_{m+2})$

In this section, we show that there does not exist any real hypersurface with recurrent normal Jacobi operator in $G_2(\mathbb{C}_{m+2})$. We first discuss the ideas of recurrence and semi-parallelism in a general setting.

Let M be a Riemannian manifold and $\mathcal{E}_j$ a Riemannian vector bundle over M with linear connection ∇^j, $j \in \{1,2\}$. It is known that $\mathcal{E}_1^* \otimes \mathcal{E}_2$ is isomorphic to the vector bundle $Hom(\mathcal{E}_1, \mathcal{E}_2)$, consisting of homomorphisms from $\mathcal{E}_1$ into $\mathcal{E}_2$. We denote by the same $\langle , \rangle$ the fiber metrics on $\mathcal{E}_1$ and $\mathcal{E}_2$ as well as that induced on $Hom(\mathcal{E}_1, \mathcal{E}_2)$. The connections ∇^1 and ∇^2 induce on $Hom(\mathcal{E}_1, \mathcal{E}_2)$ a connection $\bar{\nabla}$, given by

$$(\bar{\nabla}_X F)V = (\bar{\nabla} F)(V; X) = \nabla^2_X FV - F\nabla^1_X V$$

for any vector field X tangent to M, cross sections V in $\mathcal{E}_1$ and F in $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$.

A section F in $Hom(\mathcal{E}_1, \mathcal{E}_2)$ is said to be *recurrent* if there exists 1-form τ such that $\bar{\nabla} F = F \otimes \tau$. We may regard parallelism as a special case of recurrence, that is, the case $\tau = 0$. Let $\bar{R}$, R^1 and R^2 be the curvature tensor corresponding to $\bar{\nabla}$, ∇^1 and ∇^2 respectively. Then we have

$$(\bar{R} \cdot F)(V; X, Y) = (\bar{R}(X, Y)F)V = R^2(X, Y)FV - FR^1(X, Y)V$$

for any $X, Y \in TM$, $V \in \mathcal{E}_1$ and $F \in Hom(\mathcal{E}_1, \mathcal{E}_2)$.

We have the following result from [5]:

Lemma 4.1. *[5] Let M be a Riemannian manifold, $\mathcal{E}_j$ a Riemannian vector bundle over M, $j \in \{1,2\}$ and F a section in $Hom(\mathcal{E}_1, \mathcal{E}_2)$. If F is recurrent then F is semi-parallel.*

From Lemma 4.1 and Corollary 1.7 we obtain the following:

Corollary 4.2. *There does not exist any real hypersurface with recurrent normal Jacobi operator in $G_2(\mathbb{C}_{m+2})$, $m \geq 3$.*

As a corollary we have the following:

Corollary 4.3. *There does not exist any real hypersurface with parallel normal Jacobi operator in $G_2(\mathbb{C}_{m+2})$, $m \geq 3$.*

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