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The n-Inverses of a Matrix

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Abstract. The concept of a left n-inverse of a bounded linear operator on a complex Banach space was introduced recently. Previously, there have been results on products and tensor products of left n-inverses, and the representation of left n-inverses as the sum of left inverses and nilpotent operators was being discussed. In this paper, we give a spectral characterization of the left n-inverses of a finite (square) matrix. We also show that a left n-inverse of a matrix T is the sum of the inverse of T and two nilpotent matrices.

1. Introduction

Let $\beta_n(y, x)$ be defined using binomial formula

$$\beta_n(y, x) = (yx - 1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y^k x^k.$$

Let $B(X)$ be the algebra of all bounded linear operators on a Banach space X. Let $S, T \in B(X)$. We define the functional calculus $\beta_n(S, T)$ by

$$\beta_n(S, T) = (yx - 1)^n|_{y=S,x=T} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} S^k T^k,$$

where S is always on the left side of T. If $ST = TS$, then $\beta_n(S, T) = (ST - I)^n$ where I is the identity operator on X.

Recall that S is a left inverse of T (or T is a right inverse of S) if $ST = I$, that is, $\beta_1(S, T) = ST - I = 0$. As in Sid Ahmed [19] and Duggal and Müller [11], S is a left n-inverse of T (or T is a right n-inverse of S) if

$$\beta_n(S, T) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} S^k T^k = 0. \tag{1}$$

Since $\beta_n(y, x)$ divides $\beta_m(y, x)$ for $m \geq n$, if S is a left n-inverse of T, then S is a left m-inverse of T for $m \geq n$.

We say that S is a strict left n-inverse of T if S is a left n-inverse of T but not a left $(n-1)$-inverse of T. It is also clear that S is a left n-inverse of T on a complex Hilbert space if and only if T^* is a left n-inverse of S^*.

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Similarly we say S is an n-inverse of T if S is both a left n-inverse and a right n-inverse of T. We say T is left n-invertible if T has a strict left n-inverse and say T is n-invertible if T has an strict n-inverse.

The concept of left n-invertible operators is motivated by the n-isometries studied earlier in [1], [2], and [18] and more recently in [5], [7], [8], [9], [12], [13], and [20], on Hilbert spaces and [3], [4], [6], [10], [14], and [16] on Banach spaces. An operator T on a Hilbert space H is an n-isometry if $\beta_n(T^*, T) = 0$, that is, T^* is a left n-inverse of T.

Some basic properties of left n-inverses were observed by Sid Ahmed [19]. Results on products and tensor products of left n-invertible operators were obtained by Duggal and Müller [11]. Further structures of left n-invertible operators were discussed by the first named author [15], [17]. Applications to elementary operators of length one or generalized derivations on $B(X)$ were given in [11] and [15]. These papers focused on left n-invertible operators on infinite dimensional Banach spaces.

In this paper we study the left n-inverses of a finite square matrix. It is clear that a left n-invertible operator is invertible. We are able to describe the set of all left n-inverses (for all n) of a given invertible matrix by using its Jordan structure. But the set of all left n-inverses for a fixed n seems difficult to characterize. Most results obtained here probably do not hold for left n-invertible operators on infinite dimensional Banach spaces. However, such a study does offer insights into the infinite dimensional case, especially for algebraic operators which are left n-invertible. For simplicity, in this paper we will emphasize the finite dimensional complex Hilbert space $\mathbb{C}^N$ and all operators on $\mathbb{C}^N$ are represented as matrices with respect to the standard bases.

In Section 1, for a matrix T, let S be a left n-inverse of T, we prove a lemma about orthogonality of the generalized eigenspaces of S and T. This leads to the spectral compatibility condition between S and T, which essentially reduces the question of finding left n-inverses of a general matrix to a matrix with only one eigenvalue.

In Section 2, we investigate the order of an n-inverse (the possible values of n). Using the construction of n-inverses of T as the sum of the inverse of T and nilpotent matrices in [15], we describe the possible values of n by finding suitable nilpotent matrices. For example, we show that the maximal possible n is $2N - 1$, where T is an $N \times N$ matrix.

In Section 3, we discuss the relation between left n-inverses of T and right n-inverses of T. In particular, an example shows that a left n-inverse of T is not necessarily a right n-inverse of T for the same n, but a left n-inverse of T is always a right $(2N-1)$-inverse of T. We also show that a left n-inverse of T can be written as the sum of the inverse of T and two nilpotent matrices. Finding a left n-inverse of T is a nonlinear problem, so we describe briefly a linearization algorithm for finding a left n-inverse of T.

2. Characterization of Left n-inverses

The following simple example shows that the left 2-inverse of a matrix is not unique. Let

$$S = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then a direct calculation gives

$$S^2T^2 - 2ST + I = 0.$$

That is, S is a left 2-inverse of T for any x.

In this section we show that an n-inverse in $\mathbb{C}^N$ must take a certain form. For example, given T, an n-inverse S of T ($n > 1$), like the inverse of T, must not only have a set of reciprocal eigenvalues of T but must also have its general eigenspaces matching with those of T. Let $\sigma(T)$ denote the set of eigenvalues of T.

We first need a couple of lemmas to prove the spectral condition theorem.

Lemma 2.1. *For any two complex numbers λ and μ, the following holds:*

$$\beta_n(S,T) = \sum_{n_1+n_2+n_3=n} \binom{n}{n_1,n_2,n_3}(S-\mu I)^{n_1}T^{n_1}\mu^{n_2}(T-\lambda I)^{n_2}(\lambda\mu-1)^{n_3}.$$

Proof. Using the multinomial formula,

$$\begin{aligned}\beta_n(S,T) &= (yx-1)^n|_{y=S,x=T}\\ &= ([y-\mu]\,x+\mu\,[x-\lambda]+[\lambda\mu-1])^n|_{y=S,x=T}\\ &= \sum_{n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3}(y-\mu)^{n_1}x^{n_1}\mu^{n_2}\,(x-\lambda)^{n_2}\,(\lambda\mu-1)^{n_3}|_{y=S,x=T}\\ &= \sum_{n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3}(S-\mu I)^{n_1}T^{n_1}\mu^{n_2}(T-\lambda I)^{n_2}(\lambda\mu-1)^{n_3}.\end{aligned}$$

A more rigorous proof can be given by using induction. See the proofs of formulas in Lemma 1 and Lemma 12 in [15]. □

Remark 2.2. *When $\mu = 0$ or $\lambda\mu - 1 = 0$ in the above formula, $\mu^{n_2} = 1$ if $n_2 = 0$ and $(\lambda\mu-1)^{n_3} = 1$ if $n_3 = 0$.*

Lemma 2.3. *Let S be a left n-inverse of T. If $\lambda \in \sigma(T)$ and $\overline{\mu} \in \sigma(S^*)$ such that $\lambda\mu \neq 1$, then* $\ker(T-\lambda I)^k \perp \ker(S-\mu I)^{*l}$ *for all $k,l \geq 1$.*

Proof. We prove the lemma by induction. We first deal with the base case ($k = l = 1$). Let v_1, v_2 be such that $(T-\lambda I)v_1 = (S-\mu I)^* v_2 = 0$. Then by Lemma 2.1,

$$\begin{aligned}0 &= \langle \beta_n(S,T)v_1, v_2\rangle\\ &= \left\langle \sum_{n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3}(\lambda\mu-1)^{n_3}T^{n_1}\mu^{n_2}(T-\lambda I)^{n_2}v_1, (S-\mu I)^{*n_1}v_2\right\rangle\\ &= (\lambda\mu-1)^n\,\langle v_1, v_2\rangle\,,\end{aligned}$$

since $(T-\lambda I)^{n_2}v_1 = (S-\mu I)^{*n_1}v_2 = 0$ for all $n_1, n_2 \neq 0$. Thus if $\lambda\mu \neq 1$, then $\langle v_1, v_2\rangle = 0$.

We now fix $l = 1$ and use induction on k. Assume $ker(T-\lambda I)^k \perp ker(S-\mu I)^*$. Let $v_1 \in ker(T-\lambda I)^{k+1}$, $v_2 \in ker(S-\mu I)^*$. We will show that $\langle v_1, v_2\rangle = 0$ and hence $ker(T-\lambda I)^k \perp ker(S-\mu I)^*$ for all $k \geq 1$. Note that

$$\begin{aligned}0 &= \langle \beta_n(S,T)v_1, v_2\rangle\\ &= \left\langle \sum_{n_1+n_2+n_3=n}\binom{n}{n_1,n_2,n_3}(\lambda\mu-1)^{n_3}T^{n_1}\mu^{n_2}(T-\lambda I)^{n_2}v_1, (S-\mu I)^{*n_1}v_2\right\rangle\\ &= \left\langle \sum_{n_2+n_3=n}\binom{n}{0,n_2,n_3}(\lambda\mu-1)^{n_3}\mu^{n_2}(T-\lambda I)^{n_2}v_1, v_2\right\rangle\\ &= (\lambda\mu-1)^n\,\langle v_1, v_2\rangle\,,\end{aligned}$$

since if $n_1 \geq 1$, $(S-\mu I)^{*n_1}v_2 = 0$. Moreover, if $n_2 \geq k+1$, $(T-\lambda I)^{n_2}v_1 = 0$; and if $1 \leq n_2 \leq k$, $(T-\lambda I)^k(T-\lambda I)^{n_2}v_1 = (T-\lambda I)^{k+n_2}v_1 = 0$, so $(T-\lambda I)^{n_2}v_1 \in ker(T-\lambda I)^k$, thus $(T-\lambda I)^{n_2}v_1 \perp v_2$ and $\langle (T-\lambda I)^{n_2}v_1, v_2\rangle = 0$ by the inductive hypothesis. Thus the only term left is when $n_1 = n_2 = 0; n_3 = n$.

By symmetry, we get

$$ker(T-\lambda I) \perp ker(S-\mu I)^{*l} \text{ for } l \geq 1. \tag{2}$$

By above argument, we can assume that $ker(T-\lambda I)^k \perp ker(S-\mu I)^*$ for all $k \geq 1$. We will show that $ker(T-\lambda I)^k \perp ker(S-\mu I)^{*l}$ for any $k,l \geq 1$. The base case $l = 1$ is by assumption. Assume for fixed l,

$$ker(T-\lambda I)^k \perp ker(S-\mu I)^{*l} \text{ for all } k \geq 1. \tag{3}$$

We now show that $ker(T-\lambda I)^k \perp ker(S-\mu I)^{*l+1}$ for all $k \geq 1$ as well. We do this by using induction on k. For $k=1$, this is just (2). Now assume

$$ker(T-\lambda I)^m \perp ker(S-\mu I)^{*l+1}, \tag{4}$$

we will show

$$ker(T-\lambda I)^{m+1} \perp ker(S-\mu I)^{*l+1}.$$

Let $v_1 \in ker(T-\lambda I)^{m+1}$ and $v_2 \in ker(S-\mu I)^{*l+1}$. Then

$$\begin{aligned} 0 &= \langle \beta_n(S,T)v_1, v_2\rangle \\ &= \left\langle \sum_{n_1+n_2+n_3=n} \binom{n}{n_1,n_2,n_3}(\lambda\mu-1)^{n_3}T^{n_1}\mu^{n_2}(T-\lambda I)^{n_2}v_1, (S-\mu I)^{*n_1}v_2\right\rangle. \end{aligned} \tag{5}$$

Now, if $n_1 \geq 1$ and $n_2 \geq 0$, $(S-\mu I)^{*n_1}v_2 \in ker(S-\mu I)^{*l}$ and $(T-\lambda I)^{n_2}v_1 \in ker(T-\lambda I)^{m+1}$. Thus by (3), these terms in (5) are zero. If $n_1 = 0$ and $n_2 > 0$, $(S-\mu I)^{*n_1}v_2 \in ker(S-\mu I)^{*l+1}$ and $(T-\lambda I)^{n_2}v_1 \in ker(T-\lambda I)^m$. Then by the inductive hypothesis (4), these terms in (5) are also zero. The only term left in (5) is when $n_1 = n_2 = 0; n_3 = n$. Thus,

$$0 = \langle \beta_n(S,T)v_1, v_2\rangle = (\lambda\mu-1)^n \langle v_1, v_2\rangle.$$

By assumption, $\lambda\mu \neq 1$, thus $\langle v_1, v_2\rangle = 0$, as desired. □

Let $M_N(\mathbb{C})$ be the set of all $N \times N$ complex matrices. Let $S, T \in M_N(\mathbb{C})$.

Corollary 2.4. *Let S be a left n-inverse of T. If $\lambda \in \sigma(T)$ with (algebraic) multiplicity p, then $\frac{1}{\lambda} \in \sigma(S)$ with (algebraic) multiplicity p.*

Proof. Let S be a left n-inverse of T. We show that the eigenvalues of S and T are reciprocal to each other and they must have same multiplicities. Let $p_T(\lambda) = \prod_{i=1}^m (\lambda-\lambda_i)^{p_i}$ be the characteristic polynomial of T. Then

$$\mathbb{C}^N = ker(T-\lambda_1 I)^{p_1} \oplus \cdots \oplus ker(T-\lambda_m I)^{p_m},$$

where $\oplus$ denote the direct sum of subspaces. Now let $\mu \in \sigma(S)$. If $1/\mu \notin \sigma(T)$, then by Lemma 2.3,

$$ker(S-\mu I)^* \perp ker(T-\lambda_1 I)^{p_1} \oplus \cdots \oplus ker(T-\lambda_m I)^{p_m}.$$

This is a contradiction. Thus, $1/\mu \in \sigma(T)$, say $1/\mu = \lambda_1$. Let q_1 be the algebraic multiplicity of the eigenvalue u of S. Again by Lemma 2.3,

$$ker(S-\mu I)^{*q_1} \perp ker(T-\lambda_2 I)^{p_2} \oplus \cdots \oplus ker(T-\lambda_m I)^{p_m}.$$

Therefore $q_1 \leq N-(p_2+\cdots+p_m) = p_1$. Similarly, since T is a right n-inverse of S, we have $p_1 \leq q_1$. Hence $p_1 = q_1$. □

Corollary 2.5. *If S is an n-inverse of T, then $Det(S) = \frac{1}{Det(T)}$.*

This is a direct consequence of Corollary 2.4 since the determinant of a matrix is the product of its eigenvalues (counting multiplicities). The following lemma provides another analogue between an n-inverse of T and the inverse of T.

Lemma 2.6. *Let $S, T, P \in M_N(\mathbb{C})$. Assume P is invertible. Then S is a strict left n-inverse of T if and only if PSP^{-1} is a strict left n-inverse of PTP^{-1}.*

Proof. It suffices to notice that

$$\begin{aligned}\beta_n(PSP^{-1}, PTP^{-1}) &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(PSP^{-1})^k(PTP^{-1})^k \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}PS^kT^kP^{-1} = P\beta_n(S,T)P^{-1} = 0.\end{aligned}$$

The proof is complete. □

Now we are ready to prove the following spectral characterization of left n-inverses.

Theorem 2.7. *Given $T \in M_N(\mathbb{C})$, S is a left n-inverse of T for some n if and only if there exists an invertible matrix P such that*

$$PTP^{-1} = \begin{bmatrix} M_T(\lambda_1) & 0 & 0 & 0 \\ 0 & M_T(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_T(\lambda_m) \end{bmatrix}, \tag{6}$$

$$\text{and } PSP^{-1} = \begin{bmatrix} M_S(\frac{1}{\lambda_1}) & 0 & 0 & 0 \\ 0 & M_S(\frac{1}{\lambda_2}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_T(\frac{1}{\lambda_m}) \end{bmatrix}, \tag{7}$$

where $M_T(\lambda_i)$ and $M_S(\frac{1}{\lambda_i})$ are matrices of the same size and have eigenvalues λ_i and $\frac{1}{\lambda_i}$, respectively.

Proof. Let P be the invertible matrix such that $PTP^{-1} = J$, where J is in a block diagonal form as in (6). For example, J could be the canonical Jordan form of T. By Lemma 2.6, $\beta_n(S,T) = 0$ if and only if $\beta_n(PSP^{-1}, PTP^{-1}) = 0$. By Corollary 2.4, $\sigma(PTP^{-1}) = \{\lambda_i, 1 \le i \le m\}$ with the the multiplicity of λ_i being p_i, and $\sigma(PSP^{-1}) = \{\mu_i = 1/\lambda_i, 1 \le i \le m\}$ with the the multiplicity of μ_i being p_i. By Lemma 2.3, $ker(PSP^{-1} - \mu_i I)^{*l} \perp ker(PTP^{-1} - \lambda_j I)^k$ for $i \ne j$ and all $k, l \ge 1$, thus PSP^{-1} must be in the same block-diagonal form as in (7).

For the proof of the other direction, see Remark 3.5 after Proposition 3.4. □

Corollary 2.8. *Let $S, T \in M_N(\mathbb{C})$. If T has N distinct nonzero eigenvalues and S is a left n-inverse of T, then $S = T^{-1}$. Conversely, if the only left n-inverse of T is T^{-1}, then T has N distinct nonzero eigenvalues.*

The first part of this corollary follows from the fact that T is similar to a diagonal matrix with distinct diagonals. For the second part, see the remark after Proposition 3.8. Theorem 2.7 reduces the question of finding left n-inverses of a general matrix to a matrix with only one eigenvalue. The following observation further reduces this eigenvalue to 1.

Lemma 2.9. *Let λ be a nonzero complex number. Then S is a strict left n-inverse of λT if and only if λS is a strict left n-inverse of T.*

Proof. It is enough to notice that

$$\begin{aligned}\beta_n(S, \lambda T) &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S^k(\lambda T)^k \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\lambda^k S^k T^k = \beta_n(\lambda S, T).\end{aligned}$$

The proof is complete. □

Let us apply previous results to find all left n-inverses of a 2×2 matrix.

Example 2.10. *Find all left n-inverses of a 2×2 matrix A. If A has two distinct eigenvalues, then by Corollary 2.8, the only left n-inverse of A is A^{-1}. If A has one eigenvalue, by Theorem 2.7 and Lemma 2.9, we may assume A is A_1 or A_2 where*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let B be a left n-inverse of A_1 or A_2. Write

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By Theorem 2.7, B has eigenvalue 1. Therefore

$$(a - \lambda)(d - \lambda) - bc = (1 - \lambda)^2$$

implies that $a + d = 2$ and $ad - bc = 1$. So B must be of the form

$$B = \begin{bmatrix} a & b \\ c & 2 - a \end{bmatrix}$$

and $bc = a(2 - a) - 1$. There are two cases. In the case $bc = 0$, we have $a = 1$. Thus

$$B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \tag{8}$$

In the case $bc \neq 0$, we solve b in terms of a and c to get

$$B = \begin{bmatrix} a & \frac{a(2-a)-1}{c} \\ c & 2 - a \end{bmatrix} \tag{9}$$

Note that when $a = 1$, B in (9) reduces to a matrix in (8). So all left n-inverses of A_1 are

$$\text{strict left 2-inverses: } \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \text{ strict left 3-inverses: } \left\{ \begin{bmatrix} a & \frac{a(2-a)-1}{c} \\ c & 2 - a \end{bmatrix} \right\}$$

where both b and c are nonzero. Interestingly, these are all just strict left 2-inverses of A_2. Furthermore, by a direct verification, these are strict 2-inverses and strict 3-inverses of A_1.

3. On the Order n of Strict n-inverses

We have focused on characterizing all n-inverses for all $n \geq 1$ in the last section. Now we ask, what more can we say about n? Specifically, we ask: given an $n > 1$, is there always a strict n-inverse? Moreover, if we are given a strict k-inverse S of T, can we tell what range k must fall within? It turns out that there is an upper-bound for strict n-inverses, and for certain numbers in this range we can *always* find a strict inverse of that order.

The following result from Theorem 2 [15] is useful for studying these questions.

Lemma 3.1. *[15] Assume $S, Q \in B(X)$ are commuting and Q is a nilpotent operator of order l.*
(a) If S is a left m-inverse of T, then $S + Q$ is a left n-inverse of T where $n = m + l - 1$. Furthermore $S + Q$ is a strict left n-inverse of T if and only if $Q^{l-1}\beta_{m-1}(S, T) \neq 0$.
(b) If S is a right m-inverse of T, then $S + Q$ is a right n-inverse of T where $n = m + l - 1$. Furthermore $S + Q$ is a strict right n-inverse of T if and only if $\beta_{m-1}(T, S)Q^{l-1} \neq 0$.
(c) If S is an m-inverse of T, then $S + Q$ is an n-inverse of T where $n = m + l - 1$. Furthermore $S + Q$ is a strict n-inverse of T if and only if either $\beta_{m-1}(T, S)Q^{l-1} \neq 0$ or $Q^{l-1}\beta_{m-1}(S, T) \neq 0$.

The following corollary of the above lemma is also needed.

Lemma 3.2. *Let Q be a nilpotent operator of order l and T be an invertible operator. If $TQ = QT$, then $T^{-1} + Q$ is a strict l-inverse of T.*

Proof. Let $S = T^{-1}$. By assumption, $SQ = QS$. Note that S is a 1-inverse of T. By Lemma 3.1, $T^{-1} + Q$ is an l-inverse of T. Note that $Q^{l-1}\beta_0(S,T) = Q^{l-1} \neq 0$, so $T^{-1} + Q$ is a strict l-inverse of T. □

We also need the following lemma.

Lemma 3.3. *Let $T \in M_N(\mathbb{C})$ with only one eigenvalue $\lambda \neq 0$. Then $T = \lambda I + Q$ for some nilpotent matrix Q of order n that is equal to the degree of the minimal polynomial of T. Furthermore $\frac{1}{\lambda}I$ is a strict n-inverse of T.*

Proof. The first part is clear. By Lemma 3.1 (c), $\lambda I + Q$ is a strict n-inverse of $\frac{1}{\lambda}I$. In other words, $\frac{1}{\lambda}I$ is a strict n-inverse of $\lambda I + Q$. □

The following proposition reveals the relations between the order n of a strict left n-inverse of T and the minimal and characteristic polynomials of T.

Proposition 3.4. *Let $T \in M_N(\mathbb{C})$. Let $p(\lambda) = \prod_{i=1}^m(\lambda - \lambda_i)^{p_i}$ and $q(\lambda) = \prod_{i=1}^m(\lambda - \lambda_i)^{q_i}$ be the minimal polynomial and characteristic polynomial of T. If T has a strict left (or right) k-inverse, then $k \leq max\{p_i + q_i - 1, 1 \leq i \leq m\}$.*

Proof. Suppose S is a left k-inverse of T. We will prove the proposition by assuming $m = 2$, since the proof of the general case is similar. Then by Theorem 2.7, there exists an invertible matrix P such that

$$PTP^{-1} = \begin{bmatrix} M_T(\lambda_1) & 0 \\ 0 & M_T(\lambda_2) \end{bmatrix}, \; PSP^{-1} = \begin{bmatrix} M_S(\frac{1}{\lambda_1}) & 0 \\ 0 & M_S(\frac{1}{\lambda_2}) \end{bmatrix},$$

where $M_T(\lambda_i)$ (resp., $M_S(\frac{1}{\lambda_i})$) is a block of the size q_i with λ_i (resp., $\frac{1}{\lambda_i}$) as its only eigenvalue. Since

$$\beta_k(PSP^{-1}, PTP^{-1}) = 0 \text{ if and only if } \beta_k\left(M_S\left(\frac{1}{\lambda_i}\right), M_T(\lambda_i)\right) = 0 \text{ for } i = 1, 2,$$

we need only to look at the individual block $M_T(\lambda_i)$ and $M_S(\frac{1}{\lambda_i})$. Now, by Lemma 3.3 and the assumption on T,

$$M_T(\lambda_i) = \lambda_i I_{q_i} + P_i, M_S\left(\frac{1}{\lambda_i}\right) = \frac{1}{\lambda_i} I_{q_i} + Q_i,$$

where P_i is a nilpotent matrix of order p_i and Q_i is a nilpotent matrix of order less than or equal to q_i. By Lemma 3.3, $\frac{1}{\lambda_i}I_{q_i}$ is a strict left p_i-inverse of $M_T(\lambda_i)$. Next, by Lemma 3.1, $M_S(\frac{1}{\lambda_i}) = \frac{1}{\lambda_i}I + Q_i$ is a left $(p_i + q_i - 1)$-inverse of $M_T(\lambda_i)$ (not necessarily strict). Thus, by Lemma 2.6, S is a strict left k-inverse of T where $k \leq \max\{p_i + q_i - 1, 1 \leq i \leq m\}$. □

Remark 3.5. *Proof of other direction in Theorem 2.7. Since $p_i + q_i - 1 \leq 2N - 1$, it follows from the proof of above proposition that PSP^{-1} as in (7) is a $(2N - 1)$-inverse of PTP^{-1}.*

Now that we know there is an upper bound for the order n of strict n-inverses of T, can we always find a k-inverse of T for some k less than this upper-bound? It turns out that there are orders that we are guaranteed to find strict inverses for. For example, this upper-bound is always attainable. We first study this question assuming T has only one eigenvalue, the general case can be dealt with by using block diagonal form of T as in Theorem 2.7. Let $J_N(\lambda)$ denote a single Jordan block of size $N \times N$ with all λ on its diagonal and all 1 on its superdiagonal. A Toeplitz matrix T is of the form $T = [a_{i-j}]_{i,j=1}^N$. It is easy to verify that $SJ_N(\lambda) = J_N(\lambda)S$ if and only if S is an upper triangular Toeplitz matrix. This S is nilpotent if and only if its diagonal is zero. The following lemma tells us the order of S. Let $\{e_1, ..., e_N\}$ be the standard bases of $\mathbb{C}^N$.

Lemma 3.6. *Let $Q \in M_N(\mathbb{C})$ be an upper triangular Toeplitz matrix with zeros on the main diagonal . Then Q is nilpotent of order $\lceil \frac{N}{j} \rceil$, where the j^{th} superdiagonal is the first nonzero one, and $\lceil \frac{N}{j} \rceil$ is the smallest integer not less than $\frac{N}{j}$ ($\lceil * \rceil$ is the ceiling function).*

Proof. Let $1 \leq j \leq N-1$. Note that

$$\begin{aligned} &Q(e_1) = \cdots = Q(e_j) = 0, \\ &Q(e_{j+1}) \in \operatorname{Span}\{e_1\}, \cdots, Q(e_N) \in \operatorname{Span}\{e_1, ..., e_{N-j}\}. \end{aligned}$$

Thus,

$$Q^k(e_N) \in \operatorname{Span}\{e_1, ..., e_{N-jk}\},$$

and

$$Q^k = 0 \text{ if } N - jk \leq 0.$$

The proof is complete. □

This lemma allows us to construct nilpotent matrices which commute with Jordan blocks, which upon combing with Lemma 3.2 yields the following result. We first need the following observation, which is useful for studying n-inverses of block diagonal matrices.

Lemma 3.7. *Let*

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}.$$

If S_1 is a strict k-inverse of T_1 and S_2 is a strict l-inverse of T_2, then S is a strict n-inverse of T where $n = \max\{k, l\}$. In particular if $S_2 = T_2^{-1}$, then S is a strict k-inverse of T.

Proof. The result clearly follows from definitions and the fact that

$$\beta_n(S, T) = \begin{bmatrix} \beta_n(S_1, T_1) & 0 \\ 0 & \beta_n(S_2, T_2) \end{bmatrix}.$$

The proof is complete. □

Proposition 3.8. *Let $T \in M_N(\mathbb{C})$ with only one eigenvalue $\lambda \neq 0$. Let $B = \oplus_{i=1}^m J_{n_i}(\lambda)$ be the Jordan canonical form of T. Let*

$$H = \cup_{i=1}^m H_i \text{ where } H_i = \left\{ \left\lceil \frac{n_i}{j} \right\rceil \text{ for } 1 \leq j \leq n_i - 1 \right\}.$$

Then there is always a strict k-inverse of T for each $k \in H$.

Proof. We will prove the result for $m = 1$. The general case follows from Lemma 3.7. By Theorem 2.7, we can assume $T = J_{n_1}(\lambda)$. By Lemma 3.6, for each $k \in H_1$, there exists a nilpotent matrix Q of order k such that $QT = TQ$. Thus, by Lemma 3.2, $T^{-1} + Q$ is a strict k-inverse of T. □

The above lemma could be improved in the case $m > 1$ since we could use more general nilpotent matrices than just nilpotent matrices of block diagonal forms. Note that if $n_1 > 1$, then $\left\lceil \frac{n_1}{n_1 - 1} \right\rceil = 2$. For an invertible matrix T, T will have a strict 2-inverse as long as T has an eigenvalue with algebraic multiplicity bigger than or equal to 2.

The above result gives small values of k such that T has a strict k-inverse; the next result gives large values of k such that T has a strict k-inverse. In particular, T always has a k-inverse for the upperbound identified in Proposition 3.4.

Lemma 3.9. *Let $T \in M_N(\mathbb{C})$ and $\lambda \neq 0$. If $T = \lambda I + Q$ for some nilpotent matrix Q of order n, then T has a strict k-inverse for each $k \in \{n, n+1, \cdots, n+N-1\}$.*

Proof. By Lemma 2.9, we may assume $\lambda = 1$. Let k be such that $n \le k \le n+N-1$. Since Q is of order n, there is a vector u such that $Q^{n-1}(u) \neq 0$. Let W be an arbitrary $N \times N$ nilpotent matrix of order $l = k-n+1$. Then there exists a vector v such that $W^{l-1}(v) \neq 0$. Let P be any invertible matrix that sends v to $Q^{n-1}(u)$. Now let $U = PWP^{-1}$. It is clear that the order of U is also l. Furthermore

$$U^{l-1}Q^{n-1}(u) = PW^{l-1}P^{-1}\left[Q^{n-1}(u)\right] = PW^{l-1}(v) \neq 0. \tag{10}$$

By Lemma 3.3, I is a strict n-inverse of T. By Lemma 3.1 with $S = I$, $I + U$ is a k-inverse of T where $n+l-1 = k$. We claim in fact $I+U$ is a strict k-inverse of T. A direct computation shows that

$$\begin{aligned}\beta_{n-1}(I,T) = \beta_{n-1}(I, I+Q) &= \sum_{i=0}^{n-1}(-1)^{n-1-i}\binom{n-1}{i}I^i(I+Q)^i \\ &= (I[I+Q]-I)^{n-1} = Q^{n-1}.\end{aligned}$$

Hence

$$U^{l-1}\beta_{n-1}(I,T) = U^{l-1}Q^{n-1},$$

and by (10)

$$U^{l-1}\beta_{n-1}(I,T)(u) = U^{l-1}Q^{n-1}(u) \neq 0.$$

By Lemma 3.1, the claim is proved. □

Summarizing previous results we have the following theorem.

Theorem 3.10. *Let $T \in M_N(\mathbb{C})$. Let $p(\lambda) = \prod_{i=1}^m(\lambda-\lambda_i)^{p_i}$ and $q(\lambda) = \prod_{i=1}^m(\lambda-\lambda_i)^{q_i}$ be the minimal polynomial and characteristic polynomial of T. Also let $K = \bigcup_{i=1}^m(K_i \cup H_i)$, where*

$$K_i = \{p_i, ..., p_i+q_i-1\} \text{ and } H_i = \left\{\left\lceil\frac{p_i}{j}\right\rceil \text{ for } 1 \le j \le p_i - 1\right\}.$$

Then there is always a strict k-inverse of T for each $k \in K$.

Proof. We first assume T has only one eigenvalue λ_1. Then for each $k \in K_1$, by Lemma 3.9, T has a strict k-inverse. For each $k \in H_1$, by Proposition 3.8, T has a strict k-inverse since p_1 is just the largest n_i in Proposition 3.8. The case $m > 1$ follows from Theorem 2.7 and Lemma 3.7. □

Corollary 3.11. *Let T be a diagonalizable matrix. Let $q(\lambda) = \prod_{i=1}^m(\lambda-\lambda_i)^{q_i}$ be the characteristic polynomial of T. Then T has a strict k-inverse if and only if $k \in \{1, 2, \ldots, max\{q_i, 1 \le i \le m\}\}$.*

The proof follows directly from Proposition 3.4 and Theorem 3.10 since if T is a diagonalizable matrix, then $p_i = 1$. Examples of diagonalizable matrices include normal matrices. We illustrate the above theorem by an example.

Example 3.12. *Consider the following* 10×10 *matrix*

$$T = \begin{bmatrix}\lambda_1 I_3 & 0 \\ 0 & J_7(\lambda_2)\end{bmatrix}$$

where $J_7(\lambda_2)$ is one Jordan block of size 7. So $p_1 = 1, q_1 = 3, p_2 = 7, q_2 = 7$. It follows that

$$1 = p_1 \le k \le p_1 + q_1 - 1 = 3,$$
$$7 = p_2 \le k \le p_2 + q_2 - 1 = 13.$$

Thus, according to Theorem 3.10, we will be able to find a 1*-inverse of T (always),* 2*-inverse,* 3*-inverse,* 4*-inverse* ($4 = \left\lceil\frac{7}{2}\right\rceil$)*,* 7*-inverse,$\cdots$,* 13*-inverse. By Proposition 3.4, the maximal order is* 13*, so we are missing only* 5 *and* 6*. By ad hoc methods, we can find* 5*-inverses or* 6*-inverses of T.*

4. Some Remarks and Questions

We first remark that, although we have assumed the space our matrices act on is a finite dimensional complex Hilbert space, our results can be extended to a finite dimensional complex vector space. Let V be a finite dimensional complex vector space and $L(V)$ be the algebra of linear transformations on V. Let $S, T \in L(V)$, and let S_M, T_M be the matrix representations of S and T with respect to a (fixed) bases of V. Then S is a left n-inverse of T if and only if S_M is a left n-inverse of T_M.

In this section we make several remarks and raise some questions about left n-inverses of T versus right n-inverses of T. We then give an abstract structure theorem of left n-inverses of T, which follows directly from the concrete structure theorem of left n-inverses of T as in Theorem 2.7. Finally we state a linearization algorithm to find left n-inverses of T.

On a finite dimensional vector space, if S is a left inverse of T, then in fact S is an inverse of T. Thus we ask the following question.

Problem 4.1. *If S is a left n-inverse of T on a finite dimensional vector space, is S automatically a right n-inverse of T?*

By Example 2.10, the answer is yes for 2×2 matrices. The answer is no beyond 2×2 matrices by the following example constructed by Stepan Paul. Let

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

then

$$S^2T^2 - 2ST + I = 0,$$

but

$$T^2S^2 - 2TS + I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \neq 0.$$

Thus S is a strict left 2-inverse, but S is not a right 2-inverse of T. It turn out that S is a strict right 3-inverse of T. The key here is that

$$ST = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \neq TS = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The answer to Problem 4.1 is of course yes if $ST = TS$. The following observations made in Proposition 8 and Corollary 11 of [15] are relevant here.

Proposition 4.2. *[15] If S is an n-inverse of T and $ST = TS$, then T is invertible and $S = T^{-1} + Q$ where $Q^n = 0$ and $QT = TQ$.*

It turns out that the condition $ST = TS$ is not needed when $n = 2$.

Corollary 4.3. *[15] If $T \in B(X)$ has a 2-inverse S, then $S = T^{-1} + Q$ where $Q^2 = 0$ and $QT^{-1} = T^{-1}Q$.*

By Proposition 3.4, we know that any strict left n-inverse must be a strict right m-inverse for some m. Thus, it brings up the question: Is there a relationship between the n and m? Specifically, how large can the gap $|n - m|$ be? More generally, what are possible values of k such that there exist S and T such that S is a strict left m-inverse of T, and S is a strict right $m+k$-inverse of T? Collectively, given an invertible T, we can study the sets

$$\begin{aligned} K &= \{k : T \text{ has a strict } k\text{-inverse}\}, \\ K_l &= \{k : T \text{ has a strict left } k\text{-inverse}\}, \\ K_r &= \{k : T \text{ has a strict right } k\text{-inverse}\}. \end{aligned} \tag{11}$$

Are these three sets the same? The following is another example where a strict left n-inverse of T is not a strict right n-inverse.

Example 4.4. *Here S is a strict right 4-inverse of T, but S is a strict left 7-inverse of T.*

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & \frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A reasonable condition to impose on S and T in Problem 4.1 is that they are both upper triangular. The answer is still no as the following example shows.

Example 4.5. *Here both S and T are upper triangular. Then S is a strict right 3-inverse of T, but S is a strict left 4-inverse of T.*

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lemma 3.1 provides a method for finding a possible strict left n-inverse of T by using the sum of a known strict left l-inverse of T and a nilpotent operator commuting with this left l-inverse. The simplest known l-inverse of T is T^{-1}. This is essentially the construction in Proposition 3.8 and Lemma 3.9. Lemma 3.1 also suggests an underlying structure of the n-inverses. The following question seems natural. Let S be a left n-inverse of T. Does there exist a nilpotent matrix Q such that $S = T^{-1} + Q$ (here Q and T do not necessarily commute)? The following simple example shows that the answer is no.

Example 4.6. *Here S is a 3-inverse of T, but $S \neq T^{-1} + Q$ for any nilpotent matrix Q. Let*

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \ T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then S is a 3-inverse of T. But

$$S - T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so $S - T^{-1}$ is in fact invertible.

We consider that S might in fact be a finite sum such as $S = T^{-1} + Q_1 + \ldots + Q_t$ with each Q_i commuting with the previous one. Indeed it turns out that $t = 2$ is enough. The following result derived easily from Theorem 2.7 seems interesting.

Theorem 4.7. *Let $T \in M_N(\mathbb{C})$ be an invertible matrix. Then S is a left (or right) n-inverse for some n if and only if $S = T^{-1} + Q_1 + Q_2$ for some nilpotent matrices Q_1 and Q_2 such that Q_1 commutes with T^{-1} and Q_2 commutes with $T^{-1} + Q_1$.*

Proof. If $S = T^{-1} + Q_1 + Q_2$, by using Lemma 3.1 twice, S is a left (or right) n-inverse for some n. We will prove the other direction by considering the case in which T has two distinct eigenvalues; the general case is similar. By Theorem 2.7, there exists an invertible matrix P such that

$$PTP^{-1} = \begin{bmatrix} M_T(\lambda_1) & 0 \\ 0 & M_T(\lambda_2) \end{bmatrix}, \; PSP^{-1} = \begin{bmatrix} M_S(\frac{1}{\lambda_1}) & 0 \\ 0 & M_S(\frac{1}{\lambda_2}) \end{bmatrix}.$$

Therefore

$$PSP^{-1} = \begin{bmatrix} M_S(\frac{1}{\lambda_1}) & 0 \\ 0 & M_S(\frac{1}{\lambda_2}) \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1} I_1 + Q_{21} & 0 \\ 0 & \frac{1}{\lambda_2} I_2 + Q_{22} \end{bmatrix},$$

$$\left(PTP^{-1}\right)^{-1} = PT^{-1}P^{-1} = \begin{bmatrix} M_T(\lambda_1) & 0 \\ 0 & M_T(\lambda_2) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} I_1 - Q_{11} & 0 \\ 0 & \frac{1}{\lambda_2} I_2 - Q_{12} \end{bmatrix},$$

where I_1 and I_2 are identity matrices and Q_{11}, Q_{12}, Q_{21} and Q_{22} are nilpotent matrices. Let

$$Q_1 = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{12} \end{bmatrix}, Q_2 = \begin{bmatrix} Q_{21} & 0 \\ 0 & Q_{22} \end{bmatrix}.$$

Then

$$PSP^{-1} = PT^{-1}P^{-1} + Q_1 + Q_2, \tag{12}$$

where Q_1 commutes with $PT^{-1}P^{-1}$, and Q_2 commutes with $PT^{-1}P^{-1} + Q_1$ since

$$PT^{-1}P^{-1} + Q_1 = \begin{bmatrix} \frac{1}{\lambda_1} I_1 & 0 \\ 0 & \frac{1}{\lambda_2} I_2 \end{bmatrix}.$$

Rewrite equation (12) as

$$S = T^{-1} + P^{-1}Q_1P + P^{-1}Q_2P.$$

It is clear that $P^{-1}Q_1P$ and $P^{-1}Q_2P$ are nilpotent matrices with the desired commuting property. □

It follows from the above theorem that the following three sets B, B_l and B_r are the same for an invertible matrix T, even though we do no know if the three sets K, K_l and K_r in (11) are the same.

$$\begin{aligned} B &= \{S : S \text{ is an } k\text{-inverse of } T \text{ for some } k\}, \\ B_l &= \{S : S \text{ is a left } k\text{-inverse of } T \text{ for some } k\}, \\ B_r &= \{S : S \text{ is a right } k\text{-inverse of } T \text{ for some } k\}. \end{aligned}$$

By definition (1), finding left n-inverses of T is a nonlinear problem. Theorem 4.7 reduces this problem to finding nilpotent matrices. We briefly describe a linearization algorithm to find left n-inverses of T.

Given T, let $p(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i)^{p_i}$ be its characteristic polynomial. By Theorem 2.7, we can assume T is in its Jordan canonical form. Let S be a left n-inverse of T. Then S is similar to its Jordan canonical form. That is, $S = PDP^{-1}$ for some Jordan canonical form D. By Theorem 2.7, we require that the characteristic polynomial of D is $q(\lambda) = \prod_{i=1}^{m}(\lambda - \frac{1}{\lambda_i})^{p_i}$. Thus $\beta_n(S, T) = 0$ if and only if

$$\begin{aligned}0 &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}S^kT^k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(P^{-1}DP)^kT^k \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P^{-1}D^kPT^k = P^{-1}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}D^kPT^k.\end{aligned}$$

Equivalently

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}D^kPT^k = 0,$$

which is a linear equation in P. Therefore by choosing a D (there are only finite many choices of D), we can use the above linear equation to find an invertible P. Then $S = PDP^{-1}$ is a left n-inverse of T.

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