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Strong Convergence of the Tamed Euler Method for Stochastic Differential Equations with Piecewise Continuous Arguments and Poisson Jumps

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Abstract. In the present work, the tamed Euler method is proven to be strongly convergent for stochastic differential equations with piecewise continuous arguments and Poisson jumps, where the diffusion and jump coefficients are globally Lipschitz continuous, the drift coefficient is one-sided Lipschitz continuous, and its derivative demonstrates an at most polynomial growth. Moreover, the convergence rate is obtained.

1. Introduction

In recent years, tremendous interest has been expressed for developing stochastic differential equations (SDEs) with Poisson jumps (SDEwjs), which play an important role in numerous applications. In particular, the convergence properties of SDEwjs have been the subject of extensive study and application. Of course, the local Lipschitz condition is not sufficient to guarantee the exsitence of the global solution, but SDEwjs demonstrate unique solutions under the global Lipschitz condition when the linear growth condition is included. Moreover, the convergence of the Euler method for SDEwjs under local Lipschitz and linear growth conditions has been studied by Yu [17]. In addition, the weak convergence of numerical methods for SDEwjs has been studied [2, 3, 6, 10, 13–15], and other convergence concepts have been investigated as well, e.g., convergence in probability [16] and almost sure convergence [11, 12].

However, limited work has been conducted for the numerical analysis of SDEs with piecewise continuous arguments and Poisson jumps (SEPCAswjs). It is apparent from the equations that studies of SEPCAswjs have been motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems, and combine the properties of both differential and difference equations. Due to the stochastic factors, numerical solutions of the systems play an important role for the investigation of their analytical solutions. In the past decades, substantial effort has been devoted to the mathematical study of the convergence and stability of SDEs with piecewise continuous arguments (SEPCAs) under standard conditions. For example, Dai et al.[4] have given the sufficient condition of convergence for linear SEPCAs. In 2011, Zhang et al.[19] studied the existence of the solution and the mean-square convergence of the Euler

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method for non-linear SEPCAs under global Lipschitz and linear growth conditions. Meanwhile, the local Lipschitz condition and $p-$th moment boundedness have been applied to guarantee the strong convergence of the numerical solutions of SEPCAs [19]. The classical explicit Euler-Maruyama (EM) method has attracted considerable attention because of its simple algebraic structure, low computational cost, and an acceptable convergence rate under the global Lipschitz condition [7]. However, if the standard conditions are relaxed, some of the results are unfortunately not correct. The explicit Euler method is divergent under the condition of super-linear growth [9]. In 2012, Martin et al.[8] improved the Euler method, and imparted strong convergence for SDEs under the condition of superlinear growth. In the present work, we prove that the tamed Euler method is strongly convergent for SEPCAswjs, where the coefficients satisfy non-Lipschitz conditions. The key issue for the convergence study is the p-th moment boundedness of the numerical solutions. Moreover, we also give the rate of the convergence.

The remainder of this article is structured as follows. In Section 2, we introduce the tamed Euler method and some preliminary concepts, and formulate the main result. In Section 3, we introduce various lemmas, which are used for the proof of the main result. In Section 4, we give a detailed proof of the main theorem.

2. Numerical Method and Main Result

2.1. Setting

Throughout this paper, unless otherwise specified, we use the following notations. We define $[\cdot]$ to be the greatest-integer function. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). $\mathcal{L}^1([0,\infty); R)$ denotes a family of continuous functions that satisfy $\int_0^\infty |\mu(t)|dt < \infty$, and $\mathcal{L}^2([0,\infty); R)$ is a family of continuous functions that satisfy $\int_0^\infty |\mu(t)|^2 dt < \infty$. Moreover, let $\|\cdot\|_{L^p} = (E\|\cdot\|^p)^{\frac{1}{p}}$ denote the L^p norm.

The following 1-dimensional SEPCAs with Poisson random measure is considered in our paper

$$dx(t) = \mu(x(t^-), x([t^-]))dt + \sigma(x(t^-), x([t^-]))dW(t) + g(x(t^-), x([t^-]))dN(t), \quad 0 < t \leq T, \tag{2.1}$$

Here $x(0) = \xi, T \in \mathcal{R}^+$, $\lim_{s\to t^-} x(s) = x(t^-)$, and the drift $\mu : R \times R \to R$, diffusion coefficient $\sigma : R \times R \to R$ and the jump coefficient $g : R \times R \to R$ are assumed to be Borel measurable functions and the coefficients are sufficiently smooth. Let $W(t)$ be a one-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. We define $N(t)$ to be a scalar Poisson process with intensity λ and we require finite intensity $0 < \lambda < 1$. $\tilde{N}(t) = N(t) - \lambda t$ is referred to as compensated Poisson process which is a martingale. The process $x(t)$ is thus defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, and denotes the exact solution of the equation (2.1) with ξ is a $\mathcal{F}_0/\mathcal{B}(R)$-measurable random variable. The Wiener process and the Poisson process are mutually independent.

Definition 2.1. *An real value stochastic process $\{x(t)\}_{0\leq t\leq T}$ is called a solution of equation (2.1) if it has the following properties:*
(i)$\{x(t)\}$ is $\mathcal{F}_t-$adapted;
(ii)$\{x(t)\}$ is càdlàg;
(iii)equation (2.1) holds for every $t \in [n, n+1)$ with probability 1.
A solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$P\{x(t) = \bar{x}(t) \ \ for\ all\ 0 \leq t \leq T\} = 1.$$

Assumption 2.2. *Assume there exists $L \in [0,\infty)$, and $C > 1$, such that for every smooth function $f : R \to R$ and any $x, x_1, x_2, y, y_1, y_2 \in R$, satisfying*

$$|\sigma(x_1,y_1)-\sigma(x_2,y_2)| \le L(|x_1-x_2|+|y_1-y_2|), \tag{2.2}$$
$$|g(x_1,y_1)-g(x_2,y_2)| \le L(|x_1-x_2|+|y_1-y_2|), \tag{2.3}$$
$$|\mu(x,y_1)-\mu(x,y_2)| \le L|y_1-y_2|, \tag{2.4}$$
$$|\mu_x(x,y)| \le L(1+|x|^C), \tag{2.5}$$
$$\Big\langle x_1-x_2, \mu(x_1,y)-\mu(x_2,y)\Big\rangle \le L|x_1-x_2|^2. \tag{2.6}$$

Remark 2.3. *$\mu(x,y)$ satisfies one-sided Lipschitz condition on x and global Lipschitz condition on y. Furthermore, the p-moment of the exact solution is bounded in any finite interval $[0,T]$(see[19]).*

2.2. Existence, uniqueness and moment of the exact solution

By extending the non-jump proof of [8], it can be shown that a unique solution exists for (2.1) under our assumptions. The essential change to that proof is the inclusion of the jump term; this can be estimated with the martingle isometry for the compensated Poisson process:

$$E(\int_0^t F(s^-)d\tilde{N}(s))^2 = \lambda \int_0^t E|F(s)|^2 ds,$$

which holds for appropriate integrand functions F (in particular , non-anticipative, if random). Thereafter such terms are handled in the same way as the Ito integral terms. This leads to a solution on any bounded time interval $[0,T]$ with $E|x(t)|^2 \le C(1+E|x(0)|^2)$ for some constant $C=C(T)$.

2.3. Numerical Method

In this paper, we investigate the strong convergence of the Tamed Euler method in finite interval. Let $h=\frac{1}{m}>0$ be a given step-size with integer $m\in \mathbf{N}^+$ and grid points t_n be defined by $t_n=nh(n=1,2,\cdots)$. The independent version of the Tamed Euler method applied to (2.1) computes approximations Y_n, by setting $Y_0=\xi$ and forming

$$Y_{n+1} = Y_n + \frac{\mu(Y_n, Y_{[nh]m})}{1+|\mu(Y_n,Y_{[nh]m})|h}h + \sigma(Y_n,Y_{[nh]m})\triangle W_n + g(Y_n,Y_{[nh]m})\triangle N_n, \tag{2.7}$$

for $n=0,1,2,\cdots,Tm$ and for simplicity, we assume $T=Nh$. Here $\triangle W_n = W(t_{n+1})-W(t_n)$ and $\triangle N_n = N(t_{n+1})-N(t_n)$. Actually, we can see [18] that $\triangle N_n \sim$ Poisson(λh), $E[\triangle N_n]=\lambda h$, $Var(\triangle N_n)=\lambda h$, and $E|\triangle N_n|^i = \sum_{j=1}^{i}(\lambda h)^j \frac{1}{j!}\sum_{s=0}^{j}(-1)^{j-s}C_j^s s^i$(where $C_j^s=\frac{j!}{s!(j-s)!}$). Furthermore, let $n=km+l,(km+l\in\{0,1,2,\cdots,Tm\}, l=0,1,2,\cdots,m-1)$. The equation (2.7) becomes the following type:

$$Y_{km+l+1} = Y_{km+l} + \frac{\mu(Y_{km+l},Y_{km})}{1+|\mu(Y_{km+l},Y_{km})|h}h + \sigma(Y_{km+l},Y_{km})\triangle W_{km+l} + g(Y_{km+l},Y_{km})\triangle N_{km+l}, \tag{2.8}$$

where $\triangle W_{km+l} = W(t_{km+l+1})-W(t_{km+l})$ and $\triangle N_{km+l}=N(t_{km+l+1})-N(t_{km+l})$. We denote the piecewise constant interpolant of the Tamed Euler method solution by $Y(\underline{t})=Y_{km+l}$, $Y([\underline{t}])=Y_{km}$ for $t\in[t_{km+l},t_{km+l+1})$. We then define the "piecewise linear" interpolant by

$$\bar{Y}(t) = x_0 + \int_0^t \frac{\mu(Y(\underline{s^-}),Y([\underline{s^-}]))}{1+|\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|h}ds + \int_0^t \sigma(Y(\underline{s^-}),Y([\underline{s^-}]))dW(s) + \int_0^t g(Y(\underline{s^-}),Y([\underline{s^-}]))dN(s), \tag{2.9}$$

for $t\in[t_{km+l},t_{km+l+1})$.

2.4. Main Result

We now prove that Assumption 2.2 is sufficient condition to ensure strong convergence of the Tamed Euler method for the equation (2.1).

Theorem 2.4. *If Assumption 2.2 holds, then for every $p \geq 2$, we have*

$$(E|x(t) - \bar{Y}(t)|^p)^{\frac{1}{p}} \leq Mh^{\frac{1}{2}},$$

for $\omega \in \Omega$ and $t \in [0, T]$. Here M is independent of h.

3. Lemmas

In order to prove Theorem 2.4, the following objects are needed. First of all, let the $\mathcal{F}/\mathcal{B}(R)$-measurable mappings $Q_{km+l} : \Omega \to R, R_{km+l} : \Omega \to R, A_{km+l} : \Omega \to R$, and $C_{km+l} : \Omega \to R$ be defined as

$$\begin{aligned}
Q_{km+l}(\omega) &= Lh + sgn(Y_{km+l})\frac{\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0,0)}{|Y_{km+l}| + |Y_{km}|}\triangle W_{km+l}(\omega),\\
R_{km+l}(\omega) &= sgn(Y_{km+l})\frac{g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0,0)}{|Y_{km+l}| + |Y_{km}|}\triangle N_{km+l}(\omega),\\
A_{km+l}(\omega) &= Q_{km+l}(\omega) + R_{km+l}(\omega),\\
C_{km+l}(\omega) &= \mu(0,0)h + \sigma(0,0)\triangle W_{km+l}(\omega),
\end{aligned} \tag{3.1}$$

for every $\omega \in \Omega, km + l \in \{0, 1, \cdots, Tm\}$ and $l \in \{0, 1, \cdots, m-1\}$. Let the $\mathcal{F}/\mathcal{P}(\{0, 1, \cdots, n\})$-measurable mapping

$$\tau_{km+l}(\omega) = \max\left(\{0\} \cup \{vm + j \in \{1, 2, \cdots, km + l\} | sgn(Y_{vm+j}(\omega)) \neq sgn(Y_{vm+j-1}(\omega))\}\right),$$

for every $\omega \in \Omega, \tau_0(\omega) = 0$ and every $km + l \in \{0, 1, \cdots, Tm\}, l \in \{0, 1, \cdots, m-1\}$. Using these notations, we now define $D_{\tau_n}(\omega) : \Omega \to R$ which is $\mathcal{F}/\mathcal{B}(R)$-measurable, given by

$$D_{\tau_{km+l}}(\omega) := \alpha_{km+l}(\omega) + \beta_{km+l}\Big(\alpha_{m[\frac{km+l-1}{m}]}(\omega) + \sum_{j=1}^{[\frac{km+l-1}{m}]-1} \alpha_{jm}(\omega) \prod_{s=j+1}^{[\frac{km+l-1}{m}]} \beta_{sm}(\omega) + |Y_0| \prod_{j=1}^{[\frac{km+l-1}{m}]} \beta_{jm}(\omega)\Big), \tag{3.2}$$

where

$$\begin{aligned}
\alpha_{km+l}(\omega) =& (9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|) \exp\Big(\sum_{j=\tau_{km+l}}^{km+l-1} A_j(\omega)\Big)\\
&+ \sup_{v \in \{0,1,\cdots,km+l\}} \Big(\sum_{j=v}^{km+l-1} (sgn(Y_j) C_j(\omega) \exp\big(\sum_{s=j+1}^{km+l-1} A_s(\omega)\big))\Big)\\
&+ \sup_{v \in \{0,1,\cdots,km+l\}} \Big(\sum_{j=v}^{km+l-1} (sgn(Y_j) g(0,0)(\omega) \triangle N_j \exp\big(\sum_{s=j+1}^{km+l-1} A_s(\omega)\big))\Big),
\end{aligned} \tag{3.3}$$

$$\beta_{km+l}(\omega) = \exp\Big(\sum_{j=m[\frac{km+l-1}{m}]}^{km+l-1} A_j(\omega)\Big) + \sup_{v \in \{0,1,\cdots,km+l\}} \Big|\sum_{j=v}^{km+l-1} A_j(\omega) \exp\big(\sum_{s=j+1}^{km+l-1} A_s(\omega)\big)\Big|,$$

for every $v \in \{0, 1, \cdots, Tm\}$ and $km + l \in \{0, 1, \cdots, Tm\}, l \in \{0, 1, \cdots, m-1\}$. As usual, $\sum_{j=v}^{w-1} \alpha_j = 0$ and $\Pi_{j=v}^{w-1} \beta_j = 1$ for every $v > w - 1$.

Remark 3.1. *It is apparent from the definitions of* $\alpha_{km+l}, \beta_{km+l}$ *and* $D_{\tau_{km+l}}$ *that* $\alpha_{km+l} \geq 0, \beta_{km+l} \geq 0$, *and* $\alpha_{km+l} + \beta_{km+l} D_{\tau_{km}} = D_{\tau_{km+l}}$ *for all* $l = 1, 2, \cdots, m-1, km+l \in \{0, 1, \cdots, Tm\}$.

Next we define these events $\Omega_{km+l}, km+l \in \{0, 1, \cdots, Tm\}$ and every $m \in \mathbf{N}^+$. Let the real number $r^h = \frac{1}{h^{\frac{1}{C+1}}} \in [0, \infty)$. Let

$$\Omega_{km+l} := \{\omega \in \Omega | \sup_{j \in \{0,1,\cdots,km+l-1\}} |D_{\tau_j}(\omega)| \leq r^h, \sup_{j \in \{0,1,\cdots,km+l-1\}} |\triangle W_j(\omega)| \leq h^{\frac{1}{C+1}}, \sup_{j \in \{0,1,\cdots,km+l-1\}} |\triangle N_j(\omega)| \leq h^{\frac{1}{C+1}}\},$$

for all $l \in \{0, 1, \cdots, m-1\}$. We use the following lemmas in order to prove Theorem 2.4.

Lemma 3.2. *Let* $Q_j(\omega)$ *be given by* (3.1) *with* $j = 0, 1, \cdots$, *then we have*

$$\sup_{z \in \{-1,1\}} \| \exp(\sum_{j=v}^{u-1} z Q_j) \|_{L^p} \leq e^{LT + \frac{p}{2} L^2 T},$$

for all v, u *with* $0 \leq v \leq u \leq Tm, m \in \mathbf{N}^+$, $p \geq 1$ *and* $z \in \{-1, 1\}$.

Proof. Note that the time discrete stochastic process $\sum_{j=v}^{u-1} z \frac{\sigma(Y_j(\omega), Y_{[jh]m}(\omega)) - \sigma(0,0)}{|Y_j| + |Y_{[jh]m}|} \triangle W_j(\omega)$ is an $\mathcal{F}_{t_u}$ martingale for $z \in \{-1, 1\}$. Therefore, the time discrete stochastic process $\exp\left(\sum_{j=v}^{u-1} z \frac{\sigma(Y_j(\omega), Y_{[jh]m}(\omega)) - \sigma(0,0)}{|Y_j| + |Y_{[jh]m}|} \triangle W_j(\omega)\right)$ is a positive $\mathcal{F}_{t_u}$ sub-martingale. Using the formula,

$$E[\exp(cY)] = \exp(\frac{c^2}{2}), \tag{3.4}$$

where Y is a standard normally distributed $\mathcal{F}/\mathcal{B}(R)$-measurable mapping, we have by (2.2) for all $x, y \in \mathbf{R}, z \in \{-1, 1\}$ and $p \geq 1$

$$\begin{aligned} &E[\exp(zpLh + pz\frac{\sigma(x, y) - \sigma(0, 0)}{|x| + |y|} \triangle W_j)] \\ \leq &e^{pLh} E[\exp(pz\frac{\sigma(x, y) - \sigma(0, 0)}{|x| + |y|} \triangle W_j)] \\ \leq &e^{pLh} e^{\frac{p^2 L^2}{2} h} = e^{(pL + \frac{p^2 L^2}{2})h}, \end{aligned} \tag{3.5}$$

for all $x, y \in \mathcal{R}, z \in \{-1, 1\}$ and $p \geq 1$, which implies that

$$E[\exp(zpQ_j) | \mathcal{F}_{t_j}] \leq e^{(pL + \frac{p^2 L^2}{2})h},$$

for $j = 0, 1, \cdots, Tm$, and

$$\begin{aligned} &E[\exp(zp\sum_{v}^{u-1} Q_j)] \leq E[\exp(zp\sum_{v}^{u-2} Q_j) E[e^{pzQ_{u-1}} | \mathcal{F}_{t_{u-1}}]] \\ \leq &E[\exp(zp\sum_{v}^{u-2} Q_j)] e^{(pL + \frac{p^2 L^2}{2})h} \leq e^{(pL + \frac{p^2 L^2}{2})T}. \end{aligned} \tag{3.6}$$

Therefore, $\| \exp(\sum_{v}^{u-1} zQ_j) \|_{L^p} \leq e^{(L + \frac{pL^2}{2})T}$. This completes the proof of the Lemma 3.2. □

Lemma 3.3. *Let* $R_j(\omega)$ *be given by* (3.1), *then we have*

$$\| \sup_{km+l \in \{0,1,\cdots,Tm\}} \exp(\sum_{j=0}^{km+l-1} R_j(\omega)) \|_{L^p} \leq \frac{2p}{2p-1} e^{LT\lambda(e^{2pL} - 1)}.$$

Proof. Using the Hölder inequality and the formula (2.3), we have

$$\begin{aligned}
&\| \sup_{km+l\in\{0,1,\cdots,Tm\}} \exp(\sum_{j=0}^{km+l-1} R_j(\omega)) \|_{L^p} \\
&= \| \sup_{km+l\in\{0,1,\cdots,Tm\}} \exp(\sum_{j=0}^{km+l-1} sgn(Y_j)\frac{g(Y_j, Y_{[jh]m}) - g(0,0)}{|Y_j| + |Y_{[jh]m}|} \triangle N_j(\omega)) \|_{L^p} \\
&\leq \| \sup_{km+l\in\{0,1,\cdots,Tm\}} \exp(\sum_{j=0}^{km+l-1} L\triangle\tilde{N}_j(\omega) sgn(Y_j)) \|_{L^{2p}} \cdot \| \sup_{km+l\in\{0,1,\cdots,Tm\}} \exp(\sum_{j=0}^{km+l-1} L sgn(Y_j)h\lambda) \|_{L^{2p}},
\end{aligned} \tag{3.7}$$

According to the Doob's martingale inequality, we have

$$\begin{aligned}
&\| \sup_{km+l\in\{0,1,\cdots,Tm\}} \exp(\sum_{j=0}^{km+l-1} R_j(\omega)) \|_{L^p} \\
&\leq \frac{2p}{2p-1}\Big(E[\exp(2p\sum_{j=0}^{Tm-1} L\triangle\tilde{N}_j(\omega))]\Big)^{\frac{1}{2p}} \cdot \| \sup_{km+l\in\{0,1,\cdots,Tm\}} \prod_{j=0}^{km+l-1} \exp(Lh\lambda) \|_{L_{2p}} \\
&\leq \frac{2p}{2p-1}\Big(E[\exp(2p\sum_{j=0}^{Tm-1} L(\triangle N_j(\omega) - h\lambda)]\Big)^{\frac{1}{2p}} \cdot \| \sup_{km+l\in\{0,1,\cdots,Tm\}} \prod_{j=0}^{km+l-1} \exp(Lh\lambda) \|_{L_{2p}} \\
&\leq \frac{2p}{2p-1}\Big(E[\exp(2p\sum_{j=0}^{Tm-1} L(\triangle N_j(\omega))]\Big)^{\frac{1}{2p}},
\end{aligned} \tag{3.8}$$

Applying the formula $E[e^{uX}] = e^{\lambda(e^u-1)}$ in [5], we obtain

$$\begin{aligned}
\| \sup_{km+l\in\{0,1,\cdots,Tm\}} \exp(\sum_{j=0}^{km+l-1} R_j(\omega)) \|_{L^p} &\leq \frac{2p}{2p-1}(\prod_{j=0}^{Tm-1} e^{2pL\lambda h(e^{2pL}-1)})^{\frac{1}{2p}} \\
&\leq \frac{2p}{2p-1} e^{LT\lambda(e^{2pL}-1)}.
\end{aligned} \tag{3.9}$$

This completes the proof of the Lemma 3.3. □

Lemma 3.4. *If the Assumptions (2.2),(2.3),(2.4) and (2.5) hold, then for $\omega \in \Omega_{km+l}$ and $p \geq 1$, we have*

$$|Y_{km+l}(\omega)| \leq D_{\tau_{km+l}}(\omega). \tag{3.10}$$

Proof. Let $km + l \in \{0, 1, \cdots, Tm\}$ be fixed and arbitrary. We establish (3.10) in the case $Y_{km} \geq 0$ since the case $Y_{km} \leq 0$ is immediately follows from the case $Y_{km} \geq 0$. Now we establish by induction on $km + l \in \{0, 1, \cdots, Tm\}$ where $k \in N, l \in \{0, 1, \cdots, m-1\}$, and m is fixed. First of all, when $\tau_0(\omega) = 0$, we have

$$\begin{aligned}
|Y_0(\omega)| =& |\xi| \leq (9L + |\mu(0,0)| + |\sigma(0,0)| + |g(0,0)| + |\xi|) \\
=& (9L + |\mu(0,0)| + |\sigma(0,0)| + |g(0,0)|) \exp(\sum_{j=\tau_0}^{-1} A_j) + \sum_{j=0}^{-1}(C_j sgn(Y_j) \exp(\sum_{s=j+1}^{-1} A_s)) \\
&+ [\exp(\sum_{j=-m}^{-1} A_j) + |\sum_{j=0}^{-1} A_j \exp(\sum_{s=j+1}^{-1} A_s)|]|\xi| \\
=& \alpha_0 + \beta_0|\xi| = D_{\tau_0}.
\end{aligned} \tag{3.11}$$

for all $\omega \in \Omega_0$, which shows (3.10) in the base case $km + l = 0$. Now we suppose that (3.10) holds for $im + j$

$$|Y_{im+j}(\omega)| \leq D_{\tau_{im+j}}(\omega), \tag{3.12}$$

where $\omega \in \Omega_{im+j}, im + j \in \{0, 1, \cdots, km + l\}$. Moreover, we fix an arbitrary $\omega \in \Omega_{km+l+1} \subset \Omega_{km+l}$ and now we show (3.10) for $\omega \in \Omega_{km+l+1}$. For this we divide it into four different cases.
(i) If $Y_{km+l} \geq 0$ and $Y_{km+l+1} < 0$ hold, then we obtain $\tau_{km+l+1} = km + l + 1$. Moreover, note that

$$|Y_{im+j}(\omega)| \leq \sup_{j \in \{0,1,\cdots,km+m-1\}} D_{\tau_j}(\omega) \leq r^h, \tag{3.13}$$

holds due to the induction hypothesis and $\omega \in \Omega_{km+l+1} \subset \Omega_{im+j}$ for $im + j \in \{0, 1, \cdots, km + l\}$. Therefore, we obtain

$$\begin{aligned}
0 >& Y_{km+l+1}(\omega)\\
=& Y_{km+l}(\omega) + \frac{\mu(Y_{km+l}(\omega), Y_{km}(\omega))}{1 + |\mu(Y_{km+l}(\omega), Y_{km}(\omega))|h} h + \sigma(Y_{km+l}(\omega), Y_{km}(\omega)) \triangle W_{km+l} + g(Y_{km+l}(\omega), Y_{km}(\omega)) \triangle N_{km+l}\\
=& Y_{km+l}(\omega) + \frac{1}{1 + |\mu(Y_{km+l}(\omega), Y_{km}(\omega))|h} [(\mu(Y_{km+l}(\omega), Y_{km}(\omega))\\
&- \mu(Y_{km+l}(\omega), 0))h + (\mu(Y_{km+l}(\omega), 0) - \mu(0, 0))h + \mu(0, 0)h]\\
&+ (\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0, 0)) \triangle W_{km+l} + \sigma(0, 0) \triangle W_{km+l}\\
&+ (g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0, 0)) \triangle N_{km+l} + g(0, 0) \triangle N_{km+l}\\
\geq& \frac{1}{1 + |\mu(Y_{km+l}(\omega), Y_{km}(\omega))|h} [-|\mu(Y_{km+l}(\omega), Y_{km}(\omega)) - \mu(Y_{km+l}(\omega), 0)|h\\
&- |\mu(Y_{km+l}(\omega), 0) - \mu(0, 0)|h - |\mu(0, 0)|h]\\
&- |\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0, 0)||\triangle W_{km+l}| - |\sigma(0, 0)||\triangle W_{km+l}|\\
&- |g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0, 0)||\triangle N_{km+l}| - |g(0, 0)||\triangle N_{km+l}|\\
\geq& [-|\mu(Y_{km+l}(\omega), Y_{km}(\omega)) - \mu(Y_{km+l}(\omega), 0)|h\\
&- |\mu(Y_{km+l}(\omega), 0) - \mu(0, 0)|h - |\mu(0, 0)|h]\\
&- |\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0, 0)||\triangle W_{km+l}| - |\sigma(0, 0)||\triangle W_{km+l}|\\
&- |g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0, 0)||\triangle N_{km+l}| - |g(0, 0)||\triangle N_{km+l}|\\
\geq& - L|Y_{km}(\omega)|h - hL(1 + |Y_{km+l}|^C)|Y_{km+l}| - |\mu(0, 0)|\\
&- L|Y_{km+l}(\omega)||\triangle W_{km+l}| - L|Y_{km}(\omega)||\triangle W_{km+l}| - |\sigma(0, 0)||\triangle W_{km+l}|\\
&- L|Y_{km+l}(\omega)||\triangle N_{km+l}| - L|Y_{km}(\omega)||\triangle N_{km+l}| - |g(0, 0)||\triangle N_{km+l}|\\
\geq& - 9L - |\mu(0, 0)| - |\sigma(0, 0)| - |g(0, 0)|.
\end{aligned} \tag{3.14}$$

(ii) If $Y_{km+l} < 0$ and $Y_{km+l+1} \geq 0$ hold, then $\tau_{km+l+1} = km + l + 1$. Similarly, we also see for $\omega \in \Omega_{km+l+1}$, we have

$$\begin{aligned}
0 \leq& Y_{km+l+1}(\omega)\\
=& Y_{km+l}(\omega) + \frac{\mu(Y_{km+l}(\omega), Y_{km}(\omega))}{1 + |\mu(Y_{km+l}(\omega), Y_{km}(\omega))|h} h\\
&+ \sigma(Y_{km+l}(\omega), Y_{km}(\omega)) \triangle W_{km+l} + g(Y_{km+l}(\omega), Y_{km}(\omega)) \triangle N_{km+l}\\
=& Y_{km+l}(\omega) + \frac{1}{1 + |\mu(Y_{km+l}(\omega), Y_{km}(\omega))|h} [(\mu(Y_{km+l}(\omega), Y_{km}(\omega))\\
&- \mu(Y_{km+l}(\omega), 0))h + (\mu(Y_{km+l}(\omega), 0) - \mu(0, 0))h + \mu(0, 0)h]\\
&+ (\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0, 0)) \triangle W_{km+l} + \sigma(0, 0) \triangle W_{km+l}\\
&+ (g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0, 0)) \triangle N_{km+l} + g(0, 0) \triangle N_{km+l}
\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&\le L|Y_{km}(\omega)|h + hL(1+|Y_{km+l}|^{C})|Y_{km+l}| + |\mu(0,0)|\\
&\quad + L|Y_{km+l}(\omega)||\triangle W_{km+l}| + L|Y_{km}(\omega)||\triangle W_{km+l}| + |\sigma(0,0)||\triangle W_{km+l}|\\
&\quad + L|Y_{km+l}(\omega)||\triangle N_{km+l}| + L|Y_{km}(\omega)||\triangle N_{km+l}| + |g(0,0)||\triangle N_{km+l}|\\
&\le 9L + |\mu(0,0)| + |\sigma(0,0)| + |g(0,0)|.
\end{aligned}
$$

Therefore, when $\tau_{km+l+1} = km+l+1$, we can have

$$
\begin{aligned}
|Y_{km+l+1}(\omega)| \le& 9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|\\
\le&(9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|)\exp\big(\sum_{j=\tau_{km+l+1}}^{km+l} A_j\big)\\
&+\Big(\exp\big(\sum_{j=km}^{km+l} A_j\big) + \sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(C_j sgn(Y_j(\omega))\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big)\\
&+\sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(g(0,0)sgn(Y_j(\omega))\triangle N_j\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big)\\
&+\sup_{v\in\{0,1,\cdots,km+l+1\}}\Big|\sum_{j=v}^{km+l}\big(A_j\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\big)\Big|\Big)\Big(\alpha_{\tau_{km},km}\\
&+\sum_{j=1}^{k-1}\alpha_{jm}\Pi_{s=j+1}^{k}\beta_{sm} + |\xi|\Pi_{j=1}^{k}\beta_{jm}\Big)\Big]\\
=&D_{\tau_{km+l+1}}(\omega).
\end{aligned}
\tag{3.16}
$$

which shows that (3.11) holds in this case for $\omega \in \Omega_{km+l+1}$ and $km+l+1$.
(iii) If $Y_{km+l} > 0$ and $Y_{km+l+1} \ge 0$ hold, then $\tau_{km+l+1} = \tau_{km+l}$. And we observe

$$x(\mu(x,y) - \mu(0,y)) \le Lx^2,$$

Then, we have

$$x(\mu(x,y) - \mu(0,y) - Lx) \le 0.$$

for $x \in R$. Hence we obtain

$$
\begin{aligned}
0 \le& Y_{km+l+1}(\omega)\\
=&Y_{km+l}(\omega) + \frac{\mu(Y_{km+l}(\omega), Y_{km}(\omega))}{1+h|\mu(Y_{km+l}(\omega), Y_{km}(\omega))|}h + \sigma(Y_{km+l}(\omega), Y_{km}(\omega))\triangle W_{km+l} + g(Y_{km+l}(\omega), Y_{km}(\omega))\triangle N_{km+l}\\
=&Y_{km+l}(\omega) + \frac{1}{1+h|\mu(Y_{km+l}(\omega), Y_{km}(\omega))|}[(\mu(Y_{km+l}(\omega), Y_{km}(\omega))\\
&-\mu(0, Y_{km}(\omega)) - LY_{km+l}(\omega))h + LY_{km+l}(\omega)h + (\mu(0, Y_{km}(\omega)) - \mu(0,0))h + \mu(0,0)h]\\
&+\sigma(0,0)\triangle W_{km+l} + \Big(\frac{\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0,0)}{|Y_{km+l}| + |Y_{km}|}\Big)(|Y_{km+l}| + |Y_{km}|)\triangle W_{km+l}\\
&+g(0,0)\triangle N_{km+l} + \Big(\frac{g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0,0)}{|Y_{km+l}| + |Y_{km}|}\Big)(|Y_{km+l}| + |Y_{km}|)\triangle N_{km+l}\\
\le&|Y_{km+l}(\omega)|(1 + hL + \frac{\sigma(Y_{km+l}(\omega), Y_{km}) - \sigma(0,0)}{|Y_{km+l}| + |Y_{km}|}\triangle W_{km+l} + \frac{g(Y_{km+l}(\omega), Y_{km}) - g(0,0)}{|Y_{km+l}| + |Y_{km}|}\triangle N_{km+l})\\
&+\mu(0,0)h + \sigma(0,0)\triangle W_{km+l} + |Y_{km}(\omega)|(hL + \frac{\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0,0)}{|Y_{km+l}| + |Y_{km}|}\triangle W_{km+l}\\
&+\frac{g(Y_{km+l}(\omega), Y_{km}) - g(0,0)}{|Y_{km+l}| + |Y_{km}|}\triangle N_{km+l}) + g(0,0)\triangle N_{km+l}.
\end{aligned}
\tag{3.17}
$$

for $\omega \in \Omega_{km+l+1}$.
(iv) If $Y_{km+l} \le 0$ and $Y_{km+l+1} < 0$ hold, then $\tau_{km+l+1} = \tau_{km+l}$. For $\omega \in \Omega_{km+l+1}$, we also obtain

$$\begin{aligned}
0 \ge & Y_{km+l+1}(\omega) \\
= & Y_{km+l}(\omega) + \frac{\mu(Y_{km+l}(\omega), Y_{km}(\omega))}{1 + h|\mu(Y_{km+l}(\omega), Y_{km}(\omega))|} h + \sigma(Y_{km+l}(\omega), Y_{km}(\omega)) \triangle W_{km+l} + g(Y_{km+l}(\omega), Y_{km}(\omega)) \triangle N_{km+l} \\
= & Y_{km+l}(\omega) + \frac{1}{1 + h|\mu(Y_{km+l}(\omega), Y_{km}(\omega))|} [(\mu(Y_{km+l}(\omega), Y_{km}(\omega)) - \mu(0, Y_{km}(\omega)) - LY_{km+l}(\omega))h \\
& + LY_{km+l}(\omega)h + (\mu(0, Y_{km}(\omega)) - \mu(0,0))h + \mu(0,0)h] \\
& + \sigma(0,0)\triangle W_{km+l} + (\frac{\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0,0)}{|Y_{km+l}| + |Y_{km}|})(|Y_{km+l}| + |Y_{km}|)\triangle W_{km+l} + g(0,0)\triangle N_{km+l} \\
& + (\frac{g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0,0)}{|Y_{km+l}| + |Y_{km}|})(|Y_{km+l}| + |Y_{km}|)\triangle N_{km+l} \\
\ge & -|Y_{km+l}(\omega)|(1 + hL + sgn(Y_{km+l}) \frac{\sigma(Y_{km+l}(\omega), Y_{km}) - \sigma(0,0)}{|Y_{km+l}| + |Y_{km}|} \triangle W_{km+l} \\
& + sgn(Y_{km+l}) \frac{g(Y_{km+l}(\omega), Y_{km}) - g(0,0)}{|Y_{km+l}| + |Y_{km}|} \triangle N_{km+l}) + \mu(0,0)h \\
& - |Y_{km}(\omega)|(hL + sgn(Y_{km+l}) \frac{\sigma(Y_{km+l}(\omega), Y_{km}(\omega)) - \sigma(0,0)}{|Y_{km+l}| + |Y_{km}|} \triangle W_{km+l} \\
& + sgn(Y_{km+l}) \frac{g(Y_{km+l}(\omega), Y_{km}(\omega)) - g(0,0)}{|Y_{km+l}| + |Y_{km}|} \triangle N_{km+l}) + \sigma(0,0)\triangle W_{km+l} + g(0,0)\triangle N_{km+l}.
\end{aligned} \tag{3.18}$$

Hence, when $\tau_{km+l+1} = \tau_{km+l}$, we have

$$\begin{aligned}
|Y_{km+l+1}(\omega)| \le & |Y_{km+l}(\omega)| \exp(A_{km+l}) + |Y_{km}(\omega)| A_{km+l} \\
& + sgn(Y_{km+l}(\omega)) C_{km+l} + sgn(Y_{km+l}(\omega)) g(0,0) \triangle N_{km+l}.
\end{aligned} \tag{3.19}$$

Case 1 $\tau_{km+l} \ge km$, we have

$$\begin{aligned}
& |Y_{km+l+1}(\omega)| \\
\le & |Y_{km+l}(\omega)| \exp(A_{km+l}) + |Y_{km}(\omega)| A_{km+l} + sgn(Y_{km+l}(\omega)) C_{km+l} + sgn(Y_{km+l}(\omega)) g(0,0) \triangle N_{km+l} \\
\le & |Y_{\tau_{km+l}}(\omega)| \exp(\sum_{j=\tau_{km+l}}^{km+l} A_j) + |Y_{km}(\omega)| \sum_{j=\tau_{km+l}}^{km+l} (A_j \exp(\sum_{s=j+1}^{km+l} A_s)) + \sum_{j=\tau_{km+l}}^{km+l} \Big(C_j sgn(Y_j(\omega)) \exp(\sum_{s=j+1}^{km+l} A_s)\Big) \\
& + \sum_{j=\tau_{km+l}}^{km+l} \Big(g(0,0) sgn(Y_j(\omega)) \triangle N_j \exp(\sum_{s=j+1}^{km+l} A_s)\Big) \\
\le & (9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|) \exp(\sum_{j=\tau_{km+l}}^{km+l} A_j) \\
& + |Y_{km}(\omega)| \sum_{j=\tau_{km+l}}^{km+l} A_j \exp(\sum_{s=j+1}^{km+l} A_s)) + \sum_{j=\tau_{km+l}}^{km+l} \Big(C_j sgn(Y_j(\omega)) \exp(\sum_{s=j+1}^{km+l} A_s)\Big) \\
& + \sum_{j=\tau_{km+l}}^{km+l} \Big(g(0,0) sgn(Y_j(\omega)) \triangle N_j \exp(\sum_{s=j+1}^{km+l} A_s)\Big)
\end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&\leq (9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|)\exp\big(\sum_{j=\tau_{km+l+1}}^{km+l} A_j\big)\\
&\quad + |Y_{km}(\omega)|\Big(\exp\big(\sum_{j=km}^{km+l} A_j\big) + \sup_{v\in\{0,1,\cdots,km+l+1\}} \big|\sum_{j=v}^{km+l} A_j \exp\big(\sum_{s=j+1}^{km+l} A_s\big)\big|\Big)\\
&\quad + \sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(C_j sgn(Y_j(\omega))\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big) + \sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(\triangle N_j sgn(Y_j(\omega)) g(0,0)\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big),
\end{aligned}
$$

for $\omega \in \Omega_{km+l+1}$. By the hypothesis, we show

$$
\begin{aligned}
|Y_{km+l+1}(\omega)| \leq &(9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|)\exp\big(\sum_{j=\tau_{km+l+1}}^{km+l} A_j\big)\\
&+ D_{\tau_{km}}\Big(\exp\big(\sum_{j=km}^{km+l} A_j\big) + \sup_{v\in\{0,1,\cdots,km+l+1\}} \big|\sum_{j=v}^{km+l} A_j \exp\big(\sum_{s=j+1}^{km+l} A_s\big)\big|\Big)\\
&+ \sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(C_j sgn(Y_j(\omega))\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big)\\
&+ \sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(\triangle N_j sgn(Y_j(\omega)) g(0,0)\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big)\\
=&\alpha_{km+l+1} + \beta_{km+l+1} D_{\tau_{km}} = D_{\tau_{km+l+1}}(\omega).
\end{aligned}
\tag{3.21}
$$

Case 2 $\tau_{km+l} < km$, we have

$$
\begin{aligned}
|Y_{km+l+1}(\omega)| \leq &|Y_{km+l}(\omega)|\exp(A_{km+l}) + |Y_{km}(\omega)|A_{km+l} + sgn(Y_{km+l}(\omega))C_{km+l} + sgn(Y_{km+l}(\omega))g(0,0)\triangle N_{km+l}\\
\leq &|Y_{km}(\omega)|\exp\big(\sum_{j=km}^{km+l} A_j\big) + |Y_{km}|\sum_{j=km}^{km+l} A_j\exp\big(\sum_{s=j+1}^{km+l} A_s\big) + \sum_{j=km}^{km+l}\Big(C_j sgn(Y_j(\omega))\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\\
&+ \sum_{j=km}^{km+l}\Big(g(0,0)\triangle N_j sgn(Y_j(\omega))\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
&\leq (9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|)\exp\big(\sum_{j=\tau_{km+l+1}}^{km+l} A_j\big) + |Y_{km}(\omega)|\Big(\exp\big(\sum_{j=km}^{km+l} A_j\big)\\
&\quad + \sup_{v\in\{0,1,\cdots,km+l+1\}} \big|\sum_{j=v}^{km+l} A_j \exp\big(\sum_{s=j+1}^{km+l} A_s\big)\big|\Big)\\
&\quad + \sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(C_j sgn(Y_j(\omega))\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big)\\
&\quad + \sup_{v\in\{0,1,\cdots,km+l+1\}}\Big(\sum_{j=v}^{km+l}\Big(g(0,0)\triangle N_j sgn(Y_j(\omega))\exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big),
\end{aligned}
\tag{3.23}
$$

By the hypothesis, we obtain

$$
\begin{aligned}
|Y_{km+l+1}(\omega)| \leq & (9L + |\sigma(0,0)| + |\mu(0,0)| + |g(0,0)|) \exp\big(\sum_{j=\tau_{km+l+1}}^{km+l} A_j\big) \\
& + D_{\tau_{km}}\Big(\exp\big(\sum_{j=km}^{km+l} A_j\big) + \sup_{v\in\{0,1,\cdots,km+l+1\}} \big|\sum_{j=v}^{km+l} A_j \exp\big(\sum_{s=j+1}^{km+l} A_s\big)\big|\Big) \\
& + \sup_{v\in\{0,1,\cdots,km+l+1\}} \Big(\sum_{j=v}^{km+l} \Big(C_j sgn(Y_j(\omega)) \exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big) \\
& + \sup_{v\in\{0,1,\cdots,km+l+1\}} \Big(\sum_{j=v}^{km+l} \Big(g(0,0)\triangle N_j sgn(Y_j(\omega)) \exp\big(\sum_{s=j+1}^{km+l} A_s\big)\Big)\Big) \\
= & \alpha_{km+l+1} + \beta_{km+l+1} D_{\tau_{km}} = \alpha_{km+l+1} + \beta_{km+l+1}\Big[\alpha_{km} + \beta_{km}(\alpha_{(k-1)m} + \sum_{j=1}^{k-2} \alpha_{jm} \Pi_{s=j+1}^{k-1} \beta_{sm} + |\xi| \Pi_{j=1}^{k-1} \beta_{jm})\Big] \\
= & D_{\tau_{km+l+1}}(\omega).
\end{aligned} \tag{3.24}
$$

for all $\omega \in \Omega_{km+l+1}$. Therefore, when $\tau_{km+l+1} = \tau_{km+l}$, we get

$$
|Y_{km+l+1}(\omega)| \leq D_{\tau_{km+l+1}}(\omega). \tag{3.25}
$$

which shows that (3.11) holds for every $\omega \in \Omega_{km+l}, km + l \in \{0, 1, \cdots, Tm\}$ and every $m \in \mathbf{N}$ by induction. This completes the proof of Lemma 3.4. □

Lemma 3.5. *([8]) (Time discrete Burkholder-Davis-Gundy type inequality). Let $k \in \mathbf{N}$ and let $Z_l : \Omega \to R, l \in \{0, 1, \cdots, Tm\}, m \in \mathbf{N}^+$, be a family of mappings such that $Z_l : \Omega \to R$ is measurable for all $l \in \{0, 1, \cdots, Tm\}$ and all $m \in \mathbf{N}^+$. Then we obtain that*

$$
\| \sup_{j\in\{0,1,\cdots,km+l\}} |\sum_{l=0}^{j-1} Z_l \triangle W_j| \|_{L^p} \leq p\Big(\sum_{l=0}^{km+l-1} \| Z_l \|_{L^p}^2 \, h\Big)^{\frac{1}{2}}, \tag{3.26}
$$

for $km + l \in \{0, 1, \cdots, Tm\}, k \in N$ and $p \in [2, \infty)$.

Lemma 3.6. *Let α_j, β_j be defined by* (3.3). *If the Assumption (2.5),(2.6) and (2.7) hold, then for all $j = 1, 2, \cdots, Tm$*
(i) $\| \alpha_j \|_{L^p} \leq c_1(T, p, L, \lambda)$;
(ii) $\| \beta_j \|_{L^p} \leq c_2(T, p, L, \lambda)$;
(iii) $\| \sum_{i=0}^{[\frac{j-1}{m}]-1} \alpha_{im} \prod_{s=i+1}^{[\frac{j-1}{m}]} \beta_{sm} \|_{L^p} \leq c_3(T, p, L, \lambda)$;
(iv) $\| |\xi| \prod_{i=1}^{[\frac{j-1}{m}]} \beta_{im} \|_{L^p} \leq c_4(T, p, L, \lambda)$. where c_1, c_2, c_3, c_4 are constants and dependent on T, p, L, λ, but independent of h and j.

Proof. Let $i \leq Tm$, then we have from Lemma 3.2 and Lemma 3.3

$$
\begin{aligned}
&\|\alpha_i\|_{L^p}\\
\leq&(9L+|\sigma(0,0)|+|\mu(0,0)|+\|\xi\|_{L^{2p}}+|g(0,0)|)\,\| e^{\sum_{j=\tau_i}^{i-1}A_j}\|_{L^{2P}}+\|\sup_{v\in\{0,1,\cdots,i\}}\Big(\sum_{j=v}^{i-1}C_j sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}A_s}\Big)\|_{L^p}\\
&+\|\sup_{v\in\{0,1,\cdots,i\}}\Big(\sum_{j=v}^{i-1}g(0,0)\triangle N_j sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}A_s}\Big)\|_{L^p} \qquad (3.27)\\
\leq&(9L+|\sigma(0,0)|+|\mu(0,0)|+\|\xi\|_{L^{2p}}+|g(0,0)|)e^{(L+2pL^2)T}\cdot\| e^{\sum_{j=\tau_i}^{i-1}R_j}\|_{L^{4p}}\\
&+\|\sup_{v\in\{0,1,\cdots,i\}}\Big(\sum_{j=v}^{i-1}(\mu(0,0)hsgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}Q_s})\Big)\|_{L^{2p}}+\|\sup_{v\in\{0,1,\cdots,i\}}\Big(\sum_{j=v}^{i-1}(\mu(0,0)hsgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}R_s})\Big)\|_{L^{2p}}\\
&+\|\sup_{v\in\{0,1,\cdots,i\}}\Big(\sum_{j=v}^{i-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}Q_s}\triangle W_j)\Big)\|_{L^{2p}}+\|\sup_{v\in\{0,1,\cdots,i\}}\Big(\sum_{j=v}^{i-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}R_s}\triangle W_j)\Big)\|_{L^{2p}}\\
&+\|\sup_{v\in\{0,1,\cdots,i\}}\Big(\sum_{j=v}^{i-1}g(0,0)\triangle N_j sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}A_s}\Big)\|_{L^p}\\
\leq&(9L+|\sigma(0,0)|+|\mu(0,0)|+\|\xi\|_{L^{2p}}+|g(0,0)|)e^{(L+2pL^2)T}\cdot\frac{8p}{8p-1}\exp\left(LT\lambda(e^{8pL}-1)\right)\\
&+|\mu(0,0)|T\Big(e^{(L+pL^2)T}+\frac{4p}{4p-1}\exp\left(LT\lambda(e^{4pL}-1)\right)\Big)+\|\sup_{v\in\{0,1,\cdots,i\}}\Big|\sum_{j=0}^{i-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}Q_s}\triangle W_j)\\
&-\sum_{j=0}^{v-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}Q_s}\triangle W_j)\Big|\,\|_{L^{2p}}+\|\sup_{v\in\{0,1,\cdots,i\}}\Big|\sum_{j=0}^{i-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}R_s}\triangle W_j)\\
&-\sum_{j=0}^{v-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}R_s}\triangle W_j)\Big|\,\|_{L^{2p}}+\| e^{\sum_{s=0}^{km+l-1}A_s}\|_{L^{2p}}\cdot|g(0,0)|\sum_{j=0}^{Tm-1}\|\triangle N_j\|_{L^{4p}}\| e^{-\sum_{s=0}^{j}A_s}\|_{L^{4p}}\\
\leq&(9L+|\sigma(0,0)|+|\mu(0,0)|+\|\xi\|_{L^{2p}}+|g(0,0)|)e^{(L+2pL^2)T}\cdot\frac{8p}{8p-1}\exp\left(LT\lambda(e^{8pL}-1)\right)\\
&+|\mu(0,0)|T\Big(e^{(L+pL^2)T}+\frac{4p}{4p-1}\exp\left(LT\lambda(e^{4pL}-1)\right)\Big)\\
&+\|\Big|\sum_{j=0}^{i-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}Q_s}\triangle W_j)\Big|\,\|_{L^{2p}}+\|\sup_{v\in\{0,1,\cdots,i\}}\Big|\sum_{j=0}^{v-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}Q_s}\triangle W_j)\Big|\,\|_{L^{2p}}\\
&+\|\Big|\sum_{j=0}^{i-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}R_s}\triangle W_j)\,\|_{L^{2p}}\\
&+\|\sup_{v\in\{0,1,\cdots,i\}}\Big|\sum_{j=0}^{v-1}(\sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1}R_s}\triangle W_j)\Big|\,\|_{L^{2p}}\\
&+\| e^{\sum_{s=0}^{km+l-1}A_s}\|_{L^{2p}}\cdot|g(0,0)|\sum_{j=0}^{Tm-1}\|\triangle N_j\|_{L^{4p}}\| e^{-\sum_{s=0}^{j}A_s}|\,\|_{L^{4p}},
\end{aligned}
$$

(3.28)

Using the inequality (3.26) of the Lemma 3.5 , we have

$$
\begin{aligned}
&\| \alpha_i \|_{L^p} \\
\leq &(9L + |\sigma(0,0)| + |\mu(0,0)| + \|\xi\|_{L^{2p}} + |g(0,0)|)e^{(L+2pL^2)T} \cdot \frac{8p}{8p-1} \exp\left(LT\lambda(e^{8pL}-1)\right) \\
&+ |\mu(0,0)|T\Big(e^{(L+pL^2)T} + \frac{4p}{4p-1}\exp\left(LT\lambda(e^{4pL}-1)\right)\Big) \\
&+ 2p(\sum_{j=0}^{i-1} \| \sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1} Q_s} \|_{L^{2p}}^2 h)^{\frac{1}{2}} + 2p(\sum_{j=0}^{i-1} \| \sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1} Q_s} \|_{L^{2p}}^2 h)^{\frac{1}{2}} \\
&+ 2p(\sum_{j=0}^{i-1} \| \sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1} R_s} \|_{L^{2p}}^2 h)^{\frac{1}{2}} + 2p(\sum_{j=0}^{i-1} \| \sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1} R_s} \|_{L^{2p}}^2 h)^{\frac{1}{2}} \\
&+ \| e^{\sum_{s=0}^{km+l-1} Q_s} \|_{L^{4p}} \cdot \| e^{\sum_{s=0}^{km+l-1} R_s} \|_{L^{4p}} \cdot |g(0,0)| \sum_{j=0}^{Tm-1} \| \triangle N_j \|_{L^{4p}} \| e^{-\sum_{s=0}^{j} Q_s} \|_{L^{8p}} \cdot \| e^{-\sum_{s=0}^{j} R_s} \|_{L^{8p}} \\
\leq &(9L + |\sigma(0,0)| + |\mu(0,0)| + \|\xi\|_{L^{2p}} + |g(0,0)|)e^{(L+2pL^2)T} \cdot \frac{8p}{8p-1} \exp\left(LT\lambda(e^{8pL}-1)\right) \\
&+ |\mu(0,0)|T\Big(e^{(L+pL^2)T} + \frac{4p}{4p-1}\exp\left(LT\lambda(e^{4pL}-1)\right)\Big) \\
&+ 4p(\sum_{j=0}^{i-1} \| \sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1} Q_s} \|_{L^{2p}}^2 h)^{\frac{1}{2}} + 4p(\sum_{j=0}^{i-1} \| \sigma(0,0)sgn(Y_j(\omega))e^{\sum_{s=j+1}^{i-1} R_s} \|_{L^{2p}}^2 h)^{\frac{1}{2}} \\
&+ e^{(L+4pL^2)T}\frac{8p}{8p-1}\exp\left(LT\lambda(e^{8pL-1})\right) \cdot |g(0,0)| \sum_{j=0}^{Tm-1} \| \triangle N_j \|_{L^{4p}} \cdot e^{(L+4pL^2)T}\frac{16p}{16p-1}\exp\left(LT\lambda(e^{16pL-1})\right).
\end{aligned}
\tag{3.29}
$$

Now, we approximation the $\triangle N_j$ and we have by the properties of the Poisson jump

$$
\sum_{j=0}^{Tm-1} \| \triangle N_j \|_{L^{4p}} = \sum_{j=0}^{Tm-1} \lambda h + \sum_{j=0}^{Tm-1} \| \triangle \tilde{N}_j \|_{L^{4p}} = T\lambda + C(4p)T = TC(4p,\lambda).
$$

Therefore, we have

$$
\begin{aligned}
\| \alpha_i \|_{L^p} \leq &(9L + |\sigma(0,0)| + |\mu(0,0)| + \|\xi\|_{L^{2p}} + |g(0,0)|)e^{(L+2pL^2)T}\frac{8p}{8p-1}\exp\left(LT\lambda(e^{8pL-1})\right) \\
&+ |\mu(0,0)|T\Big(e^{(L+pL^2)T} + \frac{2p}{2p-1}\exp\left(LT\lambda(e^{2pL-1})\right)\Big) \\
&+ 4p|\sigma(0,0)|\sqrt{T}(e^{(L+pL^2)T} + \frac{4p}{4p-1}\exp\left(LT\lambda(e^{4pL}-1)\right)) \\
&+ e^{(L+2pL^2)T}\frac{8p}{8p-1}\exp\left(LT\lambda(e^{8pL}-1)\right)|g(0,0)|TC(4p,\lambda) \\
&\cdot e^{(L+4pL^2)T}\frac{16p}{16p-1}\exp\left(LT\lambda(e^{16pL}-1)\right) = c_1(T,p,L,\lambda).
\end{aligned}
$$

(ii) In view of Lemma 3.2 and Lemma 3.3, we have

$$
\begin{aligned}
\| \beta_i \|_{L^p} \le & \| e^{\sum_{j=[ih]m}^{i-1} A_j} \|_{L^p} + \| \sup_{v\in\{0,1,\cdots,i\}} \Big| \sum_{j=v}^{i-1} A_j e^{\sum_{s=j+1}^{i-1} A_s} \Big| \|_{L^p} \\
\le & \| e^{\sum_{j=[ih]m}^{i-1} Q_j} \|_{L^{2p}} \cdot \| e^{\sum_{j=[ih]m}^{i-1} R_j} \|_{L^{2p}} + \| \sup_{i\in\{0,1,\cdots,Tm\}} e^{\sum_{s=o}^{i-1} A_s} \|_{L^{2p}} \cdot \| \sup_{i\in\{0,1,\cdots,Tm\}} \sup_{v\in\{0,1,\cdots,i\}} \Big| \sum_{j=v}^{i-1} A_j e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}} \\
\le & e^{(L+pL^2)T} \frac{4p}{4p-1} \exp\left(TL\lambda(e^{4pL}-1)\right) \\
& + e^{(L+2pL^2)T} \frac{8p}{8p-1} \exp\left(TL\lambda(e^{8pL}-1)\right) \cdot [\| \sup_{i\in\{0,1,\cdots,Tm\}} \sup_{v\in\{0,1,\cdots,i\}} \Big| \sum_{j=v}^{i-1} Q_j e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}} \\
& + \| \sup_{i\in\{0,1,\cdots,Tm\}} \sup_{v\in\{0,1,\cdots,i\}} \Big| \sum_{j=v}^{i-1} R_j e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}}]
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
\le & e^{(L+pL^2)T} \frac{4p}{4p-1} \exp\left(TL\lambda(e^{4pL}-1)\right) \\
& + e^{(L+2pL^2)T} \frac{8p}{8p-1} \exp\left(TL\lambda(e^{8pL}-1)\right) \cdot [\| \sup_{i\in\{0,1,\cdots,Tm\}} \sup_{v\in\{0,1,\cdots,i\}} \Big| \sum_{j=v}^{i-1} Lhe^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}} \\
& + \| \sup_{v\in\{0,1,\cdots,i\}} | \sum_{j=v}^{i-1} \frac{\sigma(Y_j, Y_{[jh]m}) - \sigma(0,0)}{|Y_j| + |Y_{[jh]m}|} \triangle W_j e^{\sum_{s=j+1}^{i-1} A_s} | \|_{L^{2p}} \\
& + \| \sup_{i\in\{0,1,\cdots,Tm\}} \sup_{v\in\{0,1,\cdots,i\}} \Big| \sum_{j=v}^{i-1} \frac{g(Y_j, Y_{[jh]m}) - g(0,0)}{|Y_j| + |Y_{[jh]m}|} \triangle N_j e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}}] \\
\le & e^{(L+pL^2)T} \frac{4p}{4p-1} \exp\left(TL\lambda(e^{4pL}-1)\right) + e^{(L+2pL^2)T} \frac{8p}{8p-1} \exp\left(TL\lambda(e^{8pL}-1)\right) \cdot [2Lh \| \sup_{i\in\{0,1,\cdots,Tm\}} \Big| \sum_{j=0}^{i-1} e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}} \\
& + 2L \| \sup_{i\in\{0,1,\cdots,Tm\}} | \sum_{j=0}^{i-1} \triangle W_j e^{\sum_{s=j+1}^{i-1} A_s} | \|_{L^{2p}} + 2L \| \sup_{i\in\{0,1,\cdots,Tm\}} \sup_{i\in\{0,1,\cdots,Tm\}} \Big| \sum_{j=0}^{i-1} \triangle N_j e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}}] \\
\le & e^{(L+2pL^2)T} \frac{4p}{4p-1} \exp\left(TL\lambda(e^{4pL}-1)\right) \\
& + e^{(L+4pL^2)T} \frac{8p}{8p-1} \exp\left(TL\lambda(e^{8pL}-1)\right) \cdot [2Lh \| \sup_{i\in\{0,1,\cdots,Tm\}} \Big| \sum_{j=0}^{i-1} e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{2p}} \\
& + 2L2p\Big(\sum_{j=0}^{Tm-1} \| e^{-\sum_{s=0}^{j} A_s} \|_{L^{2p}}^2 h \Big)^{\frac{1}{2}} + 2L \sum_{j=0}^{Tm-1} \| \Big| e^{-\sum_{s=0}^{j} A_s} \Big| \|_{L^{4p}} \cdot \sum_{j=0}^{Tm-1} \| \Big| \triangle N_j \Big| \|_{L^{4p}}] = c_2(T,p,L,\lambda).
\end{aligned} \tag{3.31}
$$

(iii) According to the above formula, we obtain

$$
\begin{aligned}
\| | \sum_{j=0}^{[\frac{i-1}{m}]-1} \alpha_{jm}^1 \Pi_{s=j+1}^{[\frac{i-1}{m}]} \beta_{sm}^1 | \|_{L^p} \le & \sum_{j=0}^{[\frac{i-1}{m}]-1} \| \alpha_{jm}^1 \|_{L^{2P}} \| \Pi_{s=j+1}^{[\frac{i-1}{m}]} \beta_{sm}^1 | \|_{L^{2P}} \\
\le & Tc_1 c_2^T = c_3(T,p,L,\lambda).
\end{aligned} \tag{3.32}
$$

(iv) Similarly, we have

$$\begin{aligned} \| |\Pi_{j=0}^{[\frac{i-1}{m}]}\beta_{jm}|\xi| \|_{L^p} \leq & \| |\Pi_{j=0}^{[\frac{i-1}{m}]}\beta_{jm}| \|_{L^{2p}} \| |\xi| \|_{L^{2p}} \\ \leq & c_2(T,p,L)^T \| |\xi| \|_{L^{2p}} = c_4(T,p,L,\lambda). \end{aligned} \tag{3.33}$$

This completes the proof of Lemma 3.6. □

Lemma 3.7. *If the Assumptions (2.5),(2.6) and (2.7) hold, then we have*

$$\sup_{m\in\mathbf{N}^+} \mathrm{E}[\sup_{km+l\in\{0,1,\cdots,Tm\}} |D_{\tau_{km+l}}(\omega)|^p] < \infty, \tag{3.34}$$

for $\omega \in \Omega$ and $p \geq 1$.

Proof. Let $p \geq 1, k \in \mathbf{N}, m \in \mathbf{N}^+$. It comes from Hölder inequality that

$$\begin{aligned} \| |D_{\tau_{km+l}}| \|_{L_p} \leq & \| |\alpha_{km+l}| \|_{L_p} + \| |\beta_{km+l}\alpha_{[\frac{km+l-1}{m}]m}| \|_{L_p} + \| |\sum_{j=1}^{[\frac{km+l-1}{m}]-1} \beta_{km+l}\alpha_{jm}\Pi_{s=j+1}^{[\frac{km+l-1}{m}]}\beta_{sm}| \|_{L_p} \\ & + \| |\Pi_{j=1}^{[\frac{km+l-1}{m}]}\beta_{km+l}\beta_{jm}|Y_0| \|_{L_p} \\ \leq & \| |\alpha_{km+l}| \|_{L_p} + \| |\beta_{km+l}| \|_{L_{2p}} \| |\alpha_{[\frac{km+l-1}{m}]m}| \|_{L_{2p}} \\ & + \| |\beta_{km+l}| \|_{L_{2p}} \| \sup_{km+l\in\{0,1,\cdots,Tm\}} |\sum_{j=1}^{[\frac{km+l-1}{m}]-1} \alpha_{jm}\Pi_{s=j+1}^{k}\beta_{sm}| \|_{L_{2p}} \\ & + \| |\beta_{km+l}| \|_{L_{2p}} \| |\Pi_{j=1}^{[\frac{km+l-1}{m}]}\beta_{jm}|Y_0| \|_{L_{2p}} . \end{aligned} \tag{3.35}$$

We obtain from Lemma 3.6

$$\begin{aligned} \| |D_{\tau_{km+l}}| \|_{L_p} \leq & c_1(T,p,L) + c_2(T,p,L)c_1(T,p,L,\lambda) + c_2(T,p,L,\lambda)c_3(T,p,L,\lambda) \\ & + c_2(T,p,L,\lambda)c_4(T,p,L,\lambda) = c_5(T,p,L,\lambda) < \infty, \end{aligned} \tag{3.36}$$

where c_5 is independent of h. This completes the proof of Lemma 3.7. □

Corollary 3.8. *Let the Euler approximation $Y_{km+l} : \Omega \to R$ for $k,m \in \mathbf{N}, l \in \{0,1,\cdots,m-1\}$, then we have*

$$E[I_{\Omega_{km+l}}|Y_{km+l}|^p] < \infty,$$

for every $p \in [1,\infty)$.

Remark 3.9. *It is easy to see that Ω_{km+l} is decreasing set on $km+l$ by the definition of Ω_{km+l}. Hence, we have $E[I_{\Omega_{Tm}}|Y_{km+l}|^p] < \infty$.*

Lemma 3.10. *Let $\Omega_{km+l} \in \mathcal{F}$, for $m \in \mathbf{N}^+$ and $\tilde{\Omega} \in \mathcal{F}$ be given by* (3.1) *and Remark 3.1. If Assumption 2.2 holds, then we have that*

$$m^p P[(\Omega_{Tm})^c] < \infty.$$

Proof. Using the definition of the Ω_{Tm}, Chebyshev's inequality, and $0 < \lambda < 1$ we obtain for $q > 0$

$$\begin{aligned}
P[(\Omega_{Tm})^c] \leq & P[\{\omega \in \Omega : \sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)| \geq r^h\}] + P[\{\omega \in \Omega : \sup_{n\in\{0,1,\cdots,Tm-1\}} |\triangle W_n(\omega)| \geq h^{\frac{1}{C+1}}\}] \\
& + P[\{\omega \in \Omega : \sup_{n\in\{0,1,\cdots,Tm-1\}} |\triangle N_n(\omega)| \geq h^{\frac{1}{C+1}}\}] \\
\leq & E[\sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)|^q](r^h)^{-q} + Tm \cdot P[\{\omega \in \Omega : |\triangle W_1(\omega)| \geq h^{\frac{1}{C+1}}\}] \\
& + P[\{\omega \in \Omega : \sup_{n\in\{0,1,\cdots,Tm-1\}} |\triangle N_n(\omega) - \lambda h + \lambda h| \geq h^{\frac{1}{C+1}}\}] \\
\leq & E[\sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)|^q](r^h)^{-q} + Tm \cdot P[\{\omega \in \Omega : \sqrt{h}|\Delta\chi_1(\omega)| \geq h^{\frac{1}{C+1}}\}] \\
& + P[\{\omega \in \Omega : \sup_{n\in\{0,1,\cdots,Tm-1\}} |\triangle N_n(\omega) - \lambda h| \geq h^{\frac{1}{C+1}} - \lambda h\}] \\
\leq & E[\sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)|^q](r^h)^{-q} + Tm \cdot P[\{\omega \in \Omega : \sqrt{h}|\Delta\chi_1(\omega)| \geq h^{\frac{1}{C+1}}\}] \\
& + P[\{\omega \in \Omega : \sup_{n\in\{0,1,\cdots,Tm-1\}} |\triangle N_n(\omega) - \lambda h| \geq h^{\frac{1}{C+1}}(1-\lambda)\}] \\
\leq & E[\sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)|^q](r^h)^{-q} + T \cdot E[|\Delta\chi_1(\omega)|^q] m^{1-0.5q+\frac{q}{C+1}} \\
& + \frac{1}{h^{\frac{q}{C+1}}(1-\lambda)^q} TmE[|\triangle\tilde{N}_n(\omega)|^q] \\
\leq & E[\sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)|^q](r^h)^{-q} + T \cdot E[|\Delta\chi_1(\omega)|^q] m^{1-0.5q+\frac{q}{C+1}} \\
& + T \cdot m^{1-0.5q+\frac{q}{C+1}}.
\end{aligned} \tag{3.37}$$

where $\Delta\chi_1 \sim N(0,1)$. Taking $q \geq \{p(C+1) \vee \frac{2(p+1)(C+1)}{C-1}\}$, we obtain

$$\begin{aligned}
& m^p P[(\Omega_{Tm})^c] \\
\leq & m^{p-\frac{q}{C+1}} E[\sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)|^q] + T \cdot E[|\Delta\chi_1(\omega)|^q] m^{p+1-0.5q+\frac{q}{C+1}} \\
\leq & E[\sup_{n\in\{0,1,\cdots,Tm-1\}} |D_{\tau_n}(\omega)|^q] + TE[|\Delta\chi_1(\omega)|^q].
\end{aligned}$$

This completes the proof of the Lemma 3.10. □

Lemma 3.11.

$$\sup_{m\in N}[\sup_{0\leq n\leq Tm} E \parallel Y_n \parallel^p] < \infty,$$

for all $p \in [1, \infty)$.

Proof. According to the formula of the Tamed Euler method and Minkowski inequality, we have

$$\begin{aligned}
\parallel Y_n \parallel_{L^p} \leq & \parallel \xi \parallel_{L^p} + \parallel \sigma(0,0)\triangle W_n \parallel_{L^p} + \parallel g(0,0)\triangle N_n \parallel_{L^p} \\
& + \parallel \sum_{i=0}^{n-1} \frac{h\mu(Y_i, Y_{[ih]m})}{1 + |\mu(Y_i, Y_{[ih]m})|h} \parallel_{L^p}
\end{aligned} \tag{3.38}$$

$$
\begin{aligned}
&+ \| \sum_{i=0}^{n-1} \Big(\sigma(Y_i, Y_{[ih]m}) - \sigma(0,0)\Big)\triangle W_i \|_{L^p} + \| \sum_{i=0}^{n-1} \Big(g(Y_i, Y_{[ih]m}) - g(0,0)\Big)\triangle N_i \|_{L^p} \\
\le & \| \xi \|_{L^p} + p\sqrt{T}|\sigma(0,0)| + Tm + (E \| g(0,0)\triangle N_n \|^p)^{\frac{1}{p}} \\
&+ p(\sum_{i=0}^{n-1} h \| \sigma(Y_i, Y_{[ih]m}) - \sigma(0,0) \|_{L^p}^2)^{\frac{1}{2}} + \sum_{i=0}^{n-1} (E[L(|Y_i| + |Y_{[ih]m}|)]^p E|\triangle N_i|^p)^{\frac{1}{p}} \\
\le & \| \xi \|_{L^p} + p\sqrt{T}|\sigma(0,0)| + Tm + |g(0,0)|hC(p,\lambda) \\
&+ pL\sqrt{2h}(\sum_{i=0}^{n-1} h \| Y_i \|_{L^p}^2 + \| Y_{[ih]m} \|_{L^p}^2)^{\frac{1}{2}} + hLC(p,\lambda)\sum_{i=0}^{n-1}(\| Y_i \|_{L^p} + \| Y_{[ih]m} \|_{L^p}) \\
\le & \| \xi \|_{L^p} + p\sqrt{T}|\sigma(0,0)| + Tm + |g(0,0)|hC(p,\lambda) \\
&+ pL\sqrt{2h}(\sum_{i=0}^{n-1} h \| Y_i \|_{L^p}^2 + \| Y_{[ih]m} \|_{L^p}^2)^{\frac{1}{2}} + hL\sqrt{2n}C(p,\lambda)\Big(\sum_{i=0}^{n-1}(\| Y_i \|_{L^p}^2 + \| Y_{[ih]m} \|_{L^p}^2)\Big)^{\frac{1}{2}} \\
\le & \| \xi \|_{L^p} + p\sqrt{T}|\sigma(0,0)| + Tm + |g(0,0)|hC(p,\lambda) \\
&+ pL\sqrt{2h}(\sum_{i=0}^{n-1} h \| Y_i \|_{L^p}^2 + \| Y_{[ih]m} \|_{L^p}^2)^{\frac{1}{2}} + hL\sqrt{2Tm}C(p,\lambda)\Big(\sum_{i=0}^{n-1}(\| Y_i \|_{L^p}^2 + \| Y_{[ih]m} \|_{L^p}^2)\Big)^{\frac{1}{2}} \\
\le & \| \xi \|_{L^p} + p\sqrt{T}|\sigma(0,0)| + Tm + |g(0,0)|hC(p,\lambda) \\
&+ L\sqrt{2h}(\sqrt{T}C(p,\lambda) + p)\Big(\sum_{i=0}^{n-1}(\| Y_i \|_{L^p}^2 + \| Y_{[ih]m} \|_{L^p}^2)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Using the Gronwall's inequality, we obtain

$$\| Y_n \|_{L^p}^2 \le 2(\| \xi \|_{L^p} + p\sqrt{T}|\sigma(0,0)| + Tm + |g(0,0)|hC(p,\lambda))^2 e^{T8L^2(p+\sqrt{T}C(p,\lambda))^2}. \tag{3.39}$$

Therefore, it is easy to see that

$$\| Y_n \|_{L^p} \le \sqrt{2}(\| \xi \|_{L^p} + p\sqrt{T}|\sigma(0,0)| + Tm + |g(0,0)|C(p,\lambda))e^{T4L^2(p+\sqrt{T}C(p,\lambda))^2}. \tag{3.40}$$

Using the Hölder inequality and Lemma 3.7, then we have

$$
\begin{aligned}
&\sup_{m\in N} \sup_{n\in\{0,1,\cdots,Tm\}} \| I_{(\Omega_{Tm})^C} Y_n \|_{L^p} \\
\le & \sup_{m\in N} \sup_{n\in\{0,1,\cdots,Tm\}} \| mI_{(\Omega_{Tm})^C} \|_{L^{2p}} \times (\sup_{m\in N} \sup_{n\in\{0,1,\cdots,Tm\}} m^{-1} \| Y_n \|_{L^{2p}}) \\
\le & \sqrt{2}e^{T4L^2(p+\sqrt{T}C(p,\lambda))^2}(\sup_{m\in N} m^{2p} P[(\Omega_{Tm})^C])^{\frac{1}{2p}}(\| \xi \|_{L^{2p}} + p\sqrt{T}|\sigma(0,0)| + T + |g(0,0)|C(p,\lambda)) < \infty.
\end{aligned} \tag{3.41}
$$

And we know the $\sup_{m\in N} \sup_{n\in\{0,1,\cdots,Tm\}} \| I_{\Omega_{Tm}} Y_n \|_{L^p} < \infty$ from the Corollary 3.8. Hence, this completes the proof of the Lemma 3.11. □

Lemma 3.12. *If Assumption 2.1 holds, then for* $t \in [0,T]$

$$E|\bar{Y}(t) - Y(\underline{t})|^p \le M(p,\lambda)h^{\frac{p}{2}}. \tag{3.42}$$

Proof. In section 2, we know the formula of $\bar{Y}(t)$ and we have

$$
\begin{aligned}
|\bar{Y}(t) - Y(\underline{t})| = & | \int_{t_{km+l}}^t \mu(Y(\underline{s^-}), Y([\underline{s^-}]))ds + \int_{t_{km+l}}^t \sigma(Y(\underline{s^-}), Y([\underline{s^-}]))dW(s) \\
&+ \int_{t_{km+l}}^t g(Y(\underline{s^-}), Y([\underline{s^-}]))dN(s)|
\end{aligned} \tag{3.43}
$$

Taking the supremum of the above formula, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} |\bar{Y}(t) - Y(\underline{t})|^p \\
\leq& 4^{p-1}h^p \sup_{km+l\in\{0,1,\cdots,Tm\}} |\mu(Y_{km+l}, Y_{km})|^p + 4^{p-1} \sup_{km+l\in\{0,1,\cdots,Tm\}} |\int_{t_{km+l}}^{t} \sigma(Y(\underline{s^-}), Y([\underline{s^-}]))dW(s)|^p \\
&+ 4^{p-1} \sup_{km+l\in\{0,1,\cdots,Tm\}} |\int_{t_{km+l}}^{t} g(Y(\underline{s^-}), Y([\underline{s^-}]))d\tilde{N}(s)|^p + 4^{p-1} \sup_{km+l\in\{0,1,\cdots,Tm\}} |\int_{t_{km+l}}^{t} \lambda g(Y(\underline{s}), Y([\underline{s}]))ds|^p.
\end{aligned} \tag{3.44}
$$

Now, we take the expectation of the above formula, we obtain by Burkholder-Davis-Gundy

$$
\begin{aligned}
&E[\sup_{t\in[0,T]} |\bar{Y}(t) - Y(\underline{t})|^p] \\
\leq& 4^{p-1}h^p E[\sup_{km+l\in\{0,1,\cdots,Tm\}} |\mu(Y_{km+l}, Y_{km})|^p] + 4^{p-1}\widetilde{C}_p h^{\frac{p}{2}} E[\sup_{km+l\in\{0,1,\cdots,Tm\}} |\sigma(Y_{km+l}, Y_{km})|^p] \\
&+ 4^{p-1}(\widetilde{C}_p + 1)h^{\frac{p}{2}} E[\sup_{km+l\in\{0,1,\cdots,Tm\}} \lambda^p |g(Y_{km+l}, Y_{km})|^p] \\
\leq& M(p, \lambda)h^{\frac{p}{2}}.
\end{aligned} \tag{3.45}
$$

This completes the proof of the Lemma 3.12. □

4. Proof of Theorem 2.4

We can know from the (2.9)

$$
\begin{aligned}
x(t) - \bar{Y}(t) =& \int_0^t [\mu(x(s^-), x([s^-])) - \frac{\mu(Y(\underline{s^-}), Y([\underline{s^-}]))}{1 + h|\mu(Y(\underline{s^-}), Y([\underline{s^-}]))|}]ds \\
&+ \int_0^t [\sigma(x(s^-), x([s^-])) - \sigma(Y(\underline{s^-}), Y([\underline{s^-}]))]dW(s) \\
&+ \int_0^t [g(x(s^-), x([s^-])) - g(Y(\underline{s^-}), Y([\underline{s^-}]))]dN(s).
\end{aligned} \tag{4.1}
$$

Proof. According to the Itô formula, we have

$$
\begin{aligned}
&|x(t) - \bar{Y}(t)|^2 \\
=& \int_0^t 2 < x(s^-) - \bar{Y}(s^-), \mu(x(s^-), x([s^-])) - \frac{\mu(Y(\underline{s^-}), Y([\underline{s^-}]))}{1 + h|\mu(Y(\underline{s^-}), Y([\underline{s^-}]))|} > ds \\
&+ \int_0^t 2 < x(s^-) - \bar{Y}(s^-), \sigma(x(s^-), x([s^-])) - \sigma(Y(\underline{s^-}), Y([\underline{s^-}])) > dW(s) \\
&+ \int_0^t |\sigma(x(s^-), x([s^-])) - \sigma(Y(\underline{s^-}), Y([\underline{s^-}]))|^2 ds \\
&+ \int_0^t [(|x(s^-) - \bar{Y}(s^-)| + g(x(s^-), x([s^-])) - g(Y(\underline{s^-}), Y([\underline{s^-}])))^2 - |x(s^-) - \bar{Y}(s^-)|^2]dN(s)
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
=&\int_0^t 2 < x(s^-) - \bar{Y}(s^-), \mu(x(s^-), x([s^-])) - \mu(\bar{Y}(s^-), Y([\underline{s^-}])) > ds\\
&+ \int_0^t 2 < x(s^-) - \bar{Y}(s^-), \mu(\bar{Y}(s^-), Y([\underline{s^-}])) - \mu(Y(\underline{s^-}), Y([\underline{s^-}])) > ds\\
&+ 2h\int_0^t < x(s^-) - \bar{Y}(s^-), \frac{\mu(Y(\underline{s^-}), Y([\underline{s^-}]))|\mu(Y(\underline{s^-}), Y([\underline{s^-}]))|}{1 + h|\mu(Y(\underline{s^-}), Y([\underline{s^-}]))|} > ds\\
&+ \int_0^t 2 < x(s^-) - \bar{Y}(s^-), \sigma(x(s^-), x([s^-])) - \sigma(Y(\underline{s^-}), Y([\underline{s^-}])) > dW(s)\\
&+ \int_0^t |\sigma(x(s^-), x([s^-])) - \sigma(Y(\underline{s^-}), Y([\underline{s^-}]))|^2 ds\\
&+ \int_0^t [(|x(s^-) - \bar{Y}(s^-)| + g(x(s^-), x([s^-]))\\
&- g(Y(\underline{s^-}), Y([\underline{s^-}])))^2 - |x(s^-) - \bar{Y}(s^-)|^2] d\tilde{N}(s)\\
&+ \int_0^t [(|x(s^-) - \bar{Y}(s^-)| + g(x(s^-), x([s^-]))\\
&- g(Y(\underline{s^-}), Y([\underline{s^-}])))^2 - |x(s^-) - \bar{Y}(s^-)|^2] \lambda ds.
\end{aligned}
$$

By the Assumption 2.2, it is easy to know that

$$
\begin{aligned}
&\sup_{t\in[0,t_1]} |x(t) - \bar{Y}(t)|^2\\
\le&(4L^2 + 2L + 3)\int_0^{t_1} |x(s^-) - \bar{Y}(s^-)|^2 ds + 5L^2\int_0^{t_1} |x([s^-]) - Y([\underline{s^-}])|^2 ds\\
&+ h^2\int_0^T |\mu(Y(\underline{s^-}), Y([\underline{s^-}]))|^4 ds + \int_0^T |\mu(\bar{Y}(s^-), Y([\underline{s^-}])) - \mu(Y(\underline{s^-}), Y([\underline{s^-}]))|^2 ds\\
&+ 2L^2\int_0^T |\bar{Y}(s^-) - Y(\underline{s^-})|^2 ds\\
&+ \sup_{t\in[0,t_1]} |\int_0^t 2 < x(s^-) - \bar{Y}(s^-), \sigma(x(s^-), x([s^-])) - \sigma(Y(\underline{s^-}), Y([\underline{s^-}])) > dW(s)|\\
&+ \sup_{t\in[0,t_1]} |\int_0^t [(|x(s^-) - \bar{Y}(s^-)| + g(x(s^-), x([s^-])) - g(Y(\underline{s^-}), Y([\underline{s^-}])))^2 - |x(s^-) - \bar{Y}(s^-)|^2] d\tilde{N}(s)|\\
&+ \sup_{t\in[0,t_1]} |\int_0^t [(|x(s^-) - \bar{Y}(s^-)| + g(x(s^-), x([s^-])) - g(Y(\underline{s^-}), Y([\underline{s^-}])))^2 - |x(s^-) - \bar{Y}(s^-)|^2] \lambda ds|\\
\le&(9L^2 + 2L + 3)\int_0^{t_1} \sup_{u\in[0,s]} |x(u) - Y(u)|^2 ds + \int_0^T |\mu(\bar{Y}(s^-), Y([\underline{s^-}])) - \mu(Y(\underline{s^-}), Y([\underline{s^-}]))|^2 ds\\
&+ h^2\int_0^T |\mu(Y(\underline{s^-}), Y([\underline{s^-}]))|^4 ds + 2L^2\int_0^T |\bar{Y}(s^-) - Y(\underline{s^-})|^2 ds\\
&+ \sup_{t\in[0,t_1]} |\int_0^t 2 < x(s^-) - \bar{Y}(s^-), \sigma(x(s^-), x([s^-])) - \sigma(Y(\underline{s^-}), Y([\underline{s^-}])) > dW(s)|\\
&+ \sup_{t\in[0,t_1]} |\int_0^t \Big[(|x(s^-) - \bar{Y}(s^-)| + g(x(s^-), x([s^-])) - g(Y(\underline{s^-}), Y([\underline{s^-}])))^2 - |x(s^-) - \bar{Y}(s^-)|^2\Big] d\tilde{N}(s)|\\
&+ \sup_{t\in[0,t_1]} |\int_0^t \Big[(|x(s^-) - Y(s^-)| + g(x(s^-), x([s^-])) - g(Y(\underline{s^-}), Y([\underline{s^-}])))^2 - |x(s^-) - \bar{Y}(s^-)|^2\Big] \lambda ds|.
\end{aligned} \tag{4.3}
$$

Now, we estimate the $\frac{p}{2}$-th power of the above formula, and we have by fundamental inequality

$$
\begin{aligned}
&(\sup_{t\in[0,t_1]}|x(t)-\bar{Y}(t)|^2)^{\frac{p}{2}}\\
\leq& 7^{\frac{p}{2}-1}\Big[(9L^2+2L+3)^{\frac{p}{2}}(\int_0^{t_1}\sup_{u\in[0,s]}|x(u)-\bar{Y}(u)|^2ds)^{\frac{p}{2}}+(\int_0^T|\mu(\bar{Y}(s^-),Y([\underline{s^-}]))-\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|^2ds)^{\frac{p}{2}}\\
&+h^p(\int_0^T|\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|^4ds)^{\frac{p}{2}}+2^{\frac{p}{2}}L^p(\int_0^T|\bar{Y}(s^-)-Y(\underline{s^-})|^2ds)^{\frac{p}{2}}\\
&+2^{\frac{p}{2}}(\sup_{t\in[0,t_1]}|\int_0^t 2<x(s^-)-\bar{Y}(s^-),\sigma(x(s^-),x([s^-]))-\sigma(Y(\underline{s^-}),Y([\underline{s^-}]))>dW(s)|)^{\frac{p}{2}}\\
&+(\sup_{t\in[0,t_1]}|\int_0^t\Big[(|x(s^-)-\bar{Y}(s^-)|+g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}])))^2-|x(s^-)-\bar{Y}(s^-)|^2\Big]d\tilde{N}(s)|)^{\frac{p}{2}}\\
&+(\sup_{t\in[0,t_1]}|\int_0^t\Big[(|x(s^-)-\bar{Y}(s^-)|+g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}])))^2-|x(s^-)-\bar{Y}(s^-)|^2\Big]\lambda ds|)^{\frac{p}{2}}\Big].
\end{aligned}
\tag{4.4}
$$

Taking the expectation for the above formula, we obtain applying the Burkholder-Davis-Gundy's inequality, Kunita's first inequality [1], Hölder inequality, and lemmas for $t_1\in[0,T]$,

$$
\begin{aligned}
&E[\sup_{t\in[0,t_1]}|x(t)-\bar{Y}(t)|^p]\\
\leq& 7^{\frac{p}{2}-1}\Big[(9L^2+2L+3)^{\frac{p}{2}}T^{\frac{p}{2}-1}E[\int_0^{t_1}\sup_{u\in[0,s]}|x(u)-Y(u)|^p ds]\\
&+T^{\frac{p}{2}-1}E[\int_0^T|\mu(\bar{Y}(s^-),Y([\underline{s^-}]))-\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|^p ds]\\
&+T^{\frac{p}{2}-1}h^pE[\int_0^T|\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|^{2p}ds]\\
&+2^{\frac{p}{2}}L^pT^{\frac{p}{2}-1}E[\int_0^T|\bar{Y}(s^-)-Y(\underline{s^-})|^p ds]\\
&+2^{\frac{p}{2}}\tilde{C}_pE[\int_0^t|<x(s^-)-\bar{Y}(s^-),\sigma(x(s^-),x([s^-]))-\sigma(Y(\underline{s^-}),Y([\underline{s^-}]))>|^2ds]^{\frac{p}{4}}\\
&+2^{\frac{p}{2}}D_p\Big(E[\int_0^{t_1}((x(s^-)-\bar{Y}(s^-))(g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}])))\\
&+|g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}]))|^2)^2\lambda ds]^{\frac{p}{4}}\\
&+E[\int_0^{t_1}((x(s^-)-\bar{Y}(s^-))(g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}])))\\
&+|g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}]))|^2)^{\frac{p}{2}}\lambda ds]\Big)\\
&+2^{\frac{p}{2}-1}E[(\sup_{t\in[0,t_1]}\int_0^t 2(x(s^-)-\bar{Y}(s^-))(g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}]))\lambda ds)^{\frac{p}{2}}]\\
&+2^{\frac{p}{2}-1}E[(\sup_{t\in[0,t_1]}\int_0^t|g(x(s^-),x([s^-]))-g(Y(\underline{s^-}),Y([\underline{s^-}])|^2\lambda ds)^{\frac{p}{2}}]
\end{aligned}
\tag{4.5}
$$

$$\leq 7^{\frac{p}{2}-1}\Big[(9L^2+2L+3)^{\frac{p}{2}}T^{\frac{p}{2}-1}E[\int_0^{t_1}\sup_{u\in[0,s]}|x(u)-\bar{Y}(u)|^p ds]$$
$$+T^{\frac{p}{2}-1}E[\int_0^T|\mu(\bar{Y}(s^-),Y([\underline{s^-}]))-\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|^p ds]+T^{\frac{p}{2}-1}h^pE[\int_0^T|\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|^{2p}ds]$$
$$+2^{\frac{p}{2}}L^pT^{\frac{p}{2}-1}E[\int_0^T|\bar{Y}(s^-)-Y(\underline{s^-})|^p ds]$$
$$+T^{\frac{p}{4}-1}2^{\frac{p}{2}}\tilde{C}_p\Big[E[\int_0^{t_1}|\frac{1}{2}|x(s^-)-\bar{Y}(s^-)|^p ds]+E[\int_0^{t_1}2^{p-1}L^p\sup_{u\in[0,s]}|x(u)-\bar{Y}(u)|^p ds]\Big]$$
$$+2^{2p-1}D_p(T^{\frac{p}{4}-1}+1)L^p3E[\int_0^{t_1}|(x(s^-)-\bar{Y}(s^-)|^p\lambda ds)]$$
$$+2^{2p-1}T^{\frac{p}{2}}3L^pE[\int_0^{t_1}\sup_{u\in[0,s]}|x(u)-\bar{Y}(u)|^p\lambda ds)]\Big].$$

Then,we have according to the Lemma 3.12

$$\begin{aligned}E[\sup_{t\in[0,t_1]}|x(t)-\bar{Y}(t)|^p]\leq &7^{\frac{p}{2}-1}T^{\frac{p}{2}-1}\int_0^{t_1}E[\sup_{u\in[0,s]}|x(u)-\bar{Y}(u)|^p]ds+7^{\frac{p}{2}-1}T^{\frac{p}{2}-1}L^p\int_0^{t_1}E[|\bar{Y}(s^-)-Y(\underline{s^-})|^p]ds\\&+7^{\frac{p}{2}-1}T^{\frac{p}{2}-1}h^p\int_0^{t_1}E[|\mu(Y(\underline{s^-}),Y([\underline{s^-}]))|^{2p}]ds\\&+7^{\frac{p}{2}-1}T^{\frac{p}{2}-1}2^{\frac{p}{2}}L^p\int_0^{t_1}E[|\bar{Y}(s^-)-Y(\underline{s^-})|^p]ds\\&+7^{\frac{p}{2}-1}T^{\frac{p}{4}-1}2^{\frac{p}{2}}\tilde{C}_p(\frac{1}{2}+2^{p-1}L^p)\int_0^{t_1}E[\sup_{u\in[0,s]}|x(u)-\bar{Y}(u)|^p]ds\\&+7^{\frac{p}{2}-1}T^{\frac{p}{2}-1}3\cdot 2^{2p-1}L^pD_p\int_0^{t_1}E[\sup_{u\in[0,s]}|x(u)-\bar{Y}(u)|^p]ds.\end{aligned} \tag{4.6}$$

Hence, we have according to the lemmas in section 3 and the Gronwall's inequality

$$\|\sup_{t\in[0,T]}|x(t)-\bar{Y}(t)|^p\|_{L^p}\leq e^{R_1(p,T)}M_1(p,T,\lambda)h^{\frac{1}{2}}. \tag{4.7}$$

This completes the proof of the Theorem 2.4. □

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