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Fixed Points of Set-Valued F-Contractions and its Application to Non-Linear Integral Equations

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Abstract. We observe that the assumption of set-valued F-contractions (Sgroi and Vetro [13]) is actually very strong for the existence of fixed point and can be weakened. In this connection, we introduce the notion of set-valued α-F-contractions and prove a corresponding fixed point theorem in complete metric spaces. Consequently, we derive several fixed point theorems in metric spaces. Some examples are given to illustrate the new theory. Then we apply our results to establishing the existence and uniqueness of solutions for a certain type of non-linear integral equations.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

Throughout the paper the sets $\mathbb{R}, \mathbb{R}^+, \mathbb{R}_+$ and $\mathbb{N}$ denote the set of all reals, set of all nonnegative reals, set of all positive reals and set of all natural numbers respectively.

Let (X, d) be a metric space and $T\colon X \to X$ be a mapping. The mapping T is said to a Banach contraction on X if; there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X.$$

The famous Banach contraction principle states that: every Banach contraction on a complete metric space has a unique fixed point, that is, there exists a unique $x^* \in X$ such that $Tx^* = x^*$. There are plenty of generalizations of this famous principle.

In 2012, Wardowski [4] extended this principle using a particular function $F\colon \mathbb{R}_+ \to \mathbb{R}$ and introduced the notion of F-contractions. An example of Wardowski shows the validity of such generalizations in complete metric spaces.

Definition 1.1 ([4]). *Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying:*

(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

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(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the set of all functions satisfying properties (F1)-(F3), by $\mathcal{F}$.

Definition 1.2 ([4]). Let (X, d) be a metric space and $T: X \to X$ be a mapping. Then, the mapping T is called an F-contraction if, there exist a function $F \in \mathcal{F}$ and a constant $\tau > 0$ such that: for all $x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \le F(d(x, y))$$

The notion of F-contraction generalizes the notion of Banach contraction and unifies several other types of contraction mappings. For examples of function F and F-contraction we refer to [4] (see also[14]). The notion of F-contractions has several generalization and interesting consequences, see, e.g., [3, 5–8] and references therein.

Edelstein [12] generalized Banach contraction principle in ε-chainable spaces. Ran and Reurings [1] and Nieto and Rodríguez-López [10, 11] generalized the Banach contraction principle into the metric spaces equipped with a partial order. Jachymski [9] introduced the contraction in metric spaces involving some graphical structures through which he generalized and unified the fixed point results of spaces endowed with partial orders and the result of Edelstein. Recently, Samet et al. [2] generalized the results of Ran and Reurings [1] and Nieto and Rodríguez-López [10, 11], and extended the Banach contraction principle by introducing the notion of α-admissible mappings.

Nadler [15] extended the Banach contraction principle for set-valued mappings. Recently, Sgroi and Vetro [13] introduced the multi-valued F-contractions and generalized the results of Nadler [15] and Wardowski [4].

In this paper, we generalize the notion of F-contractions by introducing a new type of contractions which in addition to the function F, involve functions $\ell: \mathbb{R}_+ \to \mathbb{R}$ and α (introduced by Samet et al. [2]). We observe that the contractive condition used by Sgroi and Vetro [13] are very strong and can be weakened. A fixed point result for new contractions is also proved, which generalizes and unifies the fixed point result of Wardowski [4], Jachymski [9], Samet et al. [2] and Sgroi and Vetro [13] in metric spaces. Some examples are presented which ensure the validity of our generalization. An application to non-linear integral equations is also discussed.

We recall some definitions and results which will be needed in the sequel.

Let (X, d) be a metric space and denote by $C(X)$ the families of non-empty closed subsets of X. A mapping $T: X \to C(X)$ is called a set-valued (or multi-valued) mapping on X. Let T be a set-valued mapping on X. A point $x \in X$ is called a fixed point of T if $x \in Tx$; and it is called an end point (or stationary point) of T if $Tx = \{x\}$. It is obvious that every end point of T is a fixed point of T but converse is not true in general. Therefore, the assumption of end points is stronger than that of fixed points.

For $x, y \in X$ we put

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Definition 1.3. [13] Let (X, d) be a metric space. A set-valued (multi-valued) mapping $T: X \to C(X)$ is called an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that for all $x, y \in X$ with $y \in Tx$ there exists $z \in Ty$ for which

$$\tau + F(d(y, z)) \le F(M(x, y)) \quad \text{if} \quad d(y, z) > 0. \tag{1}$$

The following definition is well-known and was used by Sgroi and Vetro [13].

Definition 1.4. Let (X, d) be a metric space and $T: X \to C(X)$ be a mapping. The graph of T is the subset $\{(x, y): x \in X, y \in Tx\}$ of $X \times X$; we denote the graph of T by $G(T)$. Then T is a closed mapping if the graph $G(T)$ is a closed subset of $X \times X$.

Sgroi and Vetro [13] proved the following theorem.

Theorem 1.5. *Let (X, d) be a complete metric space and let $T\colon X \to C(X)$ be a closed F-contraction. Then T has a fixed point.*

We observe the following important fact about the set-valued F-contractions and their fixed points.

Remark 1.6. Let T be an F-contraction, x be a fixed point of T and if possible, let Tx be not singleton. Since $x \in Tx$, by the definition of F-contraction; there exists $z \in Tx$ such that $d(x, z) > 0$ (since Tx is not singleton) and

$$\tau + F(d(x, z)) \leq F(M(x, x)).$$

Notice that, $M(x, x) = 0$, therefore by the definition we must have $d(x, z) = 0$, that is, $z = x$. This contradiction shows that Tx is singleton. Therefore, $Tx = \{x\}$, that is, x is an end point of T.

In view of the above remark, Theorem 1.5 can be stated in a more stronger form as follows:

Theorem 1.7. *Let (X, d) be a complete metric space and let $T\colon X \to C(X)$ be a closed F-contraction. Then T has an end point.*

Denote by Ψ the family of nondecreasing functions $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{i=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}_+$, where ψ^n is the n-th iterate of ψ.

Definition 1.8. [2] Let (X, d) be a metric space and $T\colon X \to X$ be a given mapping. We say that:

(i) T is an α-ψ-contraction mapping if there exist two functions $\alpha\colon X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.

(ii) T is α-admissible if: $x, y \in X,\ \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$.

We use the function $\alpha\colon X \times X \to [0, \infty)$ and a particular type of function $\ell\colon \mathbb{R}_+ \to \mathbb{R}$ to generalize the notion of set-valued F-contractions.

2. Main Results

In this section, we introduce some definitions and prove some fixed point results.

Definition 2.1. Let $\ell\colon \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying: $\liminf_{s \to t+} \ell(s) > 0$ for every $t \in \mathbb{R}_+$. We denote the class of all such functions ℓ by $\mathcal{L}$.

Example 2.2. (a) Let $\beta\colon \mathbb{R}_+ \to (0, 1)$ be a function satisfying the condition $\limsup_{s \to t+} \beta(s) < 1$ for every $t \in \mathbb{R}_+$ and $\ell_1(t) = -\log(\beta(t))$ for all $t \in \mathbb{R}_+$. Then, $\ell_1 \in \mathcal{L}$.

(b) Let $\tau > 0$ be a fixed real number and $\ell_2(t) = \tau$ for all $t \in \mathbb{R}_+$. Then, $\ell_2 \in \mathcal{L}$.

(c) Let $\ell_3(t) = t^a$ for all $t \in \mathbb{R}_+$, where $a \in \mathbb{R}$. Then, $\ell_3 \in \mathcal{L}$.

(d) Let $\ell_4(t) = \log(1 + t)$, for all $t \in \mathbb{R}_+$. Then, $\ell_4 \in \mathcal{L}$.

Definition 2.3. Let (X, d) be a complete metric space, $\alpha\colon X \times X \to \mathbb{R}^+$ a function and let $T\colon X \to C(X)$ be a mapping. Then the mapping T is called d-α-admissible if the following condition holds:

$$\Big(\alpha(x, y) \geq 1,\ y \in Tx,\ z \in Ty,\ d(y, z) < d(x, y)\Big) \implies \alpha(y, z) \geq 1.$$

Remark 2.4. Notice that, in the above definition the admissibility of T depends on the associated metric structure with X, whereas no such metric structure were involved in the original paper of Samet et al. [2]. For instance, let $X = \mathbb{R}^+$ and d be the usual and d_1 the discrete metric on X. Define the function $\alpha\colon X \times X \to \mathbb{R}^+$ by $\alpha(x, y) = 1$ if $y \leq x$; $\alpha(x, y) = 0$ if $y > x$ and a mapping $T\colon X \to C(X)$ by $Tx = [0, 2x]$. Note that, T is not d-α-admissible. Indeed, for $x = 1, y = 1/3, z = 2/3$ we have $\alpha(x, y) \geq 1, y \in Tx, z \in Ty$ and $d(y, z) < d(x, y)$, but $\alpha(y, z) = 0$. On the other hand, T is d_1-α-admissible.

Next, we define the set-valued α-F-contractions on metric spaces.

Definition 2.5. Let (X,d) be a metric space, $\alpha\colon X\times X\to\mathbb{R}^+$ a function and let $T\colon X\to C(X)$ be a mapping. If for all $x,y\in X$ with $\alpha(x,y)\geq 1$, $y\in Tx$, there exists $z\in Ty$ such that

$$d(y,z)>0 \implies \ell(d(x,y))+\alpha(x,y)F(d(y,z))\leq F(M(x,y)) \tag{2}$$

for some $F\in\mathcal{F}$ and $\ell\in\mathcal{L}$. Then the mapping T is called a set-valued α-F-contraction.

Notice that, if $M(x,y)=0$ in the above condition then the contractive condition (2) forces $d(y,z)$ to be zero. In the further discussion, we emphasis on the existence of fixed points of the mapping T, and it is clear that if in the above definition $d(y,z)$ is zero, then T certainly has a fixed point. Therefore, in the rest of the paper we will not mind these possibilities.

Remark 2.6. If we take $\alpha(x,y)=1$ for all $x,y\in X$, then a set-valued α-F-contraction reduces into set-valued F-contraction, so the class of set-valued F-contractions is contained properly (see, Example 2.20 of this paper) in the class of set-valued α-F-contractions. Thus, the set-valued α-F-contractions are weaker type of contractions than a set-valued F-contraction. Let $\triangle$ denotes the diagonal of the product $X\times X$. Notice that, if $\alpha(x,x)\geq 1$ on $\triangle$ then one can draw same conclusions as in Remark 1.6. Therefore, the assumption of set-valued F-contractions can be weakened through the assumption of set-valued α-F-contraction only when we discard the case: $\alpha(x,x)\geq 1$ on $\triangle$.

A sequence $\{x_n\}$ in a metric space (X,d) is called a trajectory of mapping $T\colon X\to C(X)$, starting at x_1 if $x_{n+1}\in Tx_n$ for all $n\in\mathbb{N}$ and it is called an reverse trajectory if $x_n\in Tx_{n+1}$ for all $n\in\mathbb{N}$. Note that, any fixed point (or end point) of a set-valued mapping forms a trivial trajectory (as well as reverse trajectory) which starts at fixed point (or end point) itself.

Lemma 2.7. *Let (X,d) be a metric space, $T\colon X\to C(X)$ be a mapping and there exists a trajectory (or a reverse trajectory) $\{x_n\}\subset X$ such that $\lim_{n\to\infty}x_n$ exists in X. If T is a closed mapping, then T has a fixed point.*

Proof. Let $\lim_{n\to\infty}x_n=u\in X$ and $\{x_n\}$ is a trajectory (the proof for reverse trajectory is similar), then by definition $(x_n,x_{n+1})\in G(T)$ for all $n\in\mathbb{N}$. Since, $\lim_{n\to\infty}x_n=u\in X$ we have $\lim_{n\to\infty}(x_n,x_{n+1})=(u,u)\in X\times X$. Since T is closed, we have $(u,u)\in G(T)$ and so by definition, $u\in Tu$. $\square$

By $\mathcal{T}(x,T)$ we denote the class of all trajectories of T starting from some $x\in X$.

Definition 2.8. Let (X,d) be a metric space, $\alpha\colon X\times X\to\mathbb{R}^+$ a function and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called an α-sequence if $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$. By $\mathfrak{A}_\alpha$ we denote the class of all α-sequences in X. For a mapping $T\colon X\to C(X)$, we denote by $G_\alpha(T)$ the α-graph of T and

$$G_\alpha(T)=\{(x_n,x_{n+1})\in X\times X\colon x_1\in X \text{ and } \{x_n\}\in\mathfrak{A}_\alpha\cap\mathcal{T}(x_1,T)\}.$$

The mapping T is called an α-closed mapping if $G_\alpha(T)$ is a closed subset of $X\times X$.

Remark 2.9. It is obvious that the α-closeness of T depends on the structure of $\mathfrak{A}_\alpha$ and $G_\alpha(T)\subset G(T)$. Also, for $\alpha(x,y)=1$ for all $x,y\in X$ we have $\mathfrak{A}_\alpha\cap\mathcal{T}(x_1,T)=\mathcal{T}(x_1,T)$ and so $G(T)=G_\alpha(T)$. Therefore, the closed mappings are a particular case of α-closed mappings.

Example 2.10. Consider the sequence $\{x_n\}=\left\{\frac{(n-1)[2+(-1)^n]}{3n}\right\}$ and $X=[0,1]$ with usual metric $d\equiv|\cdot|$. Let $k\in[1,\infty), l\in[0,1)$ be fixed and the function α be defined by

$$\alpha(x,y)=\begin{cases} k, & \text{if } (x,y)=(x_{2n},x_{2n+2}),\ n\in\mathbb{N} \text{ or } x=y=1;\\ l, & \text{otherwise.}\end{cases}$$

Notice that, $\mathfrak{A}_\alpha = \left\{\{1\}, \left\{x_{2(n+k)}\right\} : k = 0, 1, 2, \dots\right\}$, where $\{1\}$ is the constant sequence with each term equals to 1. Define a mapping $T \colon X \to C(X)$ by

$$Tx = \begin{cases} \{x_n, x_{n+1}, x_{n+2}\}, & \text{if } x = x_n,\ n \in \mathbb{N}; \\ \{x\}, & \text{otherwise.} \end{cases}$$

It is obvious that $G_\alpha(T) = \{(1,1), (x_{2n}, x_{2n+2}) \colon n \in \mathbb{N}\}$ and so T is α-closed. On the other hand, T is not a closed mapping. Indeed, $(x_{2n+1}, x_{2n+2}) \in G(T)$ for all $n \in \mathbb{N}$ and $(x_{2n+1}, x_{2n+2}) \to (1/3, 1)$ as $n \to \infty$, but $(1/3, 1) \notin G(T)$ since $1 \notin T(1/3)$.

Remark 2.11. If T is α-closed and there exists a convergent trajectory $\{x_n\} \in \mathfrak{A}_\alpha$ of T starting from x_1, then in view of Lemma 2.7 the limit of $\{x_n\}$ is a fixed point of T.

Definition 2.12. Let (X, d) be a metric space and $\alpha \colon X \times X \to \mathbb{R}^+$ be a function. Then (X, d) is called α-complete if every Cauchy sequence in $\mathfrak{A}_\alpha$ is convergent in $X \times X$.

Remark 2.13. It is obvious that the α-completeness of X depends on the structure of $\mathfrak{A}_\alpha$. Also, for $\alpha(x, y) = 1$ for all $x, y \in X$ we have $\mathfrak{A}_\alpha = l_X$ (the class of all sequences in X) and so the α-completeness reduces into the completeness. Therefore, the completeness of metric spaces is a particular case of α-completeness of spaces.

Example 2.14. Let us replace X in Example 2.10 by $Y = (0, 1]$. Then, the space (Y, d) is not complete. But the space (Y, d) is α-complete (since $\mathfrak{A}_\alpha = \left\{\{1\}, \left\{x_{2(n+k)}\right\} : k = 0, 1, 2, \dots\right\}$ and $x_{2n} \to 1 \in Y$).

The following theorem is the main result of this paper and it generalizes the result of Sgroi and Vetro [13].

Theorem 2.15. *Let (X, d) be an α-complete metric space and $T \colon X \to C(X)$ be a set-valued α-F-contraction. Suppose the following conditions hold:*

(a) T is d-α-admissible;

(b) there exist $x_0, x_1 \in X$ such that $x_1 \in Tx_0$ and $\alpha(x_0, x_1) \geq 1$.

Then there exists a trajectory $\{x_n\} \in \mathfrak{A}_\alpha$ of T starting from x_0 such that $\{x_n\}$ is a Cauchy sequence. In addition, if T is an α-closed mapping, then T has a fixed point.

Proof. Let $x_0, x_1 \in X$ be such that $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. Since T is a set-valued α-F-contraction, there exists $x_2 \in Tx_1$ such that

$$\ell(d(x_0, x_1)) + F(d(x_1, x_2)) \leq F(M(x_0, x_1))$$

which with the definition of ℓ gives

$$F(d(x_1, x_2)) \leq F(M(x_0, x_1)) - \ell(d(x_0, x_1)) < F(M(x_0, x_1)). \tag{3}$$

By definition, we have

$$\begin{aligned} M(x_0, x_1) &= \max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\} \\ &= \max\left\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2}\right\} \\ &\leq \max\left\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_1) + d(x_1, x_2)}{2}\right\} \\ &= \max\left\{d(x_0, x_1), d(x_1, x_2)\right\}. \end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$, then by (F1) and (3), the above inequality yields a contradiction. Therefore, we must have $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1)$. As a consequence, from (3) we obtain $d(x_1, x_2) < d(x_0, x_1)$. Since T is d-α-admissible, it shows that $\alpha(x_1, x_2) \geq 1$. Also, by definition of ℓ and the inequality (3) we have

$$F(d(x_1, x_2)) \leq F(d(x_0, x_1)) - \ell(d(x_0, x_1)) < F(d(x_0, x_1)). \tag{4}$$

Again, since $\alpha(x_1, x_2) \geq 1$, $x_2 \in Tx_1$ there exists $x_3 \in Tx_2$ such that

$$\ell(d(x_1, x_2)) + F(d(x_2, x_3)) \leq F(M(x_1, x_2))$$

which with the definition of ℓ gives

$$F(d(x_2, x_3)) \leq F(M(x_1, x_2)) - \ell(d(x_1, x_2)) < F(M(x_1, x_2)).$$

Using similar arguments as the above, we obtain $d(x_2, x_3) < d(x_1, x_2)$. Since T is d-α-admissible it shows that $\alpha(x_2, x_3) \geq 1$. Also, we get

$$F(d(x_2, x_3)) \leq F(d(x_1, x_2)) - \ell(d(x_1, x_2)) < F(d(x_1, x_2)). \tag{5}$$

By repetition of this process, we obtain a sequence $\{x_n\}$ such that $x_n \in Tx_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ and the following inequality is satisfied

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \ell(d(x_{n-1}, x_n)) < F(d(x_{n-1}, x_n)).$$

Thus, $\{x_n\} \in \mathfrak{A}_\alpha$ is a trajectory of T starting from x_0.

Denote $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, then from the above inequality we have

$$F(d_n) \leq F(d_{n-1}) - \ell(d_{n-1}) < F(d_{n-1}). \tag{6}$$

Since $F \in \mathcal{F}$, the above inequality shows that $\{d_n\}$ is a decreasing sequence which is bounded below. Suppose, $\lim_{n\to\infty} d_n = \delta \geq 0$. Since $\liminf_{s\to t+} \ell(s) > 0$ for every $t \in \mathbb{R}_+$, there exist $\tau > 0$ and $\varepsilon > 0$ such that

$$\ell(s) \geq \tau \text{ for all } s \in (\delta, \delta + \varepsilon].$$

We can choose $k \in \mathbb{N}$ such that $d_n \in (\delta, \delta + \varepsilon]$ for all $n \geq k$. Therefore, it follows from the above inequality that $\ell(d_n) \geq \tau$ for all $n \geq k$. Using this inequality in (6) we obtain

$$F(d_{n+1}) \leq F(d_n) - \tau \text{ for all } n \geq k.$$

For all $n \in \mathbb{N}$, it follows from the above inequality that

$$F(d_{k+n}) \leq F(d_{k+n-1}) - \tau \leq \cdots \leq F(d_k) - n\tau, \tag{7}$$

which yields, $\lim_{n\to\infty} F(d_n) = -\infty$ and so by (F2) we have

$$\lim_{n\to\infty} d_n = 0. \tag{8}$$

By (F3), there exists $p \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^p F(d_n) = 0. \tag{9}$$

We will show that the sequence $\{x_n\}$ is a Cauchy sequence. For all $n \in \mathbb{N}$, it follows from the inequality (7) that

$$\begin{aligned} d_{k+n}^p F(d_{k+n}) - d_{k+n}^p F(d_k) &\leq d_{k+n}^p [F(d_k) - n\tau] - d_{k+n}^p F(d_k) \\ &= -n\tau d_{k+n}^p \leq 0. \end{aligned}$$

Using (8) and (9) in the above inequality we obtain $\lim_{n\to\infty} nd_n^p = 0$. Therefore, there exists $n_1 \in \mathbb{N}$ such that $nd_n^p < 1$ for all $n \geq n_1$, that is,

$$d_n < \frac{1}{n^{1/p}} \text{ for all } n \geq n_1.$$

If $m, n \in \mathbb{N}$ with $m > n \geq n_1$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= d_n + d_{n+1} + \cdots + d_{m-1} = \sum_{i=n}^{m-1} d_i \\ &\leq \sum_{i=n}^{m-1} \frac{1}{i^{1/p}}. \end{aligned}$$

Since, the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/p}}$ is convergent, it follows from the above inequality that the sequence $\{x_n\}$ is a Cauchy sequence. By α-completeness of X, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now the existence of fixed point follows from remark 2.11 and u is a fixed point of T. □

We obtain the some corollaries as a consequence of Theorem 2.15 in complete metric spaces. The following corollary is an improved set-valued version of result of Wardowski [4].

Corollary 2.16. *Let (X, d) be a complete metric space and $T: X \to C(X)$ be a mapping. If for all $x, y \in X$ with $y \in Tx$ there exists $z \in Ty$ such that*

$$d(y, z) > 0 \implies \ell(d(x, y)) + F(d(y, z)) \leq F(d(x, y)) \tag{10}$$

for some $F \in \mathcal{F}$ and $\ell \in \mathcal{L}$. Then for each $x_0 \in X$ the trajectory $\{x_n\}$ of T starting from x_0 is a Cauchy sequence. In addition, if T is a closed mapping, then T has an end point.

Proof. Taking $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.15 and using (F2), Remark 2.6, we obtain the desired result. □

The following example is a simple illustration of the above result and it also illustrates the advantage of taking the function $\ell \in \mathcal{L}$ instead constant $\tau > 0$ in the contractive condition (see the contractive condition of Wardowski [4]).

Example 2.17. Let $X = \{x_n : n \in \mathbb{N}\}$, where $x_n = 1 - \frac{1}{n}$ and define $D: X \times X \to \mathbb{R}^+$ by

$$D(x, y) = \begin{cases} \max\{x, y\}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases} \tag{11}$$

Then, (X, D) is a complete metric space. Define a mapping $T: X \to C(X)$ by

$$Tx = \begin{cases} \{x_1, x_{n-1}\}, & \text{if } x = x_n, n > 1; \\ \{x\}, & \text{otherwise.} \end{cases}$$

Define $\ell: \mathbb{R}_+ \to \mathbb{R}$ by

$$\ell(t) = \begin{cases} \log\left(\dfrac{x_n}{x_{n-1}}\right), & \text{if } t = x_n, n \in \mathbb{N}; \\ 1, & \text{otherwise} \end{cases}$$

and choose $F(t) = \log(t)$, then, T satisfies the condition (10). Notice that, the sequence $\{x_n\}$ is not convergent with respect to the metric D, so T is a closed mapping. All other conditions of Corollary 2.16 are satisfied and T has a fixed points in X. Namely, $x_1 \in Tx_1$.

On the other side, if $2 < n$, then $x_{n-1} \in Tx_n$, and $Tx_{n-1} = \{x_1, x_{n-2}\}$. If we take $x_{n-2} \in Tx_{n-1}$ then $D(x_{n-1}, x_{n-2}) > 0$, but then $D(x_{n-1}, x_{n-2}) = x_{n-1}$, and $D(x_n, x_{n-1}) = x_n$. Therefore, $F(D(x_{n-1}, x_{n-2})) = \log(x_{n-1})$ and $F(D(x_n, x_{n-1})) = \log(x_n)$ as $n \to \infty$. Therefore, we cannot obtain a constant $\tau > 0$ such that

$$\tau + F(D(x_{n-1}, x_{n-2})) \leq F(D(x_n, x_{n-1})).$$

Similarly, if we take $x_1 \in Tx_{n-1}$, we obtain same conclusion.

Corollary 2.18. *Let (X, d) be a complete metric space and $T: X \to C(X)$ be a closed mapping. Suppose, there exists a function $\beta: \mathbb{R}_+ \to (0, 1)$ such that $\limsup_{s \to t+} \beta(s) < 1$ for all $t \in \mathbb{R}_+$ and for all $x, y \in X$ with $y \in Tx$ there exists $z \in Ty$ such that*

$$d(y, z) > 0 \implies d(y, z) \leq \beta(d(x, y))d(x, y).$$

Then T has an end point.

Proof. Setting $\ell(t) = -\log(\beta(t))$, $F(t) = \log t$ for all $t \in \mathbb{R}_+$ and $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.15 and using Remark 2.6 we obtain the desired result. □

Corollary 2.19 (Sgroi and Vetro [13]). *Let (X, d) be a complete metric space and let $T: X \to C(X)$ be a closed F-contraction. Then T has an end point.*

Proof. Taking $\ell(t) = \tau$, for all $t \in \mathbb{R}_+$ and $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.15 we obtain the desired result. □

The next example shows that we actually weak the assumption of Sgroi and Vetro [13]. Also, it shows that an α-F-contraction may have a fixed point which is not an end point.

Example 2.20. Let $X = \{x_n : n \in \mathbb{N}\} \cup \{0\}$, where $\{x_n\}$ is a strictly decreasing sequence of positive numbers converging to 0 (with respect to usual metric on X). Let the function $D: X \times X \to \mathbb{R}^+$ be defined by (11) and a function $\alpha: X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } \big(x = y = 0\big) \text{ or } \big(x = x_n, y = x_{n+1}, n \in \mathbb{N}\big); \\ 0, & \text{otherwise.} \end{cases}$$

Then, (X, D) is an α-complete metric space (indeed, it is a complete metric space). Let $T: X \to X$ be the mapping defined by

$$Tx = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{0, x_n, x_{n+1}\}, & \text{if } x = x_n, n \text{ is odd}; \\ \{x_{n+1}\}, & \text{if } x = x_n, n \text{ is even}. \end{cases}$$

Then, T is D-α-admissible. Notice that, $\{x_n\} \in \mathcal{T}(x_1, T) \cap \mathfrak{A}_\alpha$ and so condition (b) of Theorem 2.15 is satisfied. Since $x_n \to 0$ (with respect to the metric D) as $n \to \infty$, therefore T is α-closed. Finally, some routine calculations shows that the mapping T is a set-valued α-F-contraction. Thus, all the conditions of Theorem 2.15 are satisfied and so we can conclude the existence of fixed point of T. On the other hand, since $Tx_{2n-1} = \{0, x_{2n-1}, x_{2n}\}$ for all $n \in \mathbb{N}$, T is not an F-contraction in the sense of Sgroi and Vetro [13].

Remark 2.21. In the above example, 0 and all the odd terms of the sequence $\{x_n\}$ are fixed points of T, but no term of the sequence $\{x_n\}$ is an end point of T. Notice that, $\alpha(x, x) \not\geq 1$ on the diagonal $\triangle$ of $X \times X$, which verifies the claim of Remark 2.6.

3. Consequences

Next, we define some concepts which are generalizations of some known ones and derive some consequences of our main result.

Let $(X, \preceq)$ be a poset and d be a metric on X. We extend some known concepts (see, Ran and Reurings [1]) concerned with single-valued mappings to the set-valued mappings. A mapping $T\colon X \to C(X)$ is called d-monotonic with respect to $\preceq$ if the following condition holds:

$$\Big(x \preceq y,\ y \in Tx,\ z \in Ty,\ d(y,z) < d(x,y)\Big) \implies y \preceq z.$$

T is called a set-valued ordered F-contraction if, for all $x, y \in X$ with $x \preceq y$, $y \in Tx$, there exists $z \in Ty$ such that

$$d(y,z) > 0 \implies \ell(d(x,y)) + F(d(y,z)) \leq F(M(x,y)) \tag{12}$$

for some $F \in \mathcal{F}$ and $\ell \in \mathcal{L}$.

Let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is called an ordered sequence if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. By $\mathfrak{A}_{\preceq}$ we denote the class of all ordered sequences in X. For the mapping T, we denote by $G_{\preceq}(T)$ the $\preceq$-graph of T and

$$G_{\preceq}(T) = \{(x_n, x_{n+1}) \in X \times X \colon x_1 \in X \text{ and } \{x_n\} \in \mathfrak{A}_{\preceq} \cap \mathcal{T}(x_1, T)\}.$$

The mapping T is called $\preceq$-closed if $G_{\preceq}(T)$ is a closed subset of $X \times X$. The space (X, d) is called $\preceq$-complete if every Cauchy sequence in $\mathfrak{A}_{\preceq}$ is convergent in $X \times X$.

Theorem 3.1. *Let $(X, \preceq)$ be a poset, d a metric on X and $T\colon X \to C(X)$ be a set-valued ordered F-contraction. Suppose the following conditions hold:*

(O1) T is d-monotonic with respect to $\preceq$;

(O2) there exist $x_0, x_1 \in X$ such that $x_1 \in Tx_0$ and $x_0 \preceq x_1$.

Then there exists a trajectory $\{x_n\} \in \mathfrak{A}_{\preceq}$ of T starting from x_0 such that $\{x_n\}$ is a Cauchy sequence. In addition, if T is $\preceq$-closed, then T has an end point.

Proof. The partial order $\preceq$ induces the function $\alpha\colon X \times X \to \mathbb{R}^+$ as follows:

$$\alpha_{\preceq}(x,y) = \begin{cases} 1, & \text{if } x \preceq y; \\ 0, & \text{otherwise.} \end{cases}$$

We shall show that all the conditions of Theorem 2.15 are satisfied with the function $\alpha_{\preceq}$ defined above. Then, it is easy to verify that the condition (12) implies that the T is a set-valued $\alpha_{\preceq}$-contraction and the d-monotonicity of T implies that T is $\alpha_{\preceq}$-admissible and so (a) of Theorem 2.15 holds. The condition (b) of Theorem 2.15 follows directly from the condition (O2) above. Notice that, $G_{\alpha_{\preceq}}(T) = G_{\preceq}(T)$, therefore T is $\alpha_{\preceq}$-closed. Further, the $\preceq$-completeness implies the $\alpha_{\preceq}$-completeness of X. Thus, all the conditions of Theorem 2.15 are satisfied and so T has a fixed point. Finally, since $x \preceq x$ for all $x \in X$, we have $\alpha_{\preceq}(x,x) = 1$ on the diagonal $\triangle$, and so by Remark 2.6 every fixed point of T is an end point of T. □

Let (X, d) be a metric space and $\mathcal{G}$ be a graph such that $V(\mathcal{G}) = X$ and $E(\mathcal{G}) \supset \triangle$ (see Jachymski [9]). Notice that, the graph $\mathcal{G}$ induces a function $\alpha\colon X \times X \to \mathbb{R}^+$ as follows:

$$\alpha_{\mathcal{G}}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(\mathcal{G}); \\ 0, & \text{otherwise.} \end{cases}$$

Since $E(\mathcal{G}) \supset \triangle$, we have $\alpha_{\mathcal{G}}(x,x) = 1$ on the diagonal $\triangle$. Therefore, we extend some known concepts (see Jachymski [1]) concerned with single-valued mappings to the set-valued mappings. A mapping $T\colon X \to C(X)$ is called d-edge preserving with respect to $\mathcal{G}$ if the following condition holds:

$$\Big((x,y) \in E(\mathcal{G}),\ y \in Tx,\ z \in Ty,\ d(y,z) < d(x,y)\Big) \implies (y,z) \in E(\mathcal{G}).$$

T is called a set-valued $\mathcal{G}$-F-contraction if, for all $x, y \in X$ with $(x, y) \in E(\mathcal{G})$, $y \in Tx$ there exists $z \in Ty$ such that

$$d(y,z) > 0 \implies \ell(d(x,y)) + F(d(y,z)) \leq F(M(x,y)) \tag{13}$$

for some $F \in \mathcal{F}$ and $\ell \in \mathcal{L}$.

Let $\{x_n\}$ be a sequence in X then the sequence $\{x_n\}$ is called a $\mathcal{G}$-sequence if $(x_n, x_{n+1}) \in E(\mathcal{G})$ for all $n \in \mathbb{N}$. By $\mathfrak{A}_{\mathcal{G}}$ we denote the class of all $\mathcal{G}$-sequences in X. For the mapping T, we denote by $G_{\mathcal{G}}(T)$ the $\mathcal{G}$-graph of T and

$$G_{\mathcal{G}}(T) = \{(x_n, x_{n+1}) \in X \times X \colon x_1 \in X \text{ and } \{x_n\} \in \mathfrak{A}_{\mathcal{G}} \cap \mathcal{T}(x_1, T)\}.$$

The mapping T is called $\mathcal{G}$-closed if $G_{\mathcal{G}}(T)$ is a closed subset of $X \times X$. The space (X, d) is called $\mathcal{G}$-complete if every Cauchy sequence in $\mathfrak{A}_{\mathcal{G}}$ is convergent in $X \times X$.

One can prove the following theorem by following similar arguments to those in Theorem 3.1.

Theorem 3.2. *Let (X, d) be a metric on X and $\mathcal{G}$ be a graph such that $V(\mathcal{G}) = X$ and $E(\mathcal{G}) \supset \triangle$ and $T \colon X \to C(X)$ be a set-valued $\mathcal{G}$-F-contraction. Suppose the following conditions hold:*

($\mathcal{G}$1) T is d-edge preserving with respect to $\mathcal{G}$;

($\mathcal{G}$2) there exist $x_0, x_1 \in X$ such that $x_1 \in Tx_0$ and $(x_0, x_1) \in E(\mathcal{G})$.

Then there exists a trajectory $\{x_n\} \in \mathfrak{A}_{\mathcal{G}}$ of T starting from x_0 such that $\{x_n\}$ is a Cauchy sequence. In addition, if T is $\mathcal{G}$-closed, then T has an end point.

Next, we state some definitions and results for set-valued cyclic type mappings and extend the assumptions and results of Kirk et al. [16] for the set-valued mappings.

In further discussion, for a nonempty subset A of a set X and a mapping $T \colon X \to C(X)$ we denote $T(A) = \bigcup_{a \in A} Ta$.

Let A and B are two nonempty closed subsets of a metric space (X, d). A mapping $T \colon A \cup B \to C(A \cup B)$ is called a set-valued cyclic-F-contraction if:

(C1) T is cyclic, that is, $Ta \in C(B)$ and $Tb \in C(A)$ for all $a \in A, b \in B$;

(C2) if $(x, y) \in C(T)$, $y \in Tx$, there exists $z \in Ty$ such that

$$d(y,z) > 0 \implies \ell(d(x,y)) + F(d(y,z)) \leq F(M(x,y)) \tag{14}$$

for some $F \in \mathcal{F}$ and $\ell \in \mathcal{L}$, where $C(T) = \Big(A \times T(A)\Big) \cup \Big(B \times T(B)\Big)$.

If T is cyclic, then $C(T)$ is called the cycle of T. Let $\{x_n\}$ be a sequence in X and T be a cyclic mapping, then the sequence $\{x_n\}$ is called a cyclic-sequence with respect to the triple (A, B, T) if it starts in $A \cup B$, that is, $x_1 \in A \cup B$; and $(x_n, x_{n+1}) \in C(T)$ for all $n \in \mathbb{N}$. By $\mathfrak{A}_C$ we denote the class of all cyclic-sequences in X. For the mapping T, we denote by $G_C(T)$ the C-graph of T and

$$G_C(T) = \{(x_n, x_{n+1}) \in C(T) \colon x_1 \in X \text{ and } \{x_n\} \in \mathcal{T}(x_1, T)\}.$$

The mapping T is called a C-closed mapping if $G_C(T)$ is a closed subset of $X \times X$. The space (X, d) is called C-complete if every Cauchy sequence in $\mathfrak{A}_C$ is convergent in $X \times X$.

Theorem 3.3. *Let A and B are two nonempty closed subsets of a C-complete metric space (X, d), where $X = A \cup B$ and $T \colon X \to C(X)$ be a set-valued cyclic-F-contraction. Suppose T is a C-closed mapping, then $A \cap B \neq \emptyset$ and T has a fixed point $u \in A \cap B$.*

Proof. The given nonempty subsets A and B of X induces the function $\alpha\colon X \times X \to \mathbb{R}^+$ as follows:

$$\alpha_C(x,y) = \begin{cases} 1, & \text{if } (x,y) \in C(T); \\ 0, & \text{otherwise.} \end{cases}$$

We shall show that all the conditions of Theorem 2.15 are satisfied with the function α_C. Then, it is easy to verify that the condition (C2) above implies that the T is a set-valued α_C-contraction. Further, if $\alpha_C(x,y) = 1$, $y \in Tx$, $z \in Ty$ then $(x,y) \in C(T)$ and by definition we have the following two possibilities:

(i) $x \in A, y \in T(A)$, $y \in Tx$, $z \in Ty$: In this case, by (C1), $y \in B$ and $z \in T(B)$ and so $(y,z) \in C(T)$, that is, $\alpha_C(y,z) = 1$.

(ii) $x \in B, y \in T(B)$, $y \in Tx$, $z \in Ty$: Then again by (C1), $y \in A$ and $z \in T(A)$ and so $(y,z) \in C(T)$, that is, $\alpha_C(y,z) = 1$.

Thus, T is d-α-admissible and so (a) of Theorem 2.15 holds. Since A, B are nonempty closed subsets of X, and $T\colon A \cup B \to C(A \cup B)$, the condition (b) of Theorem 2.15 trivially holds. Notice that, $G_{\alpha_C}(T) = G_C(T)$, therefore C-closedness implies α_C-closedness of T. The C-completeness implies the α_C-completeness of X. Thus, all the conditions of Theorem 2.15 are satisfied and T has a fixed point u (say). Finally, by (C1) if $u \in A$, then $u \in Tu \subset B$ so $u \in A \cap B$. Similarly, if $u \in B$, again we have $u \in A \cap B$. □

Let (X,d) be a metric space, $\alpha\colon X \times X \to \mathbb{R}^+$ a function and $T\colon X \to X$ be a mapping (single-valued mapping), then T is said to be an F-contraction if for all $x, y \in X$ with $\alpha(x,y) \geq 1$

$$d(Tx,Ty) > 0 \implies \ell(d(x,y)) + \alpha(x,y)F(d(Tx,Ty)) \leq F(M(x,y))$$

for some $\ell \in \mathcal{L}$ and $F \in \mathcal{F}$. The mapping T is called d-α-admissible if

$$\alpha(x,y) \geq 1,\ d(Tx,Ty) < d(x,y) \implies \alpha(Tx,Ty) \geq 1.$$

For $x_0 \in X$ the sequence $\{x_n\}$ is said to be a Picard sequence of T starting at x_0, if $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Notice that, for a set-valued mapping $S\colon X \to C(X)$ if we set $Sx = \{Tx\}$, $x \in X$, then every trajectory of S reduces into a Picard sequence of T. The space (X,d) is called α-regular if for every sequence $\{x_n\} \in \mathfrak{A}_\alpha$ such that $\lim_{n\to\infty} x_n = u \in X$, there exists $n_0 \in \mathbb{N}$ such that $\alpha(x_n,u) \geq 1$ for all $n > n_0$. For $x,y \in X$, an α-chain from x to y in X is a sequence $\{y_i\}_{i=0}^p$ such that $x = y_0, y = y_p$, $\alpha(y_i,y_{i+1}) \geq 1$, $i = 0,1,\ldots,p$ and $p \in \mathbb{N}$. A set $A \subset X$ is called α-chainable if for every two points $x, y \in A$ there exists an α-chain from x to y in X.

Finally, we give the following single-valued version of our main theorem (Theorem 2.15), which will be useful in proving the applications to the existence and uniqueness of solution of a particular type of integral equations.

Theorem 3.4. *Let (X,d) be an α-complete metric space and $T\colon X \to C(X)$ be an α-F-contraction. Suppose the following conditions hold:*

(a) T is d-α-admissible;

(b) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(c) X is α-regular.

Then there exists a Picard sequence $\{x_n\} \in \mathfrak{A}_\alpha$ of T starting from x_0 such that $\{x_n\}$ is a Cauchy sequence and converges to some $u \in X$ and u is a fixed point of T. In addition, if $Fix(T)$ the set all fixed points of T is α-chainable then T has a unique fixed point.

Proof. The existence of Picard sequence $\{x_n\} \in \mathfrak{A}_\alpha$ such that $\{x_n\}$ is a Cauchy sequence, follows if set: $S\colon X \to C(X)$ and $Sx = \{Tx\}$ for all $x \in X$ in Theorem 2.15. For the existence of fixed point, since X is α-complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. By property (c), there exists $n_0 \in \mathbb{N}$ such that

$\alpha(x_n, u) \geq 1$ for all $n > n_0$. Without loss of generality, we may assume that $d(Tx_n, Tu) > 0$ for all $n > n_0$. Because T is an α-F-contraction, for every $n > n_0$ we have

$$F(d(Tx_n, Tu)) \leq F(d(x_n, u)) - \ell(d(x_n, u)) < F(d(x_n, u)).$$

By $(F1)$ we have $d(Tx_n, Tu) < d(x_n, u)$. Now,

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) = d(u, x_n) + d(Tx_n, Tu) \\ &< d(u, x_n) + d(x_n, u). \end{aligned}$$

Since $\lim_{n\to\infty} x_n = u$, letting $n \to \infty$ in the above inequality, we obtain $d(u, Tu) = 0$, that is, $Tu = u$. Suppose $\mathrm{Fix}(T)$ is α-chainable, $u, v \in\mathrm{Fix}(T)$ and $d(u, v) > 0$. Then there exists a sequence $\{y_i\}_{i=0}^{p}$ such that $y_0 = u, y_p = v$ and $\alpha(y_i, y_{i+1}) \geq 1$ for $i = 0, 1, \dots, p$. Without loss of generality we may assume that $d(Ty_i, Ty_{i+1}) > 0$ for $i = 0, 1, \dots, p$. Since T is an α-F-contraction we have

$$F(d(Ty_i, Ty_{i+1})) \leq d(y_i, y_{i+1}) - \ell(d(y_i, y_{i+1})) < d(y_i, y_{i+1})$$

and by d-α-admissibility of T, we have $\alpha(Ty_i, Ty_{i+1}) \geq 1$. Similarly, we obtain $\alpha(T^n y_i, T^n y_{i+1}) \geq 1$ for all $n \in \mathbb{N}$ and $i = 0, 1, \dots, p$. Denote by $d_n^i = d(T^n y_i, T^n y_{i+1})$ Then for any $n \in \mathbb{N}$ and $i = 0, 1, \dots, p$ we have

$$F(d_n^i) \leq F(d_{n-1}^i) - \ell(d_{n-1}^i),$$

which with (F1) shows that $\{d_n^i\}$ is a decreasing sequence which is bounded below. Now following similar arguments to those in Theorem 2.15 we obtain $\lim_{n\to\infty} d_n^i = 0$ for $i = 0, 1, \dots, p$. Therefore,

$$d(u, v) = d(T^n u, T^n v) = d(T^n y_0, T^n y_p) \leq \sum_{i=0}^{p-1} d_n^i.$$

Letting $n \to \infty$ we obtain $d(u, v) = 0$. This contradiction shows that $u = v$, which completes the proof. □

4. Application to Integral Equations

We consider the following integral equation:

$$u(t) = \beta A(u(t)) + \gamma B(u(t)) + g(t), \ t \in [0, T], \ T > 0, \tag{15}$$

where

$$A(u(t)) = \int_0^t K_1(t, s, u(s))ds, B(u(t)) = \int_0^t K_2(t, s, u(s))ds \quad \text{and} \quad \beta, \gamma \geq 0.$$

Let $C(I, \mathbb{R})$ be the space of all continuous functions on I, where $I = [0, T]$ with the norm $\|u\| = \sup_{t\in I} | u(t) |$ and the metric $d(u, v) = \sup_{t\in I} | u(t) - v(t) |$ for all $u, v \in C(I, \mathbb{R})$. For $r > 0$ and $u \in C(I, \mathbb{R})$ we denote by $B(u, r) = \{v \in C(I, \mathbb{R})\colon d(u, v) \leq r\}$ the closed ball centered at u and of radius r.

Theorem 4.1. *Let $r > 0$ be a fixed real and the following conditions are satisfied:*

(i) $K\colon I \times I \times \mathbb{R} \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are continuous;

(ii) there exists $u_0 \in C(I, \mathbb{R})$ such that $\beta A(u_0(t)) + \gamma B(u_0(t)) + g(t) \in B(u_0, r)$;

(iii) If $v \in B(u, r)$, then

$$| K_i(t, s, u(s)) - K_i(t, s, v(s)) | \leq L_i(t, s, u(s), v(s)) | u(s) - v(s) |, \ i = 1, 2$$

for all $t, s \in I, u, v \in \mathbb{R}$ and for some continuous functions $L_1, L_2\colon I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$.

If there exists $\tau > 0$ such that $L_i(t, s, u(s), v(s))(\beta + \gamma)T \leq e^{-\tau}$, $i = 1, 2$ for all $s, t \in I$, then the integral equation (15) *has a unique solution.*

Proof. Define a function $\alpha_r\colon C(I,\mathbb{R})\times C(I,\mathbb{R})\to\mathbb{R}^+$ by

$$\alpha_r(u,v)=\begin{cases}1, & \text{if } v\in B(u,r);\\ 0, & \text{otherwise.}\end{cases}$$

Note that $(C(I,\mathbb{R}),d)$ is a complete metric space. Define $T\colon C(I,\mathbb{R})\to C(I,\mathbb{R})$ by

$$T(u)t=\beta A(u(t))+\gamma B(u(t))+g(t),\ t\in I.$$

If $\alpha_r(u,v)=1$, that is, $v\in B(u,r)$, then by the definition of T and (iii) we have

$$\begin{aligned}
d(Tu,Tv) &= \sup_{t\in I}\mid\beta A(u(t))+\gamma B(u(t))-\beta A(v(t))-\gamma B(v(t))\mid\\
&= \sup_{t\in I}\left|\beta\int_0^t[K_1(t,s,u(s))-K_1(t,s,v(s))]ds\right.\\
&\quad\left.+\gamma\int_0^t[K_2(t,s,u(s))-K_2(t,s,v(s))]ds\right|\\
&\le \sup_{t\in I}\left\{\beta\int_0^t\mid K_1(t,s,u(s))-K_1(t,s,v(s))\mid ds\right.\\
&\quad\left.+\gamma\int_0^t\mid K_2(t,s,u(s))-K_2(t,s,v(s))\mid ds\right\}\\
&\le \sup_{t\in I}\left\{\beta\int_0^t L_1(t,s,u(s),v(s))\mid u(s)-v(s)\mid ds\right.\\
&\quad\left.+\gamma\int_0^t L_2(t,s,u(s),v(s))\mid u(s)-v(s)\mid ds\right\}\\
&\le d(u,v)\sup_{t\in I}\left\{\beta\int_0^t L_1(t,s,u(s),v(s))ds\ +\gamma\int_0^t L_2(t,s,u(s),v(s))ds\right\}\\
&\le d(u,v)\sup_{t\in I}\left\{\beta\int_0^t\frac{e^{-\tau}}{(\beta+\gamma)T}ds\ +\gamma\int_0^t\frac{e^{-\tau}}{(\beta+\gamma)T}ds\right\}\\
&\le d(u,v)\sup_{t\in I}\left\{\frac{e^{-\tau}}{T}t\right\}.
\end{aligned}$$

This implies that

$$d(Tu,Tv)\le e^{-\tau}d(u,v). \tag{16}$$

Because $\alpha_r(u,v)=1$, by passing logarithm we obtain from the inequality (16) that

$$\tau+\alpha_r(u,v)\ln(d(Tu,Tv))\le\ln(d(u,v)).$$

Thus, T is an α_r-F-contraction with $F(t)=\ln t$ and $\ell(t)=\tau$ for all $t\in\mathbb{R}_+$. Now we shall show that T is d-α_r-admissible. If $\alpha_r(u,v)=1$, that is, $v\in B(u,r)$, then by the inequality (16) we have

$$d(Tu,Tv)\le e^{-\tau}d(u,v)<d(u,v)\le r.$$

This implies that $Tv\in B(Tu,r)$, that is, $\alpha_r(Tu,Tv)=1$. Therefore, T is d-α_r-admissible. We shall show that $C(I,\mathbb{R})$ is α_r-regular. Suppose $\{x_n\}$ be a convergent sequence in $C(I,\mathbb{R})$ and $\lim_{n\to\infty}x_n=u$. Then there exists n_0 such that $d(x_n,x_m)<r$ for all $m>n>n_0$, that is,

$$x_m\in B(x_n,r)\quad\text{for all}\quad m>n>n_0.$$

Since $B(x_n, r)$ is closed, letting $m \to \infty$ in the above inequality we obtain $u \in B(x_n, r)$ for all $n > n_0$, that is, $\alpha_r(x_n, u) = 1$ for all $n > n_0$. Thus, $C(I, \mathbb{R})$ is α_r-regular. By Theorem 3.4, T has a fixed point which is a solution of integral equation (15).

Finally, it remains to show the uniqueness of solution. We shall show that the space $C(I, \mathbb{R})$ is α_r-chainable. Suppose $u, v \in C(I, \mathbb{R})$ and $R = d(u, v) > 0$. If $R \leq r$, then we have $v \in B(u, r)$, that is, $\alpha(u, v) = 1$, and so $\{y_i\}_{i=0}^{1}$, where $y_0 = u, y_1 = v$ is an α_r-chain from u to v. If $r < R$, then there exists $p \in \mathbb{N}$ such that $\frac{R}{p} \leq r$. Now consider the sequence $\{y_i\}_{i=0}^{p}$ such that $y_i = u + \frac{i}{p}(v - u) \in C(I, \mathbb{R})$. Then for $i = 0, 1, \ldots, p$ we have

$$d(y_i, y_{i+1}) = \sup_{t \in I} \{| \; y_i - y_{i+1} \; |\} = \sup_{t \in I} \left\{ | \; \frac{u}{p} - \frac{v}{p} \; | \right\} = \frac{d(u, v)}{r} = \frac{R}{p} \leq r.$$

So, $y_{i+1} \in B(x_i, r)$ for $i = 0, 1, \ldots, p$. Therefore, by definition of α_r, the sequence $\{y_i\}_{i=0}^{p}$ is an α_r-chain from u to v. Therefore, $C(I, \mathbb{R})$ is α_r-chainable and so $\mathrm{Fix}(T)$ is also α_r-chainable. Thus, by Theorem 3.4 the solution of integral equation (15) is unique. □

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