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A Note on the Group Inverses of Block Matrices Over Rings

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Abstract. Suppose R is an associative ring with identity 1. The purpose of this paper is to give some necessary and sufficient conditions for the existence and the representations of the group inverse of the block matrix $\begin{pmatrix} AX+YB & A \\ B & 0 \end{pmatrix}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under some conditions. Some examples are given to illustrate our results.

1. The first section

The Drazin (group) inverse of 2×2 block matrices have numerous applications in many areas, especially in singular differential and difference equations and finite Markov chains (see[1,4,5,7,15]). In 1979, Campbell and Meyer proposed a problem to find a concrete expression for Drazin (group) inverse of block matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square but need not to be the same size(see[1]). Although the problem has not been solved completely yet, ones have got results on group inverses under some special conditions in [2,3,6,9-11,16-18,21].

The purpose of this paper is to extend some recent results on the group inverse of block matrices, as for basic ring and matrix type.

In [23], Li et al.investigate the group inverse of the block matrix $\begin{pmatrix} AX+YB & A \\ B & 0 \end{pmatrix}$ over skew filed under the conditions $A^\sharp$ exists, $XA = AX$, and $r(A) = r(AX)$. This generalizes Theorem 1.1 of [8]. In this paper, the sufficient and necessary condition for the existence of the group inverse of the above partitioned matrix over any ring is characterized.

In [13], Bu et al.investigated the group inverse of the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ over skew filed under the conditions A is invertible and $(D - CA^{-1}B)^\sharp$ exists. In [19], Deng et al.studied the group inverse of the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ over complex Banach spaces under the conditions A and $S = D - CA^\sharp B$ are group invertible, $A^\pi B = 0$ and $S^\pi C = 0$. In this paper, we will investigate the group inverse of the above block matrix over rings under weaker conditions. We should pointed that the methods of solving problems are quite different from those of [13, 14, 19, 20].

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Let R, K be an associative ring with identity 1 and skew field, respectively. $R^{m\times n}$ be the set of all $m\times n$ matrices over R. We denote $R^{m\times 1}$ and $R^{1\times n}$ by R^m and $R^{(n)}$, respectively. For $A\in R^{n\times n}$, if there exists a matrix $X\in R^{n\times n}$ such that $AXA=A$, $XAX=X$ and $AX=XA$, then X is called the group inverse of A and can be denoted by $A^{\sharp}$. For $A\in R^{n\times m}$, if X satisfies only $AXA=A$, then A is called regular and X is called a {1}-inverse or regular inverse of A. In this case, denote the set of all {1}-inverse of A by $A\{1\}$. Let $A^{(1)}$ be any {1}-inverse of A, denote $I-A^{(1)}A$ and $I-AA^{(1)}$ by $A^{\pi l}$ and $A^{\pi r}$ respectively. If $A\in K^{m\times n}$, $r(A)$ denotes the rank of A. If A is group invertible, we denote $I-AA^{\sharp}$ by A^{π}, where I is an identity matrix of order n.

In this paper, for $A\in R^{m\times n}$, we also denote by $R(A)=\{Ax|x\in R^n\}$ and $R_r(A)=\{xA|x\in R^{(m)}\}$ the range and the row range of A, respectively.

2. Some Lemmas

In the next section we will use the following results.

Lemma 2.1. [22] *Let $A\in R^{n\times n}$, the followings are equivalent:*
(i) $A^{\sharp}$ exists;
(ii) $A=A^2X$ and $A=YA^2$ for some $X,Y\in R^{n\times n}$. In this case, $A^{\sharp}=YAX=AX^2=Y^2A$;
(iii) $R(A)=R(A^2), R_r(A)=R_r(A^2)$.

Lemma 2.2. *Let $A,X\in R^{n\times n}$. If $AX=XA$, $A^{\sharp}$ exists, then $A^{\sharp}X=XA^{\sharp}$.*

Proof. The proof of this Lemma is similar to that of Lemma 3.1 in [24], so we omit it here. □

Lemma 2.3. *Let $A,X\in R^{n\times n}$. If $AX=XA$, $A^{\sharp}$ exists, $R(A)\subset R(AX)$ and $R_r(A)\subset R_r(XA)$, then $(XAA^{\sharp})^{\sharp}$ exists and the following equalities hold.*
(i) $XA^2(XAA^{\sharp})^{\sharp}=A^2$;
(ii) $(XAA^{\sharp})^{\sharp}A^2X=A^2$;
(iii) $(XAA^{\sharp})^{\sharp}AA^{\sharp}=(XAA^{\sharp})^{\sharp}$;
(iv) $AA^{\sharp}(XAA^{\sharp})^{\sharp}=(XAA^{\sharp})^{\sharp}$.

Proof. Since $R(A)\subset R(AX)$, there exists a matrix Y over R such that $A=AXY$. Using Lemma 2, we have $XAA^{\sharp}=XA^{\sharp}A=XA^{\sharp}AXY=XAA^{\sharp}XAA^{\sharp}Y$, i.e., $R(XAA^{\sharp})\subset R[(XAA^{\sharp})^2]$, so $R(XAA^{\sharp})=R[(XAA^{\sharp})^2]$.

Similarly, using $R_r(A)\subset R_r(XA)$ we also get $R_r(XAA^{\sharp})=R_r[(XAA^{\sharp})^2]$. By Lemma 1, $(XAA^{\sharp})^{\sharp}$ exists.

(i) Notice that $R_r(A)\subset R_r(XA)$, then there exists a matrix $Z\in R^{n\times n}$ such that $A=ZXA$. Hence

$$\begin{aligned}XA^2(XAA^{\sharp})^{\sharp}&=A^4X[A^{\sharp}]^2(XAA^{\sharp})^{\sharp}=ZXA^4X[A^{\sharp}]^2(XAA^{\sharp})^{\sharp}\\&=ZA^2[(XAA^{\sharp})]^2(XAA^{\sharp})^{\sharp}=ZA^2(XAA^{\sharp})\\&=ZXA^2=ZXAA=A^2\end{aligned}$$

(ii) Similarly, using $R(A)\subset R(AX)$ we can obtain that $(XAA^{\sharp})^{\sharp}A^2X=A^2$.

(iii) By (i), we have

$$\begin{aligned}(XAA^{\sharp})^{\sharp}AA^{\sharp}&=(XAA^{\sharp})^{\sharp}[A^{\sharp}]^2A^2=(XAA^{\sharp})^{\sharp}[A^{\sharp}]^2XA^2(XAA^{\sharp})^{\sharp}\\&=(XAA^{\sharp})^{\sharp}XAA^{\sharp}(XAA^{\sharp})^{\sharp}=(XAA^{\sharp})^{\sharp}\end{aligned}$$

(iv) Similarly, from (ii), we can prove (iv). □

Lemma 2.4. *Let $A\in R^{m\times n}, B\in R^{n\times m}, R(BAB)=R(B)$ and $R_r(BAB)=R_r(B)$, then $(AB)^{\sharp}$ and $(BA)^{\sharp}$ exist and the following equalities hold.*
(i) $(AB)^{\sharp}=A[(BA)^{\sharp}]^2B$;
(ii) $(BA)^{\sharp}=B[(AB)^{\sharp}]^2A$;
(iii) $(AB)^{\sharp}A=A(BA)^{\sharp}$;
(iv) $(BA)^{\sharp}B=B(AB)^{\sharp}$.

Proof. The proof of this Lemma is similar to those of Lemma 2.2 and 2.3 in [12]. □

3. Main Results

We begin with the following theorem.

Theorem 3.1. *Let* $M = \begin{pmatrix} AX+YB & A \\ B & 0 \end{pmatrix}$, *where* $A, B, X, Y \in R^{n\times n}$, $A^\sharp$ *exists,* $XA = AX$, $R(A) \subset R(AX)$ *and* $R_r(A) \subset R_r(XA)$, *then we have*

(i) $M^\sharp$ *exists if and only if* $R(B) = R(BSB)$, $R_r(B) = R_r(BSB)$, *where* $S = A^\pi Y - (XAA^\sharp)^\sharp$;

(ii) If $M^\sharp$ *exists, then* $M^\sharp = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, *where*

$$M_1 = (SB)^\sharp A^\pi + (SB)^\pi A^\sharp (XAA^\sharp)^\sharp - (SB)^\pi ZB(SB)^\sharp A^\pi,$$
$$M_2 = -(SB)^\sharp (XAA^\sharp)^\sharp + (SB)^\pi A^\sharp [(XAA^\sharp)^\sharp]^2 + (SB)^\pi ZB(SB)^\sharp [(XAA^\sharp)^\sharp]^2,$$
$$M_3 = -B(SB)^\sharp A^\sharp (XAA^\sharp)^\sharp + B[(SB)^\sharp]^2 A^\pi + B(SB)^\sharp ZB(SB)^\sharp A^\pi,$$
$$M_4 = -B(SB)^\sharp A^\sharp [(XAA^\sharp)^\sharp]^2 - B[(SB)^\sharp]^2 (XAA^\sharp)^\sharp - B(SB)^\sharp ZB(SB)^\sharp (XAA^\sharp)^\sharp,$$
$$Z = A^\sharp [(XAA^\sharp)^\sharp]^2 + A^\sharp (XAA^\sharp)^\sharp Y.$$

Proof. (*i*) The "only if" part.
Note that

$$M = \begin{pmatrix} YB & A \\ B & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}, \tag{1}$$

Using Lemma 3, we know that $(XAA^\sharp)^\sharp$ exists, and

$$M^2 = \begin{pmatrix} YBSB & XA^2+YBA \\ BSB & BA \end{pmatrix} \begin{pmatrix} I & 0 \\ A^\sharp YB & I \end{pmatrix} \begin{pmatrix} I & 0 \\ A^\sharp (XAA^\sharp)^\sharp B & I \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}, \tag{2}$$

$$M^2 = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} XAYB+AB & XA^2 \\ BYB & BA \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \tag{3}$$

By Lemma 1, there exist matrices $\overline{X}$ and $\overline{Y}$ over R such that $M = M^2\overline{X}$ and $\overline{Y}M^2 = M$.
Let

$$\overline{X} = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^\sharp (XAA^\sharp)^\sharp B & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^\sharp YB & I \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}, \tag{4}$$

$$\overline{Y} = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix}. \tag{5}$$

From Eqs. (1), (2), (4) and $M = M^2\overline{X}$, we can obtain following equations

$$YBSBX_1 + (XA^2+YBA)X_3 = YB, \tag{6}$$

$$YBSBX_2 + (XA^2+YBA)X_4 = A, \tag{7}$$

$$BSBX_1 + BAX_3 = B, \tag{8}$$

$$BSBX_2 + BAX_4 = 0. \tag{9}$$

It follows, from Eqs. (1), (3), (5) and $\overline{Y}M^2 = M$, that we have

$$Y_1(XAYB+AB) + Y_2BYB = YB, \tag{10}$$

$$Y_1XA^2 + Y_2BA = A, \tag{11}$$

$$Y_3(XAYB + AB) + Y_4BYB = B, \tag{12}$$

$$Y_3XA^2 + Y_4BA = 0. \tag{13}$$

Instead of (8), we have

$$YBSBX_1 + YBAX_3 = YB.$$

Substituting it into (6), we have $XA^2X_3 = 0$, so $(XAA^\sharp)^\sharp A^2XX_3 = 0$. By Lemma 3 (*ii*), it follows by $A^2X_3 = 0$, so $AX_3 = 0$.

Substituting $AX_3 = 0$ into (8), we get $BSBX_1 = B$, i.e., $R(BSB) = R(B)$.

From (13), by Lemma 3 (*i*), we can get

$$-Y_4BAA^\sharp YB = Y_3XA^2A^\sharp YB = Y_3XAYB, \tag{14}$$

and

$$-Y_4BAA^\sharp(XAA^\sharp)^\sharp B = Y_3XA^2A^\sharp(XAA^\sharp)^\sharp B = Y_3AB. \tag{15}$$

Substitute (14) into (12), we have

$$-Y_4BAA^\sharp YB + Y_3AB + Y_4BYB = B. \tag{16}$$

Substitute (15) into (16), by Lemma 3 (*iv*), we have

$$-Y_4BAA^\sharp YB - Y_4BAA^\sharp(XAA^\sharp)^\sharp B + Y_4BYB = B,$$

that is $Y_4BSB = B$, which implies $R_r(B) = R_r(BSB)$. This completes proof of "only if" part.

In what follows, we give the proof of "if" part.

It follows from Lemma 2.4 that $R(BSB) = R(B)$ and $R_r(BSB) = R_r(B)$ imply $(SB)^\sharp$ and $(BS)^\sharp$ exist. Let

$$X_1 = (SB)^\sharp,\ \ X_2 = -(SB)^\sharp(XAA^\sharp)^\sharp,\ \ X_3 = 0,\ \ X_4 = A^\sharp(XAA^\sharp)^\sharp$$

and

$$Y_1 = (SB)^\pi A^\sharp(XAA^\sharp)^\sharp,\ \ Y_2 = (SB)^\sharp S,\ \ Y_3 = -B(SB)^\sharp A^\sharp(XAA^\sharp)^\sharp,\ \ Y_4 = (BS)^\sharp.$$

We can easily obtain that $R(BSB) = R(B)$ implies $BSB(SB)^\sharp = B$.

We claim that X_1, X_2, X_3 and X_4 satisfy the Eqs. (6)-(9). Next, we verify the claim by computation separately.

(1) $YBSBX_1 + (XA^2 + YBA)X_3 = YBSB(SB)^\sharp = YB;$

(2) $YBSBX_2 + (XA^2 + YBA)X_4$
$= -YBSB(SB)^\sharp AA^\sharp(XAA^\sharp)^\sharp + (XA^2 + YBA)A^\sharp(XAA^\sharp)^\sharp$
$= -YBAA^\sharp(XAA^\sharp)^\sharp + XA(XAA^\sharp)^\sharp + YBAA^\sharp(XAA^\sharp)^\sharp = A;$

(3) $BSBX_1 + BAX_3 = BSB(SB)^\sharp = B;$

(4) $BSBX_2 + BAX_4$
$= -BSB(SB)^\sharp AA^\sharp(XAA^\sharp)^\sharp + BAA^\sharp(XAA^\sharp)^\sharp$
$= -BAA^\sharp(XAA^\sharp)^\sharp + BAA^\sharp(XAA^\sharp)^\sharp = 0.$

From $S = A^\pi Y - (XAA^\sharp)^\sharp$, we have $SB - YB = -AA^\sharp YB - AA^\sharp(XAA^\sharp)^\sharp B$. By Lemma 2.4, we know $B(SB)^\sharp = (BS)^\sharp B$.

By Lemma 3 and computations, we also can verify Y_1, Y_2, Y_3 and Y_4 are the solutions of (10)-(13). This shows that there exist matrices $\overline{X}$ and $\overline{Y}$ over R such that $M = M^2\overline{X} = \overline{Y}M^2$ hold. Hence, by Lemma 1, $M^\sharp$ exists.

(*ii*) By Lemma 1 and Lemma 3, the expression of $M^\sharp$ can be obtained from $M^\sharp = \overline{Y}M\overline{X}$. □

Example for Theorem 3.1: Let $R = \mathbb{Z}/(6)$, $M = \begin{pmatrix} AX+YB & A \\ B & 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 4 \\ 4 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}.$$

$$AX = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \qquad YB = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}.$$

It is easy to verity $AX = XA$, $R(A) \subset R(AX)$ and $R_r(A) \subset R_r(XA)$. By a direct computation, we know that $(XAA^\sharp)^\sharp$ and $(SB)^\sharp$ exist.

Further, we have

$$A^\sharp = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \; A^\pi = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}, \; (XAA^\sharp)^\sharp = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, \; SB = \begin{pmatrix} 4 & -4 \\ 2 & 0 \end{pmatrix}.$$

$$(SB)^\sharp = \begin{pmatrix} 0 & 2 \\ -4 & 2 \end{pmatrix}, \quad (SB)^\pi = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}.$$

Clearly, $R(B) = R(BSB)$ and $R_r(B) = R_r(BSB)$. By Theorem 1, $M^\sharp$ exists and

$$M^\sharp = \begin{pmatrix} -4 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 \\ 4 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \end{pmatrix}.$$

The following corollary follow Theorem 1.

Corollary 3.2. [23, *Theorem*1], *Let* $M = \begin{pmatrix} AX+YB & A \\ B & 0 \end{pmatrix}$, *where* $A, B, X, Y \in K^{n\times n}$, $A^\sharp$ *exists,* $XA = AX$, $r(A) = r(AX)$, *then*

(i) $M^\sharp$ *exists if and only if* $r(B) = r(BSB)$, *where* $S = A^\pi Y - (XAA^\sharp)^\sharp$;

(ii) If $M^\sharp$ *exists, then the representation of* $M^\sharp$ *is the same as in Theorem 1.*

Proof. When $R = K$, it is easy to see that

$$r(A) = r(AX) \Longleftrightarrow R(A) \subset R(AX) \text{ and } R_r(A) \subset R_r(XA);$$

$$r(B) = r(BSB) \Longleftrightarrow R(B) \subset R(BSB) \text{ and } R_r(B) \subset R_r(BSB).$$

Whence the corollary is easily proved. □

Next, we consider the generalizations of some results in [13, 14, 19, 20].

Theorem 3.3. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in R^{(n+m)\times(n+m)}$, *where* $A \in R^{n\times n}$, $A^\sharp$ *exists, and* $A^\pi B = 0$, *Let* $S = D - CA^\sharp B$. *If* $S^\sharp$ *exists, then*

(1) $M^\sharp$ *exists if and only if* $P = A^2 + BS^\pi C$ *is regular,* $P^{\pi r}A = AP^{\pi l} = 0$ *and* $S^\pi CA^\pi = 0$, *for some* $P^{(1)} \in P\{1\}$;

(2) If $M^\sharp$ *exists, then* $M^\sharp = M_1M_2$, *where*

$$M_1 = \begin{pmatrix} AP^{(1)}(BS^\sharp CA^\sharp + I)AP^{(1)} & -AP^{(1)}B(S^\sharp)^2 \\ S^\pi CP^{(1)}(I + BS^\sharp CA^\sharp)AP^{(1)} & \\ -S^\sharp CA^\sharp AP^{(1)} & (S^\sharp)^2 - S^\pi CP^{(1)}B(S^\sharp)^2 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} A - BS^\sharp CA^\pi & BS^\pi \\ CA^\pi & S \end{pmatrix}.$$

Proof. (1): The "Only if " part.

Since $M^\sharp$ exists, we have that Lemma 1, there exist matrices X and Y over R such that $M = M^2X = YM^2$. By computations, $A^\pi B = 0$, and $P = A^2 + BS^\pi C$, we have

$$M = \begin{pmatrix} A & 0 \\ S^\pi C & S \end{pmatrix} \triangle_2 = \triangle_1 M_2, \qquad M^2 = \triangle_1 \begin{pmatrix} P & 0 \\ 0 & S^2 \end{pmatrix} \triangle_2,$$

where $\triangle_1$, $\triangle_2$ are the following invertible matrices,

$$\triangle_1 = \begin{pmatrix} I & BS^\sharp \\ CA^\sharp & I + CA^\sharp BS^\sharp \end{pmatrix}, \quad \triangle_2 = \begin{pmatrix} I & A^\sharp B \\ S^\sharp C & I + S^\sharp CA^\sharp B \end{pmatrix},$$

and

$$\triangle_1^{-1} = \begin{pmatrix} I + BS^\sharp CA^\sharp & -BS^\sharp \\ -CA^\sharp & I \end{pmatrix}, \quad \triangle_2^{-1} = \begin{pmatrix} I + A^\sharp BS^\sharp C & -A^\sharp B \\ -S^\sharp C & I \end{pmatrix}.$$

From $M^\sharp$ exists, we have M^2 is group invertible, so is also regular. By $M^2 = \triangle_1 diag(P, S^2)\triangle_2$, it is easy to see that P is regular. Let

$$X = \triangle_2^{-1} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \triangle_1^{-1},$$

by the equations $M = M^2X$ and $M = YM^2$, we have

$$\begin{pmatrix} P & 0 \\ 0 & S^2 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} A - BS^\sharp CA^\pi & BS^\pi \\ CA^\pi & S \end{pmatrix}$$

and

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & S^2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ S^\pi C & S \end{pmatrix}.$$

From above the two equations, we have

$$PX_1 = A - BS^\sharp CA^\pi, \tag{17}$$

$$PX_2 = BS^\pi, \tag{18}$$

$$S^2X_3 = CA^\pi, \tag{19}$$

$$S^2X_4 = S, \tag{20}$$

$$Y_1P = A, \tag{21}$$

$$Y_2S^2 = 0, \tag{22}$$

$$Y_3P = S^\pi C, \tag{23}$$

$$Y_4S^2 = S. \tag{24}$$

From (17), we get $PX_1A^\sharp A = A$, so $PP^{(1)}A = A$, i.e., $P^{\pi r}A = 0$. By (21), we have $AP^{(1)}P = A$, i.e., $AP^{\pi l} = 0$. Using (19), we get $S^\pi CA^\pi = 0$.

The "if " part.

Let

$$X_1 = P^{(1)}A(I - A^\sharp BS^\sharp CA^\pi),\ X_2 = P^{(1)}BS^\pi,\ X_3 = (S^\sharp)^2CA^\pi,\ X_4 = S^\sharp$$

and

$$Y_1 = AP^{(1)},\ Y_2 = 0,\ Y_3 = S^\pi CA^\sharp AP^{(1)},\ Y_4 = S^\sharp.$$

Note that, by $A^\pi B = 0$, $P^{\pi r}A = AP^{\pi l} = 0$ and $S^\pi CA^\pi = 0$, it is easy to verify X_1, X_2, X_3, X_4 and Y_1, Y_2, Y_3, Y_4 satisfy the Eqs.(17) – (20) and (21) – (24), respectively. That implies $M = M^2X = YM^2$ have solution X, Y, so $M^\sharp$ exists.

(2) By Lemma 1, we can compute that

$$\begin{aligned} M^\sharp &= Y^2M = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \triangle_1^{-1} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \triangle_1^{-1} \triangle_1 M_2 \\ &= \begin{pmatrix} AP^{(1)} & 0 \\ S^\pi CA^\sharp AP^{(1)} & S^\sharp \end{pmatrix} \begin{pmatrix} I + BS^\sharp CA^\sharp & -BS^\sharp \\ -CA^\sharp & I \end{pmatrix} \begin{pmatrix} AP^{(1)} & 0 \\ S^\pi CA^\sharp AP^{(1)} & S^\sharp \end{pmatrix} \begin{pmatrix} A - BS^\sharp CA^\pi & BS^\pi \\ CA^\pi & S \end{pmatrix} \\ &= M_1 M_2. \end{aligned}$$

□

Example for Theorem 3.3: Let $\mathbb{Z}$ be the integer ring, and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix over $\mathbb{Z}/(6\mathbb{Z})$, where

$$A = \begin{pmatrix} 2 & 0 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 4 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 2 & 0 \\ 4 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}.$$

Then

$$A^\sharp = \begin{pmatrix} 2 & 0 \\ 2 & 4 \end{pmatrix}, \quad A^\pi = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad A^\pi B = 0, \quad S = D - CA^\sharp B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$S^\sharp = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S^\pi = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = A^2 + BS^\pi C = \begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix}.$$

Let $P^{(1)} = \begin{pmatrix} 4 & 0 \\ -4 & 4 \end{pmatrix}$. Therefore $P^{\pi r}A = AP^{\pi l} = 0$, $S^\pi CA^\pi = 0$. By Theorem 2, we have

$$M^\sharp = \begin{pmatrix} -2 & 2 & -2 & 2 & 2 \\ -2 & 2 & -2 & 2 & 0 \\ -2 & 2 & 2 & 2 & 0 \\ -1 & -2 & 2 & 1 & -2 \\ 0 & -1 & 0 & 2 & 1 \end{pmatrix}.$$

By computations, from Theorem 2, we can obtain the following corollaries.

Corollary 3.4. [13, *Theorem*3.1][14, *Corollary*3.1], $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in R^{(n+m)\times(n+m)}$, *where* $A \in R^{n\times n}$ *is invertible and* $S = D - CA^{-1}B$ *is group invertible, then*

(i) $M^\sharp$ *exists if and only if* $P = A^2 + BS^\pi C$ *is invertible;*

(ii) If $M^\sharp$ *exists, then* $M^\sharp = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, *where*

$$M_1 = AP^{-1}(A + BS^\sharp C)P^{-1}A,$$
$$M_2 = AP^{-1}(A + BS^\sharp C)P^{-1}BS^\pi - AP^{-1}BS^\sharp,$$
$$M_3 = S^\pi CP^{-1}(A + BS^\sharp C)P^{-1}A - S^\sharp CP^{-1}A,$$
$$M_4 = S^\pi CP^{-1}(A + BS^\sharp C)P^{-1}BS^\pi - S^\sharp CP^{-1}BS^\pi - S^\pi CP^{-1}BS^\sharp + S^\sharp.$$

Proof. When A is invertible, we have $A^\pi = P^{\pi r} = P^{\pi l} = 0$. Further, $PP^{(1)} = I = P^{(1)}P$, this implies P is invertible and $P^{(1)} = P^{-1}$. Whence the corollary is easily proved. □

Corollary 3.5. [19, *Theorem*12], $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in R^{(n+m)\times(n+m)}$, *where* $A \in R^{n\times n}$ *and* $S = D - CA^{\sharp}B$ *are group invertible,* $A^{\pi}B = 0$ *and* $S^{\pi}C = 0$ *,then* $M^{\sharp}$ *exists and*

$$M^{\sharp} = \begin{pmatrix} A^{\sharp} + A^{\sharp}BS^{\sharp}CA^{\sharp} & -A^{\sharp}BS^{\sharp} \\ -S^{\sharp}CA^{\sharp} & S^{\sharp} \end{pmatrix} \begin{pmatrix} I - A^{\sharp}BS^{\sharp}CA^{\pi} & A^{\sharp}BS^{\pi} \\ S^{\sharp}CA^{\pi} & I \end{pmatrix}.$$

Proof. Note that $S^{\pi}C = 0$, $P = A^2$ and $P^{(1)} = (A^{\sharp})^2$, the proof immediately follows from Theorem 2. □

Corollary 3.6. [20, *Theorem*5], $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in R^{(n+m)\times(n+m)}$, *where* $A \in R^{n\times n}$ *and* $S = D - CA^{\sharp}B$ *are group invertible,* $A^{\pi}B = 0$, $CA^{\pi} = 0$ *and* $S^{\pi}C = 0$, *then* $M^{\sharp}$ *exists and*

$$M^{\sharp} = \begin{pmatrix} A^{\sharp} + A^{\sharp}BS^{\sharp}CA^{\sharp} & A^{\sharp}(I + BS^{\sharp}CA^{\sharp})A^{\sharp}BS^{\pi} - A^{\sharp}BS^{\sharp} \\ -S^{\sharp}CA^{\sharp} & S^{\sharp}(I - C(A^{\sharp})^2BS^{\pi}) \end{pmatrix}.$$

Proof. Note that $CA^{\pi} = 0$, the proof immediately follows from Corollary 3. □

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