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Stability Results for Fixed Point Sets of $\alpha_* - \psi$ Contractive Multivalued Mappings

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Abstract. In this paper, we established a stability result for fixed point sets associated with a sequence of multivalued mappings which belong to class of functions obtained by a multivalued extension of certain generalized contraction mapping. Certain other aspects of these mappings are already studied in the existing literatures. We also construct an illustrative example.

1. Introduction and Preliminaries

The concept of stability is associated with the investigation of limiting behaviors. It is not a single notion. Several concepts of stability appear corresponding to the various situations arising in the studies of both continuous and discrete dynamical systems[15, 17]. Our purpose in this paper is to establish a stability result for fixed point sets associated with a sequence of uniformly convergent multivalued mappings. Such a sequence of fixed point sets is said to be stable when it converges to the corresponding fixed point set of the limiting function. This convergence is understood with respect to the Hausdorff metric.

When a fixed point for a mapping exists, it need not be unique. In this sense the fixed point sets are naturally associated with mappings and their study falls in the domain of multivalued analysis. Also the multivalued mappings often have more fixed points. As an instance, we can mention the case of Nadler's theorem[13, 14] which is the setvalued extension of the Banach contraction mapping principle. Unlike the Banach's result, the fixed point of Nadler's contraction is not unique. The consideration of multivalued mappings provide us normally with a larger fixed point sets which sometimes have very interesting structures. Stability result of fixed point sets for multivalued mapping have appeared in a large number of papers[3, 9, 10, 12, 16]. Such stability was also discussed in the paper of Nadler[13, 14]. More recent references are[4–7]. It may be mentioned that there are other interesting studies related to the limits of sequence of mappings, as, for instance, the preservance of chaotic properties in the limit under uniform convergence has been discussed in [2]. In this paper we consider $\alpha_* - \psi$ contractive multivalued mappings which are defined by Asl et al[1] as a multivalued extensions of a generalized contraction known as $\alpha - \psi$

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contraction [8] . There is a good number of works on $\alpha - \psi$ contractions and its generalizations[4, 6]. We show that a uniformly convergent sequence of $\alpha_* - \psi$ multivalued contractions has stable fixed point sets. The result is supported with an example.

Let (X, d) be a metric space and $CL(X)$ be the family of all nonempty closed subsets of X. The Hausdorff metric H induced by d is defined by

$$H(A, B) = \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in B} d(x, B)\right\},$$

where $A, B \in CL(X)$ and $d(x, B) = \inf_{y\in B} d(x, y)$.

Note that H is a metric on $CB(X)$ (the family of all closed and bounded subsets of X). On $CL(X)$, H satisfies all the properties of the metric except that $H(A, B)$ can be infinite if either A or B is unbounded.

Let $T : X \to CL(X)$ be a multivalued mapping. A point $z \in X$ is a fixed point of T if $z \in Tz$.

Definition 1.1. *[1]. Let (X, d) be a metric space and $T : X \to CL(X)$ a mapping. The mapping T is called an $\alpha_* - \psi$ contractive multivalued mapping if for all $x, y \in X$*

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y)). \tag{1.1}$$

where

1. *$\alpha_* : 2^X \times 2^X \to [0, \infty)$ be any function defined as $\alpha_*(A, B) = \inf\{\alpha(x, y) : x \in A$ and $y \in B\}$;*
   Therefore, $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$.
   and
2. *$\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function with $\sum \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$, in which $\alpha : X \times X \to [0, \infty)$ is any function.*

Definition 1.2. *[1]. Let (X, d) be a metric space and $T : X \to CL(X)$ be a mapping. The mapping T is called an α_*-admissible if $\alpha(x, y) \geq 1 \Longrightarrow \alpha_*(Tx, Ty) \geq 1$. Where $\alpha : X \times X \to [0, \infty)$ be any function and α_* is defined above.*

Recently, Asl et al.[1] obtained the following theorem.

Theorem 1.1. *Let (X, d) be a metric space and $T : X \to CL(X)$ α_*-admissible and $\alpha_* - \psi$ contractive multivalued mapping. Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.*

2. Main Results

We begin with the following lemma.

Lemma 2.1. *Let X be a metric space and $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x_n, y_n) \geq 1 \Rightarrow \alpha(a, b) \geq 1, \text{ whenever } x_n \to a \text{ and } y_n \to b \text{ as } n \to \infty. \tag{2.1}$$

Suppose $\{T_n\}$ is a sequence of $\alpha_* - \psi$ contractive multivalued mapping on X which are α_*-admissible with the same α and ψ. If $T_n \to T$ as $n \to \infty$ uniformly then the limit mapping T is α_*-admissible where α and ψ are the same as for the sequence $\{T_n\}$.

Proof. Let $\alpha(x, y) \geq 1$, for some $x,\ y \in X$. Suppose $a \in Tx$ and $b \in Ty$ be arbitrary. Since $T_n \to T$ uniformly, there exist two sequences $\{x_n\}$ in $\{T_n x\}$ and $\{y_n\}$ in $\{T_n y\}$ such that $x_n \to a$ and $y_n \to b$ as $n \to \infty$.

Since $\alpha(x, y) \geq 1$ and each T_n is α_*-admissible, it follows from Definition 1.1 that

$$\alpha_*(T_n x, T_n y) \geq 1.$$

Hence $\alpha(x_n, y_n) \geq 1$ for all $n \in \mathbb{N}$ by (2.1), $\alpha(a, b) \geq 1$. Thus we have,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(a, b) \geq 1 \text{ for all } a \in Tx \text{ and for all } b \in Ty.$$

Hence, $\alpha(x, y) \geq 1$ implies that $\alpha_*(Tx, Ty) \geq 1$. Hence the limit mapping T is α_*-admissible. □

Now onwards, $\psi : [0, \infty) \to [0, \infty)$ is a strictly increasing mapping, with the additional condition that $\Phi(t) = \sum \psi^n(t) < \infty$ with $\Phi(t) \to 0$ as $t \to 0$.

Theorem 2.1. *Let X be a complete metric space and $T_i : X \to CB(X),\ i = 1, 2$, be $\alpha_* - \psi$ contractive multivalued mapping and α_*-admissible with the same α and ψ. Suppose that the following conditions hold:*

(i) *For any $x \in F(T_1)$, we have $\alpha(x, y) \geq 1$ whenever $y \in T_2 x$, and for any $x \in F(T_2)$, we have $\alpha(x, y) \geq 1$ whenever $y \in T_1 x$;*

(ii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$, where $x_{n+1} \in T_i x_n,\ i = 1, 2$.*

Then $H(F(T_1), F(T_2)) \leq \Phi(\psi(k))$ where $k = \sup_{x \in X} H(T_1 x, T_2 x)$.

Proof. By Theorem 1.1, $F(T_1)$ and $F(T_2)$ are nonempty. Let $q > 1$ be any number. Pick $x_0 \in F(T_1)$. We choose $x_1 \in T_2 x_0$ such that $d(x_0, x_1) \leq qk$. Since T_2 is α_*-admissible $\alpha(x_0, x_1) \geq 1$ implies that $\alpha_*(T_2 x_0, T_2 x_1) \geq 1$. Let $q_0 > 1$ be any number, choose $x_2 \in T_2 x_1$ such that

$$d(x_1, x_2) \leq q_0 H(T_2 x_0, T_2 x_1) \leq q_0 \alpha_*(T_2 x_0, T_2 x_1) H(T_2 x_0, T_2 x_1) \leq q_0 \psi(d(x_0, x_1)) \leq q_0 \psi(qk).$$

Since $\alpha_*(T_2 x_0, T_2 x_1) \geq 1$, therefore $\alpha(x_1, x_2) \geq 1$ and α_*-admissibility of the mapping T_2 implies that $\alpha_*(T_2 x_1, T_2 x_2) \geq 1$.

Since ψ is strictly increasing function, we have $\psi(d(x_1, x_2)) < \psi(q_0 \psi(qk))$. Set $q_1 = \dfrac{\psi(q_0 \psi(qk))}{\psi(d(x_1, x_2))}$. For $x_2 \in T_2 x_1$, we choose $x_3 \in T_2 x_2$ such that

$$d(x_2, x_3) \leq q_1 H(T_2 x_1, T_2 x_2) \leq q_1 \alpha_*(T_2 x_1, T_2 x_2) H(T_2 x_1, T_2 x_2) \leq q_1 \psi(d(x_1, x_2)) \leq \psi(q_0 \psi(qk)).$$

Now $\alpha_*(T_2 x_1, T_2 x_2) \geq 1$, therefore $\alpha(x_2, x_3) \geq 1$. Again α_*-admissibility of T_2 is implies that $\alpha_*(T_2 x_2, T_2 x_3) \geq 1$. Again, since ψ is strictly increasing function, we get

$$\psi(d(x_2, x_3)) < \psi^2(q_0 \psi(qk)).$$

Set $q_2 = \dfrac{\psi^2(q_0 \psi(qk))}{\psi(d(x_1, x_2))}$. Now, for $x_3 \in T_2 x_2$, we choose $x_4 \in T_2 x_3$ such that

$$d(x_3, x_4) \leq q_2 H(T_2 x_2, T_2 x_3) \leq q_2 \alpha_*(T_2 x_2, T_2 x_3) H(T_2 x_2, T_2 x_3) \leq q_2 \psi(d(x_2, x_3)) \leq \psi^2(q_0 \psi(qk)).$$

Continuing in this manner we construct a sequence $\{x_n\}$ such that

$$d(x_n, x_{n+1}) \leq \psi^{n-1}(q_0 \psi(qk)),$$

where $x_{n+1} \in T_2 x_n$ and $\alpha_*(T_2 x_n, T_2 x_{n+1}) \geq 1$.

Let $m > n > 1$. By the triangle inequality

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(q_0\psi(qk)) < \infty,$$

and $\{x_n\}$ is a Cauchy sequence in X. Since X is complete $x_n \to z \in X$ for some $z \in X$. Since $\alpha(x_n, y_n) \geq 1$ and $x_n \to z$ by the hypothesis $\alpha(x_n, z) \geq 1$. Thus $\alpha_*(T_2x_n, T_2z) \geq 1$. Now,

$$d(x_{n+1}, T_2z) \leq \alpha_*(T_2x_n, z)H(T_2x_n, T_2z) \leq \psi(d(x_n, z)).$$

Making $n \to \infty$, we get $d(z, T_2z) \leq \psi(0)$. By the definition of ψ we have $\psi(0) = 0$. Hence $z \in F(T_2)$.

Again, by the triangle inequality

$$\begin{aligned} d(x_0, z) &\leq \sum_{i=0}^{n} d(x_i, x_{i+1}) + d(x_{n+1}, z) \\ &\leq \sum_{i=0}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=0}^{\infty} \psi^{i-1}(q_0\psi(qk)) \\ &\leq \sum_{i=0}^{\infty} \psi^{n-1}(q_0\psi(qk)) = \Phi(q_0\psi(qk)). \end{aligned}$$

Thus, given arbitrary $x_0 \in F(T_1)$, we can find $z \in F(T_2)$ for which

$$d(x_0, z) \leq \Phi(q_0\psi(qk)).$$

Reversing the roles of T_1 and T_2, we conclude that for each $y_0 \in F(T_2)$, there exists $y_1 \in T_1y_0$ and $w \in F(T_1)$ such that, $d(y_0, w) \leq \Phi(q_0\psi(qk))$. Hence

$$H(F(T_1), F(T_2)) \leq \Phi(q_0\psi(qk)).$$

Letting $q_0 \to 1,\ q_1 \to 1$ we get the required result. □

Now we present our stability result.

Theorem 2.2. *Let X be a complete metric space. Let $\{T_n\}$ be a sequence of $\alpha_* - \psi$ contractive multivalued mappings, uniformly convergent to a $\alpha_* - \psi$ contractive multivalued mappings T. Suppose that the following hold:*

(i) $\alpha(x_n, y_n) \geq 1 \Rightarrow \alpha(x, y) \geq 1$, *whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$;*

(ii) *For all $n \geq 1$, for any $x \in F(T_n)$, we have $\alpha(x, y) > 1$ whenever $y \in Tx$ and for any $x \in F(T)$, we have $\alpha(x, y) > 1$ whenever $y \in T_nx$.*

Then

$$\lim_{n\to\infty} H(F(T_n), F(T)) = 0,$$

that is, the fixed point sets of T_n are stable.

Proof. By Lemma 2.1, T is α_*-admissible. Let $k_n = \sup_{x\in X} H(T_nx, Tx)$. Since $\{T_n\}$ converges to T uniformly on

X,

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \sup_{x\in X} H(T_n x, Tx) = 0.$$

Now, from Theorem 2.1, we get

$$H(F(T_n), F(T)) \leq \Phi(\psi(k_n))$$

Since $\psi(t)$ and $\Phi(t) \to 0 \quad$ as $t \to 0$, we have.

$$\lim_{n\to\infty} H(F(T_n), F(T)) \leq \lim_{n\to\infty} \Phi(\psi(k_n)) = 0.$$

This proves the theorem. □

Example 2.1. *Let $X = \mathbb{R}$. $d(x, y) = |x - y|$. Define $T_n : \mathbb{R} \to CL(\mathbb{R})$ by*

$$T_n x = \begin{cases} \{1 + \frac{1}{n}, \frac{1}{4x} + \frac{1}{n}\}, & \text{if } x > 1; \\ \{\frac{1}{n}, \frac{1}{n} + \frac{x}{16}\}, & \text{if } 0 < x \leq 1; \\ \{0\}, & \text{if } x = 0; \\ \{2, 3\}, & \text{otherwise.} \end{cases}$$

Let the mapping $\alpha : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be given by

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x, y \in (0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

By the definition of α_* we said that each T_n is α_*−admissible. $T_n \to T$ as $n \to \infty$. The T is given by

$$Tx = \begin{cases} \{1, \frac{1}{4x}\}, & \text{if } x > 1; \\ \{0, \frac{x}{16}\}, & \text{if } 0 < x \leq 1; \\ \{2, 3\}, & \text{otherwise.} \end{cases}$$

T is α_*−admissible. We define $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{1}{2} t.$$

Each T_n is $\alpha_* - \psi$ contraction, and T is also $\alpha_* - \psi$ contraction. Let $x, y \in (0, 1]$;

$$\begin{aligned} H(T_n x, T_n y) &= \max\{\sup_{x\in Tx} d(x, Ty), \sup_{y\in Ty} d(y, Tx)\} \\ &= \max\{\inf\{|\frac{x}{16}|, |\frac{x}{16} - \frac{y}{16}|\}, \inf\{|\frac{y}{16}|, |\frac{y}{16} - \frac{x}{16}|\}\} \\ &= |\frac{x}{16} - \frac{y}{16}|. \end{aligned}$$

Therefore $\alpha_*(x, y) H(T_n x, T_n y) \leq \psi(d(x, y))$.

$F(T_1) = \{0, 1\}$ and $F(T_n) = \{0\}$ for $n \geq 2$. $F(T) = \{0\}$. Hence

$$H(F(T_n), F(T)) \to 0 \quad \text{as } n \to \infty.$$

3. Conclusion

We obtain the result here under the assumption of uniform convergence. The proof of the theorem necessarily uses this concept. It remains to be seen whether the requirement of uniform convergence can be relaxed. This can be treated as an open problem.

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