Filomat 31:11 (2017), 3443–3452
https://doi.org/10.2298/FIL1711443O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Behavior of Meromorphic Functions at the Boundary of the Unit Disc

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Abstract. In this paper, a boundary version of the Schwarz lemma for meromorphic functions is investigated. The modulus of the angular derivative of the meromorphic function $I^n f(z) = \frac{1}{z} + 2^n c_0 + 3^n c_1 z + 4^n c_2 z^2 + ...$ that belongs to the class of $\mathcal{M}$ on the boundary point of the unit disc has been estimated from below.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

First, let us set the following standard notations: $\mathbb{C}$ denotes the complex numbers, $\mathbb{E} := \{z \in \mathbb{C} : \ |z| < 1\}$ is the unit disc and $\mathbb{T} := \partial\mathbb{E} = \{z \in \mathbb{C} : \ |z| = 1\}$ is the unit circle. Now, let us start by recalling the classical form of the Schwarz lemma ([6], p.329).

Lemma 1.1 (Schwarz lemma). *Let $f : \mathbb{E} \to \mathbb{E}$ be holomorphic function and $f(0) = 0$. Then $\left|f(z)\right| \leq |z|$ for all z and $\left|f'(0)\right| \leq 1$, with equality in either case if and only if is a rotation.*

For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2], [20]). The basic tool in proving our results is the following lemma due to Jack ([2]).

Lemma 1.2 (Jack's lemma). *Let $f(z)$ be holomorphic function in the unit disc $\mathbb{E}$ with $f(0) = 0$. If $\left|f(z)\right|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then*

$$z_0 f'(z_0) = \ k f(z_0),$$

where $k \geq 1$ is a real number.

Let $\mathcal{A}$ denote the class of functions

$$f(z) = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + ...$$

2010 *Mathematics Subject Classification*. Primary 30C80; Secondary 32A10, 30D30

Keywords. Schwarz lemma on the boundary, Holomorphic function, Meromorphic function, Jack's lemma.
Received: 21 November 2016; Accepted: 21 February 2017
Communicated by Miodrag Mateljević

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that are holomorphic in the punctured disc $\mathbb{U} = \{z \in \mathbb{C} : \ 0 < |z| < 1\}$. Define

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = \frac{1}{z} + 2c_0 + 3c_1 z + 4c_2 z^2 + ... = \frac{\left(z^2 f(z)\right)'}{z},$$

$$I^2 f(z) = I^1\left(I^1 f(z)\right),$$

and for $n = 1, 2, 3, ...$

$$I^n f(z) = I^1\left(I^{n-1} f(z)\right) = \frac{1}{z} + 2^n c_0 + 3^n c_1 z + 4^n c_2 z^2 + ... = \frac{1}{z} + \sum_{k=2}^{\infty} k^n c_{k-2} z^{k-2}$$

and

$$I^{n+1} f(z) = \frac{1}{z} + 2^{n+1} c_0 + 3^{n+1} c_1 z + 4^{n+1} c_2 z^2 + ... = \frac{1}{z} + \sum_{k=2}^{\infty} k^{n+1} c_{k-2} z^{k-2}. \tag{1.1}$$

Also, $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\Re\left(-z^2\left(I^{n+1} f(z)\right)'\right) > 0, \ \ |z| < 1, \tag{1.2}$$

where $I^{n+1} f(z)$ is defined by (1.1).

Let $f(z) \in \mathcal{M}$ and consider the function

$$\phi(z) = \frac{1 + z^2\left(I^n f(z)\right)'}{1 - z^2\left(I^n f(z)\right)'}. \tag{1.3}$$

The function $\phi(z)$ is holomorphic in $\mathbb{E}$ and $\phi(0) = 0$. That is,

$$\phi(z) = \frac{z^2}{2}\left(3^n c_1 + 2.4^n c_2 z + ...\right).$$

Differentiating (1.3) we obtain

$$z^2\left(I^n f(z)\right)'' + 2z\left(I^n f(z)\right)' = \frac{2\phi'(z)}{\left(1 + \phi(z)\right)^2}.$$

We can easily confirm the identity

$$z\left(I^n f(z)\right)' = I^{n+1} f(z) - 2I^n f(z). \tag{1.4}$$

Differentiating (1.4) we take

$$z^2\left(I^n f(z)\right)'' = z\left(I^{n+1} f(z)\right)' - 3z\left(I^n f(z)\right)'.$$

Therefore, we have

$$-z^2\left(I^{n+1} f(z)\right)' = -z^2\left(I^n f(z)\right)' - \frac{2z\phi'(z)}{\left(1 + \phi(z)\right)^2}.$$

Now, let us show that the function $\left|\phi(z)\right| < 1$ in $\mathbb{E}$. If there exists a point $z_0 \in \mathbb{E}$ such that

$$\max_{|z| \le |z_0|}\left|\phi(z)\right| = \left|\phi(z_0)\right| = 1,$$

then Jack's lemma gives us that $\phi(z_0) = e^{i\theta}$ and $z_0\phi'(z_0) = k\phi(z_0)$. Thus, we have

$$\begin{aligned}
-z_0^2\left(I^{n+1}f(z_0)\right)' &= -z_0^2\left(I^n f(z_0)\right)' - \frac{2z_0\phi'(z_0)}{\left(1+\phi(z_0)\right)^2} \\
&= \frac{1-\phi(z_0)}{1+\phi(z_0)} - \frac{2k\phi(z_0)}{\left(1+\phi(z_0)\right)^2} \\
&= \frac{1-e^{i\theta}}{1+e^{i\theta}} - \frac{2ke^{i\theta}}{(1+e^{i\theta})^2}.
\end{aligned}$$

and

$$\Re\left(-z_0^2\left(I^{n+1}f(z_0)\right)'\right) = \Re\left(\frac{1-e^{i\theta}}{1+e^{i\theta}} - \frac{2ke^{i\theta}}{(1+e^{i\theta})^2}\right).$$

Therefore, we obtain

$$\begin{aligned}
\Re\left(-z_0^2\left(I^{n+1}f(z_0)\right)'\right) &= \Re\left(\frac{1-(\cos\theta+i\sin\theta)}{1+\cos\theta+i\sin\theta} - \frac{2ke^{i\theta}}{(1+e^{i\theta})^2}\right) \\
&= \Re\left(\frac{1-(\cos\theta+i\sin\theta)}{1+\cos\theta+i\sin\theta} - 2k\frac{1}{2+e^{i\theta}+e^{-i\theta}}\right) \\
&= \Re\left(\frac{1-(\cos\theta+i\sin\theta)}{1+\cos\theta+i\sin\theta} - \frac{2k}{2\,(1+\cos\theta)}\right) \\
&= -k\Re\left(\frac{1}{1+\cos\theta}\right) \le 0.
\end{aligned}$$

This contradict (1.2). Thus, there is no point $z_0 \in \mathbb{E}$ such that $\left|\phi(z_0)\right| = 1$ for all $z \in \mathbb{E}$. Consequently, we conclude that $\left|\phi(z)\right| < 1$ for $|z| < 1$. Thus, by the Schwarz lemma, we obtain

$$|c_1| \le \frac{2}{3^n}. \tag{1.5}$$

Moreover, the equality in (1.5) occurs for the solution of equation

$$\left(I^n f(z)\right)' = \frac{z^2-1}{z^2(1+z^2)},$$

with the condition at $z = 0$

$$\lim_{z\to 0} z^2 f(z) = 0.$$

In particular, for $n = 1$, we have

$$\left(I^1 f(z)\right)' = \frac{z^2-1}{z^2(1+z^2)}, \tag{1.6}$$

with the condition at $z = 0$

$$\lim_{z\to 0} z^2 f(z) = 0.$$

Thus, from (1.6), we obtain

$$f(z) = \frac{1}{z} + \frac{2}{3}z - \frac{2}{15}z^3 + \frac{2}{35}z^5 - \ldots$$

The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [21] and then rediscovered and partially improved by Osserman in 2000 [17].

Lemma 1.3. *Let $f(z) = c_p z^p + \ldots$ be a holomorphic function self-mapping of $\mathbb{E}$, that is $\left|f(z)\right| < 1$ for all $z \in \mathbb{E}$. Assume that there is a $b \in \mathbb{T}$ so that f extend continuously to b, $\left|f(b)\right| = 1$ and $f'(b)$ exists. Then*

$$\left|f'(b)\right| \geq p + \frac{1 - \left|c_p\right|}{1 + \left|c_p\right|}. \tag{1.7}$$

The equality in (1.7) *holds if and only if f is of the form*

$$f(z) = -z^p \frac{\gamma - z}{1 - \gamma z}, \ \ \forall z \in E,$$

for some constant $\gamma \in (-1, 0]$.

Corollary 1.4. *Under the hypotheses of Lemma 1.3, we have*

$$\left|f'(b)\right| \geq p, \tag{1.8}$$

with equality only if f is of the form

$$f(z) = z^p e^{i\theta},$$

where θ is a real number.

The following Lemma 1.5 and Corollary 1.6, known as the Julia-Wolff lemma, is needed in the sequel [19].

Lemma 1.5 (Julia-Wolff lemma). *Let f be a holomorphic function in $\mathbb{E}$, $f(0) = 0$ and $f(\mathbb{E}) \subset \mathbb{E}$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \mathbb{T}$, $\left|f(b)\right| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq \left|f'(b)\right| \leq \infty$.*

Corollary 1.6. *The holomorphic function f has a finite angular derivative $f'(b)$ if and only if f' has the finite angular limit $f'(b)$ at $b \in \mathbb{T}$.*

Inequality (1.8) and its generalizations have important applications in geometric theory of functions (see, e.g., [6], [19]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [2], [4], [5], [10], [11], [17], [18], [20] and references therein).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + \ldots$, with a zero set $\{z_k\}$ (see [4]).

S. G. Krantz and D. M. Burns [9] and D. Chelst [3] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([12], [13], [14], [15] and [16]).

Also, M. Jeong [8] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [7] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section, the modulus of the angular derivative of the meromorphic function $I^n f(z) = \frac{1}{z} + 2^n c_0 + 3^n c_1 z + 4^n c_2 z^2 + \ldots$ that belongs to the class of $\mathcal{M}$ on the boundary point of the unit disc has been estimated from below.

Theorem 2.1. *Let $f(z) \in \mathcal{M}$. Assume that, for some $b \in \mathbb{T}$, $(I^n f(z))'$ has angular limit $(I^n f(z))'_{z=b}$ at b and $(I^n f(z))'_{z=b} = 0$. Then we have the inequality*

$$\left|(I^n f(z))''_{z=b}\right| \geq 1. \tag{2.1}$$

The equality in (2.1) *occurs for the solution of equation*

$$(I^n f(z))' = \frac{z^2-1}{z^2(1+z^2)},$$

with the condition at $z = 0$

$$\lim_{z\to 0} z^2 f(z) = 0.$$

Proof. Consider the function

$$\phi(z) = \frac{1 + z^2\,(I^n f(z))'}{1 - z^2\,(I^n f(z))'}.$$

$\phi(z)$ is a holomorphic function in the unit disc E and $\phi(0) = 0$. From the Jack's lemma and since $f(z) \in \mathcal{M}$, we have $\left|\phi(z)\right| < 1$ for $|z| < 1$. Also, we have $\left|\phi(b)\right| = 1$ for $b \in \mathbb{T}$.

For $p = 2$, from (1.8), we obtain

$$\begin{aligned} 2 \quad \leq \quad \left|\phi'(b)\right| = &\left| \frac{\left(2b\,(I^n f(z))'_{z=b} + b^2\,(I^n f(z))''_{z=b}\right)\left(1 - b^2\,(I^n f(z))'_{z=b}\right)}{\left(1 - b^2\,(I^n f(z))'_{z=b}\right)^2} \right. \\ &\left. + \frac{\left(2b\,(I^n f(z))'_{z=b} + b^2\,(I^n f(z))''_{z=b}\right)\left(1 + b^2\,(I^n f(z))'_{z=b}\right)}{\left(1 - b^2\,(I^n f(z))'_{z=b}\right)^2} \right|, \end{aligned}$$

$$2 \leq \left|b^2\,(I^n f(z))''_{z=b} + b^2\,(I^n f(z))''_{z=b}\right| = 2\left|(I^n f(z))''_{z=b}\right|$$

and

$$\left|(I^n f(z))''_{z=b}\right| \geq 1.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$(I^n f(z))' = \frac{z^2-1}{z^2(1+z^2)}.$$

Then, we have

$$(I^n f(z))'' = \frac{2z\left(z^4+z^2\right) - \left(4z^3+2z\right)\left(z^2-1\right)}{(z^4+z^2)^2},$$

and

$$\left|(I^n f(z))''_{z=1}\right| = 1$$

□

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|(I^n f(z))''_{z=b}\right| \geq \frac{1}{2}\left(\frac{6 + 3^n\,|c_1|}{2 + 3^n\,|c_1|}\right). \tag{2.2}$$

The inequality (2.2) *is sharp with equality for the solution of equation*

$$(I^n f(z))' = \frac{z^3 + az^2 - az - 1}{z^5 + az^4 + az^3 + z^2},$$

with the condition at $z = 0$

$$\lim_{z\to 0} z^2 f(z) = 0,$$

where $a = \frac{3^n}{2}\,|c_1|$ *is an arbitrary number from* $[0, 1]$ (*see* (1.5)).

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. For $p = 2$, using the inequality (1.7) for the function $\phi(z)$, we obtain

$$2 + \frac{1-|a_2|}{1+|a_2|} \leq \left|\phi'(b)\right| = 2\left|(I^n f(z))''_{z=b}\right|$$

Since

$$|a_2| = \frac{\left|\phi''(0)\right|}{2} = \frac{3^n}{2}|c_1|,$$

then we have

$$2 + \frac{1-\frac{3^n}{2}|c_1|}{1+\frac{3^n}{2}|c_1|} \leq 2\left|(I^n f(z))''_{z=b}\right|,$$

$$2 + \frac{2-3^n|c_1|}{2+3^n|c_1|} \leq 2\left|(I^n f(z))''_{z=b}\right|$$

$$\left|(I^n f(z))''_{z=b}\right| \geq \frac{1}{2}\left(2 + \frac{2-3^n|c_1|}{2+3^n|c_1|}\right)$$

and

$$\left|(I^n f(z))''_{z=b}\right| \geq \frac{1}{2}\left(\frac{6+3^n|c_1|}{2+3^n|c_1|}\right).$$

To show that the inequality (2.2) is sharp, take the holomorphic function

$$\phi(z) = \frac{1+z^2(I^n f(z))'}{1-z^2(I^n f(z))'} = z^2\frac{z+a}{1+az}$$

Then

$$\phi'(z) = 2\frac{2z(I^n f(z))' + z^2(I^n f(z))''}{\left(1-z^2(I^n f(z))'\right)^2} = \frac{\left(3z^2+2az\right)(1+az) - a\left(z^3+az^2\right)}{(1+az)^2},$$

$$\phi'(1) = 2\frac{2(I^n f(z))'_{z=1} + (I^n f(z))''_{z=1}}{\left(1-(I^n f(z))'_{z=1}\right)^2} = \frac{3+a}{1+a}$$

and

$$\left|(I^n f(z))''_{z=1}\right| = \frac{1}{2}\left(\frac{3+a}{1+a}\right).$$

Since $a = \frac{3^n}{2}|c_1|$ is satisfied with equality. □

Theorem 2.3. *Let $f(z) \in \mathcal{M}$. Assume that, for some $b \in \mathbb{T}$, $(I^n f(z))'$ has angular limit $(I^n f(z))'_{z=b}$ at b and $(I^n f(z))'_{z=b} = 0$. Then we have the inequality*

$$\left|(I^n f(z))''_{z=b}\right| \geq 1 + \frac{(2-3^n|c_1|)^2}{4-3^{2n}|c_1|^2+4^{n+1}|c_2|}. \tag{2.3}$$

The inequality (2.3) *is sharp with equality for the solution of equation*

$$(I^n f(z))' = \frac{z^2-1}{z^2(1+z^2)},$$

with the condition at $z = 0$

$$\lim_{z\to 0} z^2 f(z) = 0.$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. By the maximum principle for each $z \in E$, we have $\left|\phi(z)\right| \leq \left|z^2\right|$. So,

$$\psi(z) = \frac{\phi(z)}{z^2}$$

is a holomorphic function in E and $\left|\psi(z)\right| < 1$ for $|z| < 1$.

In particular, we have

$$\left|\psi(0)\right| = \frac{3^n}{2}\left|c_1\right| \leq 1 \tag{2.4}$$

and

$$\left|\psi'(0)\right| = 4^n\left|c_2\right|.$$

Moreover, it can be seen that

$$\frac{b\phi'(b)}{\phi(b)} = \left|\phi'(b)\right| \geq \left|\left(b^2\right)'\right| = \frac{b\left(b^2\right)'}{b^2}.$$

The function

$$\begin{aligned}
\Upsilon(z) &= \frac{\psi(z)-\psi(0)}{1-\overline{\psi(0)}\psi(z)} = \frac{\frac{3^n}{2}c_1 + 4^n c_2 z + ... - \frac{3^n}{2}c_1}{1-\overline{\frac{3^n}{2}c_1}\left(\frac{3^n}{2}c_1 + 4^n c_2 z + ...\right)} \\
&= \frac{4^n c_2 z + ...}{1-\overline{\frac{3^n}{2}c_1}\left(\frac{3^n}{2}c_1 + 4^n c_2 z + ...\right)} = \frac{4^n c_2 z}{1-\frac{3^{2n}}{4}\left|c_1\right|^2} + ...
\end{aligned}$$

is a holomorphic in the unit disc E, $|\Upsilon(z)| < 1$ for $|z| < 1$, $\Upsilon(0) = 0$ and $|\Upsilon(b)| = 1$ for $b \in \mathbb{T}$.

From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1+|\Upsilon'(0)|} &\leq |\Upsilon'(b)| = \frac{1-\left|\psi(0)\right|^2}{\left|1-\overline{\psi(0)}\psi(b)\right|^2}\left|\psi'(b)\right| \leq \frac{1+\left|\psi(0)\right|}{1-\left|\psi(0)\right|}\left|\psi'(b)\right| \\
&= \frac{1+\left|\psi(0)\right|}{1-\left|\psi(0)\right|}\left\{\left|\phi'(b)\right| - 2\right\}.
\end{aligned}$$

Since

$$\Upsilon'(z) = \frac{1-\left|\psi(0)\right|^2}{\left(1-\overline{\psi(0)}\psi(z)\right)^2}\psi'(z),$$

$$|\Upsilon'(0)| = \frac{\left|\psi'(0)\right|}{1-\left|\psi(0)\right|^2} = \frac{4^n\left|c_2\right|}{1-\frac{3^{2n}}{4}\left|c_1\right|^2} = \frac{4^{n+1}\left|c_2\right|}{4-3^{2n}\left|c_1\right|^2},$$

we take

$$\begin{aligned}
\frac{2}{1+\frac{4^{n+1}|c_2|}{4-3^{2n}|c_1|^2}} &\leq \frac{1+\frac{3^n}{2}\left|c_1\right|}{1-\frac{3^n}{2}\left|c_1\right|}\left\{2\left|\left(I^n f(z)\right)''_{z=b}\right| - 2\right\} \\
&= \frac{2+3^n\left|c_1\right|}{2-3^n\left|c_1\right|}\left\{2\left|\left(I^n f(z)\right)''_{z=b}\right| - 2\right\},
\end{aligned}$$

$$\frac{2\left(4-3^{2n}|c_1|^2\right)}{4-3^{2n}|c_1|^2+4^{n+1}|c_2|}\frac{2-3^n|c_1|}{2+3^n|c_1|}\leq 2\left|(I^n f(z))''_{z=b}\right|-2$$

and

$$\frac{2\left(2-3^n|c_1|\right)^2}{4-3^{2n}|c_1|^2+4^{n+1}|c_2|}+2\leq 2\left|(I^n f(z))''_{z=b}\right|$$

Therefore, we obtain

$$\left|(I^n f(z))''_{z=b}\right|\geq 1+\frac{\left(2-3^n|c_1|\right)^2}{4-3^{2n}|c_1|^2+4^{n+1}|c_2|}.$$

To show that the inequality (2.3) is sharp, take the holomorphic function

$$(I^n f(z))'=\frac{z^2-1}{z^2(1+z^2)}.$$

Then

$$(I^n f(z))''=\frac{2z\left(z^4+z^2\right)-\left(4z^3+2z\right)\left(z^2-1\right)}{\left(z^4+z^2\right)^2}$$

and

$$\left|(I^n f(z))''_{z=1}\right|=1.$$

Since $|c_1|=\frac{2}{3^n}$, (2.3) is satisfied with equality. □

If $z^2\left(I^n f(z)\right)'$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following Theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{M}$ and $z^2\left(I^n f(z)\right)'$ has no zeros in E except $z=0$ and $c_1>0$. Assume that, for some $b\in\mathbb{T}$, $(I^n f(z))'$ has angular limit $(I^n f(z))'_{z=b}$ at b and $(I^n f(z))'_{z=b}=0$. Then we have the inequality*

$$\left|(I^n f(z))''_{z=b}\right|\geq 1-\frac{1}{2}\frac{3^n|c_1|\ln^2\left(\frac{3^n}{2}c_1\right)}{3^n|c_1|\ln\left(\frac{3^n}{2}c_1\right)-4^n|c_2|}. \tag{2.5}$$

In addition, the equality in (2.5) *occurs for the solution of equation*

$$(I^n f(z))'=\frac{z^2-1}{z^2(1+z^2)}$$

with the condition at $z=0$

$$\lim_{z\to 0}z^2 f(z)=0.$$

Proof. Let $c_1>0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $z^2\left(I^n f(z)\right)'$ has no zeros in E except $E-\{0\}$, we denote by $\ln\psi(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln\psi(0)=\ln\left(\frac{3^n}{2}c_1\right)<0.$$

The auxiliary function

$$\Phi(z)=\frac{\ln\psi(z)-\ln\psi(0)}{\ln\psi(z)+\ln\psi(0)}$$

is a holomorphic in the unit disc E, $|\Phi(z)|<1$,$\Phi(0)=0$ and $|\Phi(b)|=1$ for $b\in\mathbb{T}$.

For $p=1$, from (1.7), we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|} &\leq |\Phi'(b)|=\frac{\left|2\ln\psi(0)\right|}{\left|\ln\psi(b)+\ln\psi(0)\right|^2}\left|\frac{\psi'(b)}{\psi(b)}\right| \\ &= \frac{-2\ln\psi(0)}{\ln^2\psi(0)+\arg^2\psi(b)}\left\{\left|\phi'(b)\right|-2\right\}.\end{aligned}$$

Replacing $\arg^2\omega(b)$ by zero, then we take

$$\frac{1}{1-\frac{4^n|c_2|}{\frac{3^n}{2}|c_1|2\ln\left(\frac{3^n}{2}c_1\right)}}\leq\frac{-1}{\ln\left(\frac{3^n}{2}c_1\right)}\left\{2\left|(I^nf(z))''_{z=b}\right|-2\right\},$$

$$-\frac{3^n|c_1|\ln^2\left(\frac{3^n}{2}c_1\right)}{3^n|c_1|\ln\left(\frac{3^n}{2}c_1\right)-4^n|c_2|}\leq 2\left|(I^nf(z))''_{z=b}\right|-2,$$

$$-\frac{1}{2}\frac{3^n|c_1|\ln^2\left(\frac{3^n}{2}c_1\right)}{3^n|c_1|\ln\left(\frac{3^n}{2}c_1\right)-4^n|c_2|}\leq\left|(I^nf(z))''_{z=b}\right|-1$$

and

$$\left|(I^nf(z))''_{z=b}\right|\geq\left(1-\frac{1}{2}\frac{3^n|c_1|\ln^2\left(\frac{3^n}{2}c_1\right)}{3^n|c_1|\ln\left(\frac{3^n}{2}c_1\right)-4^n|c_2|}\right).$$

Thus, we obtain the inequality (2.5) with an obvious equality case. □

The following inequality (2.6) is weaker, but is simpler than (2.5) and does not contain the coeffient c_2.

Theorem 2.5. *Under the same assumptions as in Theorem 2.4, we have*

$$\left|(I^nf(z))''_{z=b}\right|\geq 1-\frac{1}{4}\ln\left(\frac{3^n}{2}c_1\right). \tag{2.6}$$

In addition, the equality in (2.6) *occurs for the solution of equation*

$$(I^nf(z))'=\frac{z^2-1}{z^2(1+z^2)}$$

with the condition at $z=0$

$$\lim_{z\to 0}z^2f(z)=0.$$

Proof. Let $c_1>0$. Using the inequality (1.8) for the function $\Phi(z)$, we obtain

$$1\leq|\Phi'(b)|=\frac{\left|2\ln\psi(0)\right|}{\left|\ln\psi(b)+\ln\psi(0)\right|^2}\left|\frac{\psi'(b)}{\psi(b)}\right|=\frac{-2\ln\psi(0)}{\ln^2\psi(0)+\arg^2\psi(b)}\left\{\left|\phi'(b)\right|-2\right\}.$$

Replacing $\arg^2 \varphi(b)$ by zero, then

$$1 \leq |\Phi'(b)| \leq \frac{-2}{\ln\left(\frac{3^n}{2}c_1\right)}\left\{2\left|(I^n f(z))''_{z=b}\right| - 2\right\},$$

$$1 \leq \frac{-2}{\ln\left(\frac{3^n}{2}c_1\right)}\left\{2\left|(I^n f(z))''_{z=b}\right| - 2\right\}$$

and

$$\left|(I^n f(z))''_{z=b}\right| \geq 1 - \frac{1}{4}\ln\left(\frac{3^n}{2}c_1\right).$$

Therefore, we obtain the inequality (2.6) with an obvious equality case. □

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